Evaluation of Cascade Angular and Radial Distributions from Their Moments

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The amount of information contained in the moments of the angular and radial distribution functions of an electron-photon cascade is investigated. Methods of reconstructing the distribution functions from their moments are analyzed. It is shown that under the Landau approximation the moments determine accurately the angular distribution for all values of $E\theta/E_s$ greater than 0.5, and the radial distribution for all values of Er/E_s greater than 3.

1. PHYSICAL SIGNIFICANCE OF THE MOMENTS

THE evaluation of the radial and angular distribution functions for the particles in an electron-photon cascade has proved to be an extremely difficult problem. Six different calculations have been made of the angular distribution[1–7] and three of the radial distribution.[1,4,7] The performance of these calculations has required the use of rather severe approximations such as highly approximate cascade cross section and, in all cases except that of reference 6, the Landau approximation,[6] while the length of the calculations involved has restricted their application to a few discrete values of the "age" parameter s. On the other hand, the calculation of the moments of these distributions is a straightforward process[8–10] when the only approximations made are the use of the asymptotic cross sections for bremsstrahlung and pair production and the assumption that all angles involved are small.

As the moments of the angular and radial distribution functions can be obtained so very much more easily than can the functions themselves, and as the use of fewer approximations enables them to be evaluated to greater precision than the functions, it is obviously worthwhile paying considerable attention to the problem of extracting from the moments as much information as possible concerning the functions.

The second moments, both angular and radial, have an immediate significance in that they give us a rough estimate of the average spread of the shower particles. It is possible, in principle, to measure the second moments experimentally, but we must note two difficulties here. Firstly, if one wishes to make a measurement of the second radial moment of the particles in a shower, without being given any information as to the shape of the radial distribution, then it is necessary to spread the measuring equipment over an area many times greater than the expected mean square spread. This is because the relatively few particles in the "tail" of the distribution may make quite a large contribution to the moment. Secondly, one must measure only those particles whose energies are an order of magnitude greater than the characteristic scattering energy $E_s = 21$ Mev, for only at these energies will particles in the tail of the distribution still satisfy the small-angle approximation. Obviously these sources of error become much worse when we try to measure the higher moments.

Another factor which limits the applicability of the higher moments of the distributions is that they depend upon, not only the lower moments, but the higher moments of the elastic scattering cross section as well, and these are not at all well known as they depend quite sensitively upon the charge distribution within the nucleus. In the Landau approximation these moments are simply put equal to zero. Therefore the higher angular and radial moments of the distributions as calculated for various models such as those of Molière,[1] Belenky,[2] and Kalos and Blatt[5] under the Landau approximation have little or no direct physical significance at all.

Thus we see that the moments are of very little practical interest in themselves, and their usefulness must be assessed on their ability to provide us with information concerning the distribution functions. It is the purpose of this paper firstly to discover just how much information about the distribution functions is contained in the moments, and secondly to find a simple way of evaluating the functions from their moments.

Where actual numerical values of the moments are required we shall use values computed under the Landau approximation. As shown elsewhere,[6] this approximation is considerably less accurate than has hitherto been supposed, even at small angles, and it becomes extremely inaccurate at large angles where, as we shall see, the moments are most important. However, as more accurate values of the moments are not available, we have no choice but to use the Landau

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[1] G. Molière in *Cosmic Raditaion*, edited by W. Heisenberg (Dover Publications, New York, 1946), first edition.
[2] S. Belenky, J. Phys. U.S.S.R. **8**, 347 (1944).
[3] J. Nishimura and K. Kamata, Progr. Theoret. Phys. (Japan) **6**, 262 (1951).
[4] L. Eyges and S. Fernbach, Phys. Rev. **82**, 123 (1951).
[5] M. H. Kalos and J. M. Blatt, Australian J. Phys. **7**, 543 (1954).
[6] B. A. Chartres and H. Messel, Phys. Rev. **104**, 517 (1956).
[7] J. Nishimura and K. Kamata, Progr. Theoret. Phys. (Japan) **6**, 628 (1951).
[8] H. S. Green and H. Messel, Phys. Rev. **88**, 331 (1952).
[9] B. A. Chartres and H. Messel, Proc. Phys. Soc. (London) **A67**, 158 (1954).
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approximation. It is hoped that this paper will assist in the assessment of the usefulness of the moments and will indicate the desirability, or otherwise of repeating their calculation without the Landau approximation.

2. MOMENT GENERATING FUNCTION

The most direct way of recovering the distribution function from its moments is by use of the moment generating function, that is, the Hankel transform of the distribution function (or Fourier transform in the case of the distribution of projected angles). This method is of particular interest because it is the only method which sets out to define the distribution function completely in terms of its moments alone. In those cases when it is applicable it also provides a proof that the moments are sufficient in themselves to uniquely determine the function.

Consider a function

$$f(x)xdx, \tag{1}$$

its even moments

$$f_n=\int_0^\infty x^{2n}f(x)xdx, \tag{2}$$

and its Hankel transform

$$g(y)=\int_0^\infty J_0(xy)f(x)xdx=\sum_{n=0}^{\infty}\frac{(-y^2)^n f_n}{4^n(n!)^2}. \tag{3}$$

If we are given all the moments f_n we can, in principle, construct the function $g(y)$ from its power series and invert the transform to find $f(x)$.

$$f(x)=\int_0^\infty J_0(xy)g(y)ydy. \tag{4}$$

Firstly, we must note that we cannot simplify this procedure by inverting the series expansion of g term by term. Doing this gives us a purely formal expansion of $f(x)$ in a series of derivatives of the delta function.

In the rather unlikely case that the series in equation (3) can be summed analytically we have no trouble in finding $f(x)$, but otherwise we must consider the restriction imposed on this method by the possibility of this series having only a finite radius of convergence. The radius of convergence y_0 is defined by

$$y_0^2=4\lim_{n\to\infty}\left[(n+1)^2\frac{f_n}{f_{n+1}}\right]. \tag{5}$$

So it is apparent that we must know the asymptotic behavior of f_n quite accurately before we can estimate the radius of convergence. Let us consider some particular examples.

To find what the radius of convergence would be if we applied this method to the reconstruction of the track-length angular distribution of the electron-photon cascade, we shall use the Tamm-Belenky model.[11] This model gives for the track-length angular distribution under the Landau approximation[2]:

$$f(x)=e^{-x}, \tag{6}$$

where

$$x=2q^{\frac{1}{2}}E\theta/E_s, \quad q=2.289.$$

The moments of this distribution are

$$f_n=(2n+1)!, \tag{6a}$$

whence

$$y_0=1. \tag{7}$$

Now although the series, Eq. (3), can in principle be summed for all values of y less than y_0, a large number of terms will be required if y is of the same order of magnitude as y_0. We can, then, say that a calculation of the Hankel transform $g(y)$ by means of the power series is feasible only for y much less than y_0. The inverse of this function can therefore be found fairly accurately for values of x much greater than $1/y_0$. Using the Tamm-Belenky moments again we see that the track-length angular distribution can be found only in the range

$$E\theta\gg E_s/2\sqrt{q}, \tag{8}$$

that is, at values of the angle much greater than the half-width of the distribution. In other words, only the tail of the distribution can be found in this way.

If we do not use the Landau approximation, the higher angular and radial moments are increased. We can no longer make a simple estimate of the radius of convergence because it depends upon the asymptotic form of the higher moments of the elastic scattering cross section. But as the true moments increase with n much more rapidly than they do under the Landau approximation we can be certain that the radius of convergence will be decreased. Consequently the minimum value of θ for which the track-length angular distribution can be found by this technique will be even further out in the tail of the distribution.

To consider the possibility of recovering the radial distribution in this way, we shall use the interpolation formula for the track-length radial moments given by Kalos[12] who has for the nth radial moment

$$\langle(Er)^{2n}\rangle=1\cdot 10(0.09372E_s^2)^n[(2n)!]^2. \tag{9}$$

The radius of convergence is therefore zero; so this method can yield us no information at all about the track-length radial distribution.

This is a result which has important implications. The fact that this series expansion of the Hankel transform of the radial distribution has zero radius of convergence means that this transformed function has a singularity of some type at the origin which prevents us from expressing it as a power series. It follows that the first few terms of such a series do not give even a crude approximation to the form of this function at the origin. Now, in his calculation of the radial distribution of the

[11] I. Tamm and S. Belenky, J. Phys. U.S.S.R. 1, 177 (1939).

[12] M. H. Kalos, Ph.D. thesis, University of Illinois, 1952 (unpublished).

electron-photon cascade, Molière[1] commenced his numerical evaluation of the transformed function at the origin, using the first few terms of the power series expansion to define it there. This means that this transformed function is completely wrong at the origin, thereby invalidating his radial distribution at large values of r. The agreement between his radial distribution and that of Nishimura and Kamata[7] at small values of r indicates that this initial error was submerged in the numerical development of the function towards smaller values of r.

3. USE OF THE MELLIN TRANSFORM

As the moment-generating-function method, the only available method which sets out to define the distribution function in terms of its moments alone, is unsatisfactory for the angular distribution and entirely useless for the radial distribution, we are forced to resort to a method which uses, in addition to the moments, some further knowledge of the behavior of the function. This additional information normally consists of a knowledge (or guess) of the general over-all behavior of the distribution, and is typically expressed in the form of a "trial function" which is to be suitably modified to bring it into agreement with the known moments.

Two different approaches based on this general pattern are typified by the method suggested by Green and Messel[13,14] in which the trial function is brought into agreement with n moments by multiplying it by a polynomial of degree n, and the method used by Spencer[15,16] for the problems of electron and x-ray diffusion in which he uses a series of trial functions each of the same form and differing only by a scaling factor. In problems where the general form of the distribution is well known and the trial function is expected to be fairly accurate by itself, these methods are quite suitable. But for our present purpose they suffer from the disadvantage that they cannot tell us to what extent the distribution so obtained is dependent upon the particular choice of trial function or of the method of modifying it. Hence they cannot help us in our endeavor to assess the amount of information that is actually contained in the moments.

A method of reconstructing the function from its moments that is suitable for our purposes must have the following properties:

(1) There should be no need to postulate a knowledge of the general behavior of the function, apart, of course, from the fact that it is continuous and monotonically decreasing. This means that, in so far as a trial function is required, it should be derivable from the moments themselves.

(2) It should be possible to assess the contribution made to the distribution function by each individual moment. This is necessary to enable us to find out how many moments should be evaluated and to see what degree of accuracy is required in the calculation of each moment.

The method about to be described satisfies these criteria.

Define the Mellin transform

$$h(s)=\int_0^\infty x^{2s}f(x)xdx, \qquad (10)$$

which has the inverse

$$f(x)=1/(\pi i)\int_{s_0-i\infty}^{s_0+i\infty} x^{-2(s+1)}h(s)ds. \qquad (11)$$

For integral values n of s we have

$$h(n)=f_n, \qquad (10a)$$

so a knowledge of the moments defines the Mellin transform at positive integral values of its argument.

In order to carry out the inverse transformation, Eq. (11), it is necessary first to extend our knowledge of the function to other values of s. We have two choices of approach here. We can either fit the known values of the function by the adjustment of suitable parameters in a function (or series of functions) whose inversion can be performed analytically, or we can perform a straight out interpolation of the function between the integral values and find the inversion by the saddle point method. The first method is of the type used by Green and Messel and Spencer which we have already discussed and classed as unsuitable for the problem in hand. We shall now see that the saddle point method is highly satisfactory and is furthermore very informative as to the actual physical significance of the moments. The application of this method to cascades is due to Nishimura and Kamata,[17] but they give no discussion of its accuracy nor of its range of application.

The interpolation that is required for the saddle point method needs to be done rather accurately as the distribution function varies inversely as the square root of the second derivative of the transformed function. However an accurate interpolation is facilitated by the fact that interpolation functions, obtained from the moments themselves, already exist—see Eqs. (6) and (9). These functions play the same role in this method that the "trial function" plays in other methods, so we see that we are essentially using only the moments alone and are not assuming any additional knowledge of the behavior of the distribution.

The interesting thing about the saddle point method, and it is this that makes it peculiarly suitable to our problem, is that it derives a particular point of the dis-

[13] H. S. Green and H. Messel, Phys. Rev. **87**, 738 (1952).
[14] H. S. Green and H. Messel, Quart. Appl. Math. **11**, 403 (1954).
[15] L. V. Spencer, Phys. Rev. **88**, 793 (1952).
[16] L. V. Spencer, Phys. Rev. **98**, 1597 (1955).
[17] J. Nishimura and K. Kamata, Progr. Theoret. Phys. (Japan) **7**, 185 (1952).

tribution from a corresponding point of the Mellin transformed function, and consequently from a particular moment. Thus in deriving the function in this way we automatically discover which of the moments are the most important ones in determining the function in any particular range of angles or radii.

We write

$$h(s)=\sigma(s)e^{h_0(s)}, \tag{12}$$

where $\exp[h_0(s)]$ is the interpolation function (assumed given for all s) and $\sigma(s)$ is the correction function, known only for integral values of s but assumed to be a slowly varying continuous function.

Now the saddle point method of evaluating an integral consists of writing

$$1/(\pi i)\int_{s_0-i\infty}^{s_0+i\infty}\exp[H(s)]ds=(2/\pi)^{\frac{1}{2}}\frac{\exp[H(s)]}{[H''(s)]^{\frac{1}{2}}}, \tag{13}$$

where s is chosen so that

$$H'(s)=0. \tag{13a}$$

If, then, we neglect the derivatives of $\sigma(s)$ as being small compared to those of the interpolation function we have

$$f(x)=1/(\pi i)\int_{s_0-i\infty}^{s_0+i\infty}x^{-2(s+1)}\sigma(s)\exp[h_0(s)]ds$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{x^{-2(s+1)}h(s)}{[h_0''(s)]^{\frac{1}{2}}}, \tag{14}$$

with

$$h_0'(s)=2\ln x. \tag{14a}$$

To perform the calculation we choose integral values of s, for which $h(s)=f_s$ is known, calculate x from Eq. (14a) and $f(x)$ from Eq. (14). If further values of $f(x)$ are required we need only interpolate for $h(s)$ to the same degree of accuracy that is required in $f(x)$. It must be noted that the accuracy of this method depends throughout upon the availability of an accurate interpolation function.

When the inverse transform of the interpolation formula can be found analytically the calculation becomes even simpler. If $f_0(x)$ is the inverse transform of $\exp[h_0(s)]$, that is, if $f_0(x)$ is the distribution function whose moments are given exactly by the interpolation formula, then Eq. (14) reduces to

$$f(x)=f_0(x)\sigma(s), \tag{15}$$

where s and x are still connected by Eq. (14a).

The physical principle underlying the above described method can be best appreciated by the following argument.[18] Suppose we have obtained an approximation to the distribution function, denoted by $f_0(x)$. We wish to modify this function so as to make its moments agree with the known values of the true function. Let us use Eq. (15) for this modification, i.e., we wish to find the function $\sigma(s)$. Let us look at the expression for the $(2n)$th moment of $f_0(x)$.

$$f_{0,n}=\int_0^\infty x^{2n}f_0(x)xdx=\exp[h_0(n)]. \tag{16}$$

Now $f_0(x)$ is a monotonically decreasing function and x^{2n} is a monotonically increasing function of x, so ideally we should find that the integrand in Eq. (16) has a sharp maximum at some value of x, say at $x=x_n$. Then the magnitude of the $(2n)$th moment depends almost entirely upon the value of $f_0(x)$ at and near the point x_n. Consequently the $(2n)$th moment of $f_0(x)$ can be adjusted simply by multiplying the function at, and near, the point x_n by the ratio of the true moment and the moment of $f_0(x)$. That is, we define $f(x)$ by Eq. (15), with $\sigma(s)$ defined by Eq. (12), but now instead of Eq. (14a) we relate s and x by $x=x_s$.

If now we replace n in Eq. (16) by s and then differentiate with respect to s, we obtain

$$h_s'(s)=\int_0^\infty 2\ln x x^{2s}f_0(x)xdx\bigg/\int_0^\infty x^{2s}f_0(x)xdx. \tag{17}$$

If the integrand in Eq. (16) has a sharp maximum, then the integrands in both the numerator and denominator of Eq. (17) have sharp maxima at very nearly the same value of x. Hence Eq. (17) reduces to Eq. (14a) and the two methods are equivalent. The requirement that the integrand in Eq. (16) have a sharp maximum is exactly equivalent to the underlying approximation inherent in the saddle point method, which is that the integrand should have a sharp minimum along the real axis. This is a fairly rigorous requirement, but in practice it does not appear to affect the accuracy of the method except for the cases $n=0$ and, perhaps $n=1$, as we shall see later.

We can now also see what role the moment interpolation function [or the equivalent trial function $f_0(x)$] plays. It determines the relation between s and x, as seen by Eqs. (14a) and (17). Thus the interpolation function tells us at what value of x the distribution makes the most contribution to the $(2n)$th moment, and then the moment itself is used to adjust the value of the function at this point.

We have calculated both the track-length angular and radial distributions of an electron-photon cascade by the above method. The *projected* angular distribution was chosen because the numerical results of a calculation of this function by Kalos and Blatt[5] were readily available for comparison. This calculation used "super-simple"[19] cross sections so we have used as "exact" moments the moments of the super-simple model as calculated by Kalos.[12] For the moment interpolation

[18] I am indebted to Dr. J. M. Blatt for the suggestion of this approach.

[19] F. L. Friedman, Massachusetts Institute of Technology Technical Report No. 31, 1949 (unpublished).

function we have used the formula

$$\exp[h_0(n)] = (2n)!/(4q_1)^n, \qquad (18)$$

where

$$q_1 = 2.145.$$

This expression was deduced from the projected moments of the Tamm-Belenky model (renormalized to agree with the super-simple model—hence the new value for the parameter q) which are greater than Eq. (18) by the factor

$$2[(n+1)/\pi]^{\frac{1}{2}}.$$

The expression we have used does not give as good a second moment but yields much more accurate higher moments. For the variable x we have used

$$x = E\theta/E_s. \qquad (19)$$

The results of this calculation of the projected angular distribution are given in Table I and Fig. 1. The solid line in Fig. 1 is the projected track-length angular distribution calculated by Kalos and Blatt. The values calculated by us from the moments are entered as circled dots. It can be seen that our calculation agrees almost exactly with that of Kalos and Blatt. We also see that these few discrete points, although widely spaced, are sufficient to define the angular distribution at all points except in the region,

$$E\theta/E_s < \tfrac{1}{2}.$$

The remarkably close agreement between the angular distribution of Kalos and Blatt and our distribution obtained from the moments provides a verification both of the accuracy of the Kalos and Blatt distribution[20] and of the validity of the method we have used to recover the distribution from the moments.

This method of reconstructing the distribution function from its moments must fail at small values of the variable, because in this region the integrand in the Mellin transform inversion integral does not have a sufficiently sharp minimum to allow the saddle point method to be used.

For the calculation of the track-length *radial* distribution (not projected) by this method, we have used for the moment interpolation formula Kalos' expression

$$\exp[h_0(n)] = 1.10(0.09372)^n[(2n)!]^2, \qquad (20)$$

where

$$x = Er/E_s, \qquad (21)$$

while for the "exact" moments we have used the moments of Approximation-A.[4,12] The results of this calculation are recorded in Table II and Fig. 2. The full line in Fig. 2 is the track-length radial distribution as calculated by Eyges and Fernbach[4] who also used a method of reconstructing the function from its moments. It can be seen that our points agree perfectly with their curve. The broken line is a continuation of the Eyges and Fernbach curve drawn so as to pass through our points.

The agreement between our points at $n=0$ and $n=1$ with the result of Eyges and Fernbach is certainly

TABLE I. Calculation under the Landau approximation of the projected track-length angular distribution of the super-simple model from its moments.

n	"Exact" moment	Interpolation formula	$\sigma(n)$	$E\theta/E_s$	$f(E\theta/E_s)$
0	1.000	1.000	1.00	0.192	0.812
1	0.300	0.233	1.29	0.859	0.160
2	0.406	0.326	1.25	1.539	2.27×10^{-2}
3	1.191	1.140	1.04	2.221	2.77×10^{-3}
4	5.940	7.440	0.798	2.904	3.06×10^{-4}

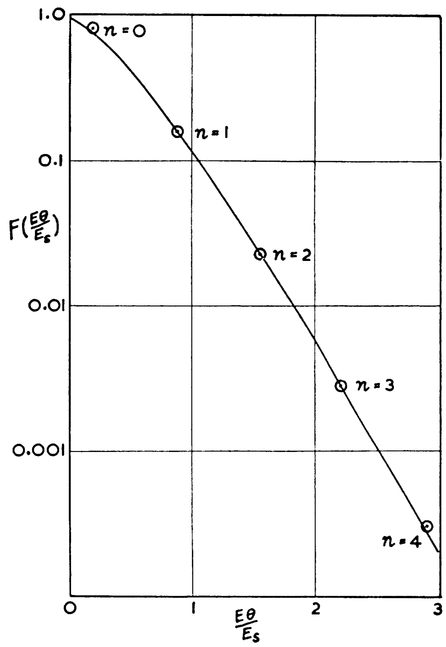

FIG. 1. Calculation under the Landau approximation of the projected track-length angular distribution of the super-simple model from its moments. Each circled dot is the value of the function as calculated from the $(2n)$th moment with the specified value of n. The curve is the distribution as calculated by Kalos and Blatt.[5]

TABLE II. Calculation under the Landau approximation of the track-length radial distribution of Approximation-A from its moments.

n	"Exact" moment	Interpolation formula	$\sigma(n)$	Er/E_s	$f(Er/E_s)$
0	1.00	1.10	0.91	0.096	23.6
1	0.72	0.41	1.76	1.94	2.29×10^{-2}
2	7.20	5.55	1.30	6.22	7.43×10^{-5}
3	4.93×10^{2}	4.68×10^{2}	1.05	13.0	4.46×10^{-7}
4	1.38×10^{5}	1.38×10^{5}	1.00	22.1	4.01×10^{-9}
5	1.04×10^{8}	1.05×10^{8}	0.99	33.7	4.43×10^{-11}
6	1.71×10^{11}	1.71×10^{11}	1.00	47.8	5.32×10^{-13}

[20] The distribution is, of course, invalidated by its dependence on the Landau approximation. By "accuracy" here we mean that there has been no additional error introduced in its derivation.

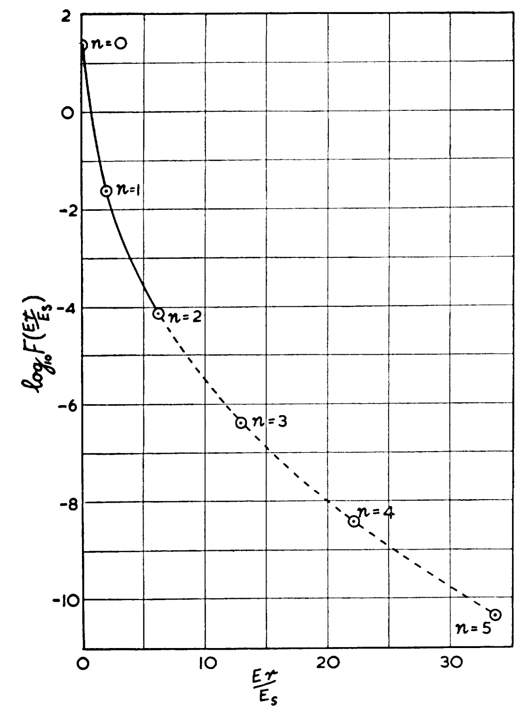

FIG. 2. Calculation under the Landau approximation of the track-length radial distribution of electrons in an electron-photon cascade from its moments. Each circled dot is the value of the function as calculated from the $(2n)$th moment with the specified value of n. The full line is the distribution as calculated by Eyges and Fernbach.[4]

heatening, but is not sufficient to guarantee the accuracy of the function in this region. Our value at $n=0$ is very doubtful because the saddle point method is not at all accurate here. Furthermore, the rather rapid fluctuation of the moment correction function $\sigma(n)$ between $n=0$ and $n=1$ throws doubt upon the accuracy of both these points. The agreement with Eyges and Fernbach does not allay these doubts because their calculation, being also based upon the moments, is probably open to similar sources of error. However our calculation is valid for all values of Er/E_s greater than about 3, although it must be noted that the particle density at this distance falls to 10^{-3} of its value at the origin so that the behavior of the radial distribution at these large distances is not likely to be of very great physical interest. The distance defined by $Er/E_s=3$ is, for an electron of energy 100 Mev at sea level in air, equal to 175 meters.

It is apparent that this method of obtaining the distribution function from its moments is not nearly as successful with the radial distribution as it is with the angular distribution. Not only is the region of small values of the variable where the function is not accurately determined much greater in the radial distribution than it is in the angular distribution, but also the discrete points at which the distribution is well defined are much further apart. Note that, in going from one point to the next, the angular distribution changes by roughly a factor of 10 whereas the radial distribution moves in steps of 100 to 1.

4. DISCUSSION

We have studied the problem of recovering the cascade angular and radial distribution functions from their moments with particular emphasis on finding out how much information about the functions is actually contained in the moments. We investigated the use of the Hankel (or Fourier) transform as the only method which allows, in principle, of a complete and exact determination of the function from the moments when *all* of the moments are known. However it is only capable of doing this when the power series expansion of the transformed function has an infinite radius of convergence. When the radius of convergence is finite, as in the case of the angular distribution, this method will yield information of the function only in the "tail" of the distribution. If the radius of convergence is zero, as in the case of the cascade radial distribution, the method fails entirely.

On the other hand, we found that the use of the closely related Mellin transform gave a very practical and simple method. It requires a knowledge of a fairly accurate interpolation function for the moments, but this is available for both the cascade angular and radial moments. It is an extremely simple method in its application and has the advantage that it is not limited, as is the Hankel transform method, to any particular class of function. However we find that the same conditions which render the Hankel transform method useless also limit the applicability of the Mellin transform method to points in the extreme tail of the distribution.

In the course of our analysis of these methods we have discovered a true measure of the actual amount of information about the distribution that is contained in the moments. We have seen that each moment contains information about the value of the function only at and near the value of x at which the maximum contribution to that moment arises. Thus the higher moments yield information only about the tail of the distribution. It follows that as the physical significance of the distribution fades away as we go further into the tail so must the importance of the higher moments become less and less. In other words, only a small number of the moments are worth calculating.

Analyses of the importance of the moments which are based entirely upon the Hankel (or Fourier) transform method tend to yield an opposite conclusion, namely that as the behavior of the distribution near the origin depends upon the behavior of the tail of the transformed function, so must it depend upon the values of the higher moments. In fact this reasoning leads one to the conclusion that a knowledge of *all* the moments yields a knowledge of the distribution at *all* values of the variable. But we can now see that this reasoning only applies when the distribution is of such a form that the Hankel transformed function has an infinite radius of convergence. As neither the cascade angular or radial distributions satisfy this criterion, we find that *there is no reason to believe that a knowledge of their moments can ever yield the behavior of the distributions at small values of the variables.*

The calculations we have performed of the track-length angular and radial distributions of the electron-photon cascade under the Landau approximation indicate that the Mellin transform method of reconstructing a function from its moments can be used for the evaluation of these distributions for all values of the angular variable $E\theta/E_s$ greater than 0.5 and of the radial variable Er/E_s greater than about 3. As this is a very simple calculation, it indicates that any future direct evaluation of the angular or radial distribution functions need only be carried out for smaller values of the variables; at the larger values the functions are much more easily obtained from their moments. However, it must be pointed out that these estimates were made using values of the moments calculated under the Landau approximation. The increase in the higher moments that will ensue when this approximation is dropped will increase the minimum angle or radius at which the distribution can be obtained from its moments. The actual magnitude of this increase cannot as yet be estimated.

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Charged-Pion Production in Lithium†*

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This article reports the measurement of charged-pion production cross sections in Li^6 and Li^7. The measurements were made on 40- and 52-Mev pions emitted at 90° to a proton beam of 242-Mev energy. The results show a higher π^+/π^- ratio for Li^6 than for Li^7; this observation can be qualitatively explained by the Pauli exclusion principle.

I. INTRODUCTION

WE have measured the absolute cross sections for the production of charged pions by 242-Mev protons on Li^6 and Li^7. The pions were emitted at 90° to the proton beam and measurements were made at pion energies of 40 and 52 Mev in the laboratory system.

The charge-independence prediction[1] for the production of pions by protons on nuclei of isotopic spin 0 is given by the Watson relation

$$2\sigma_0=\sigma_+ + \sigma_-,$$

where σ_0 is the differential production cross section for neutral pions, and σ_+ and σ_- are the corresponding differential cross sections for positive and negative pions. Since this experiment determines $\sigma_+ + \sigma_-$, a measurement of σ_0 at the same proton energy would constitute a test of the charge independence hypothesis. This prediction assumes, however, that equivalent final nucleon states are available for each of the three modes of meson production. At proton energies not greatly in excess of threshold, the details of nuclear structure may obscure the charge-independence prediction. This experiment was undertaken to obtain new data on pion production in low-Z materials and to explore the utility of the Watson relation as a test of charge independence at the energies available with the Rochester synchrocyclotron.

II. EXPERIMENTAL ARRANGEMENT

The internal proton beam bombarded a 3 in.$\times\frac{3}{16}$ in. $\times\frac{1}{8}$ in. Li target located at a radius of 59 in. in the synchrocyclotron. Pions emitted in the median plane at 90° to the proton beam followed curved trajectories in the fringing field of the cyclotron. Floating-wire measurements were used to design a channel that defined the solid angle and energy interval of the emitted pions. The pions were detected in a scintillation crystal telescope by using pulse-height analysis to separate the pions from the background. The proton beam was measured absolutely using the $C^{12}(p,pn)C^{11}$ reaction from the carbon in a 2-mil polyethylene foil attached to the target.

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‡ Now at RCA Laboratories, Princeton, New Jersey.
[1] K. M. Watson, Phys. Rev. **85**, 852 (1952); A. M. L. Messiah, Phys. Rev. **86**, 430 (1952); J. M. Luttinger, Phys. Rev. **86**, 581 (1952).