Classical Boltzmann Theory of Cyclotron Resonance for Warped Surfaces*

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A classical Boltzmann model of cyclotron resonance with an energy-independent collision time is developed. An approximate solution of the Boltzmann equation is obtained by expanding the perturbed distribution function $\Phi(p,\Theta,\phi)$, in a Fourier series in ϕ, where the spherical coordinates are defined with the polar axis along the magnetic field direction. The solution yields a fundamental cyclotron resonance absorption line, which shows line shape anisotropy, as well as a shift in resonance peak with magnetic field direction. The theory also indicates resonance absorption at harmonics of the fundamental cyclotron resonance frequency, due to the warping of the constant-energy surfaces. The results are applied to a calculation of line shapes and harmonic intensities for heavy holes in silicon and germanium.

INTRODUCTION

CYCLOTRON resonance experiments in semiconductors[1–7] have indicated that a classical description of most of the observed phenomena is reasonable. Electrons in the conduction bands of germanium and silicon can be treated as particles moving in classical orbits, described by constant energy surfaces consisting of ellipsoids of revolution.[2,4,6] Two kinds of holes occur for the valence bands of both germanium and silicon. Each kind of hole can be treated as a classical particle, whose motion can be described in terms of warped spherical constant energy surfaces.[3,5] These models for the energy surfaces have been used to calculate the frequencies of cyclotron resonance motion in a dc magnetic field. By fitting the theoretical expressions for the frequencies to the experimental results, the shapes of the constant-energy surfaces for carriers in germanium and silicon have been determined.[1–7]

In this paper, we shall consider a perturbation treatment of the classical model of cyclotron resonance for warped surfaces, using the Boltzmann transport equation. The application of the Boltzmann equation to dc galvanomagnetic effects in metals and semiconductors has a long history.[8] Jones and Zener,[9] Blochinzev and Nordheim,[10] Davis,[11] and others have discussed the theory of magnetoresistance and Hall effect for ellipsoidal energy surfaces. Jones[12] and Blackman[13] have used such calculations to explain the anisotropy of the de Haas-Van Alphen effect in bismuth. More recently, Meiboom and Abeles[14] and Shibuya[15] used an ellipsoidal model to explain the anisotropy of dc Hall effect and magnetoresistance in n-type germanium.

Margenau,[16] Allis,[17] and others have developed the general Boltzmann theory of rf conductivity and applied it to ionized gases. London,[18] Pippard,[19] and Reuter and Sondheimer[20] have developed Boltzmann theories of rf conductivity for superconductors and metals. More recently, Luttinger and Goodman[21] and McClure[22] have considered the general theory of conductivity for warped surfaces. We shall discuss a Boltzmann treatment of cyclotron resonant absorption of energy, for slightly warped spherical energy surfaces, assuming an energy-independent collision time. The expressions obtained will be applied to the valence bands of germanium and silicon. The results have been compared with experimental data in a previous paper.[23]

I. BOLTZMANN EQUATION

The Boltzmann equation is written in a slightly modified version of the notation of Wilson[24]:

$$[e\mathbf{E}+(e/c)\mathbf{v}\times\mathbf{H}]\nabla_p f+\mathbf{v}\cdot\nabla_r f+\partial f/\partial t=-(f-f_0)/\tau. \quad (1)$$

Here, f is the distribution function, $f(\mathbf{p},\mathbf{r},t;\,\mathbf{H},\mathbf{E})$, normalized so that $\int f d\mathbf{p}=n$, the number of carriers per cc; ∇_p and ∇_r are, respectively, the gradients in momentum space and coordinate space; $\mathbf{v}\equiv\nabla_p\mathcal{E}(\mathbf{p})$; and f_0 is the distribution function in the absence of $\mathbf{E}$ and $\mathbf{H}$. τ is the collision time, which may in general be a function of $\mathcal{E}$.

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We will assume a uniform distribution of carriers in coordinate space, so that $\nabla_r f=0$. $\mathbf{H}$ is assumed constant, f is written in the form $f=f_0-\Phi\cdot(\partial f_0/\partial\mathcal{E})$, and $\mathbf{E}$ and Φ are assumed to have an $e^{j\omega t}$ dependence.

If these are substituted in Eq. (1), and a term of order $E[\Phi(\partial f_0/\partial\mathcal{E})]$ is neglected compared to one of order $E[f_0]$, we obtain

$$\tau e\mathbf{E}\cdot\mathbf{v}-\tau(e/c)(\mathbf{v}\times\mathbf{H})\cdot\nabla_p\Phi-(1+j\omega\tau)\Phi=0. \quad (2)$$

The assumption that the perturbed part of the distribution function, $\Phi(\partial f_0/\partial\mathcal{E})$, is small compared to f_0, holds only if $\mathbf{E}$ is small. This means, physically, that Eq. (2) holds at resonance, only if carriers gain little energy between collisions, and remain on essentially the same constant-energy surface in momentum space.

The current due to the perturbed distribution function is given by an integral over momentum space:

$$\mathbf{J}=e\int f\mathbf{v}d\mathbf{p}=-e\int(\partial f_0/\partial\mathcal{E})\Phi\mathbf{v}d\mathbf{p}. \quad (3)$$

Equations (2) and (3) form the basis for the discussion of cyclotron resonance for warped spherical constant-energy surfaces.

II. WARPED SURFACES

The energy-momentum relation for a warped constant-energy surface is taken to be of the form

$$\mathcal{E}=(p^2/2m)[1+g(\mathbf{p}/p)], \quad (4)$$

where m is an average effective mass, and $g(\mathbf{p}/p)$ is a small, nonspherical term, which is a function of angle only. We introduce spherical coordinates (p,Θ,ϕ) in momentum space, with the direction of $\mathbf{H}$ as the z direction. If we now assume τ independent of energy, and write $\Phi=\chi(\Theta,\phi)pe\tau/m$, then substitution into Eq. (2) yields

$$\mathbf{u}\cdot\mathbf{E}+\omega^0\tau[(1+R)D_\phi+QD_\Theta]\chi-(1+j\omega\tau)\chi=0, \quad (5)$$

where

$$\mathbf{u}(\Theta,\phi)=(m/p)\nabla_p\mathcal{E}=(m/p)\mathbf{v},\quad \omega^0=eH/mc,$$
$$\mathbf{w}(\Theta,\phi)=\mathbf{u}-\hat{p}=[g\hat{p}+\tfrac{1}{2}p\nabla_p g],\quad D_\phi=\partial/\partial\phi,$$
$$Q(\Theta,\phi)=(w_x\sin\phi-w_y\cos\phi)(1/\sin\Theta),$$
$$D_\Theta=\sin\Theta(\sin\Theta+\cos\Theta\partial/\partial\Theta),$$
$$R(\Theta,\phi)=(w_x\cos\phi+w_y\sin\phi)(1/\sin\Theta),\quad \hat{p}\equiv\mathbf{p}/p.$$

An approximate solution of Eq. (5) is obtained, by expanding χ in a Fourier series in ϕ, as

$$\chi=\sum\chi_n(\Theta)\cos n\phi+\chi_{-n}(\Theta)\sin n\phi.$$

This leads to a set of linear equations, in which only χ_n and χ_{-n} are coupled, to first order in g. Substitution of χ into Eq. (3) gives the current, and an approximate expression for the power absorption per unit volume:

$$P=\tfrac{1}{2}\operatorname{Re}(\mathbf{J}\cdot\mathbf{E})$$
$$=\frac{3}{8}\frac{Ne^2}{m}\tau\sum_n\mathbf{E}\cdot\left\{\int\frac{(1+j\omega\tau)(\mathbf{S}_n\mathbf{u}_n+\mathbf{S}_{-n}\mathbf{u}_{-n})\sin\Theta d\Theta}{(1+j\omega\tau)^2+[n\omega^0(1+R_0)]^2\tau^2}\right\}\cdot\mathbf{E}, \quad (6)$$

where N is the number of carriers per cc, and

$$\mathbf{S}\equiv\mathbf{u}(1+g)^{-5/2}\left[\frac{1}{4\pi}\int(1+g)^{-\frac{3}{2}}\sin\Theta d\Theta d\phi\right]^{-1}.$$

In this expression, the subscripts (n) and $(-n)$ refer, respectively, to $\cos n\phi$ and $\sin n\phi$ harmonic components, and $n=0$ refers to a constant term. The quantities appearing under the integral sign are functions of Θ, and of magnetic field direction.

For $\omega\tau\ll1$, resonance occurs for the fundamental cyclotron current $(n=1)$, when

$$\omega=\omega^0(1+R_0). \quad (7)$$

To first order in the warping, this resonance condition is the same as that obtained from the Shockley integral for the cyclotron frequency.[25]

Equation (6) also contains contributions to the current for $n=2, 3, \cdots$. These correspond to resonant absorption of energy at harmonics of the fundamental cyclotron frequency.[26]

III. HOLES IN GERMANIUM AND SILICON

(a) Energy Surfaces

The expression of Eq. (6) for cyclotron resonance in the case of slightly warped spherical energy surfaces, does not depend on the explicit form of the energy-momentum relationship for carriers. However, theoretical considerations of Dresselhaus, Kip and Kittel,[7] Shockley,[27] and others and experimental observations[5,7] indicate that the top of the valence band in both germanium and silicon consists of a pair of warped spherical energy surfaces, degenerate at $k=0$, and given by an expression of the form

$$\mathcal{E}=-\frac{h^2}{2m_0}\times\{Ak^2\pm[B^2k^4+c^2(k_x^2k_y^2+k_y^2k_z^2+k_z^2k_x^2)]^{\frac{1}{2}}\}, \quad (8)$$

in the cubic axis system, where the $(\pm)$ sign refers to the two different bands.

For purposes of a Boltzmann calculation, we shall write

$$\mathcal{E}=-(p^2/2m)[1+g(\mathbf{p}/p)], \quad (9)$$

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where

$$\mathbf{p} \equiv \hbar \mathbf{k},$$

$$m = m_0[A \pm (B^2 + \tfrac{1}{6}C^2)^{\frac{1}{2}}]^{-1},$$

$$g(\mathbf{p}/p) = \pm (B^2 + \tfrac{1}{6}C^2)^{\frac{1}{2}}[A \pm (B^2 + \tfrac{1}{6}C^2)^{\frac{1}{2}}]^{-1}$$

$$\times \left\{ \left[1 - \frac{C^2}{2(B^2 + \frac{1}{6}C^2)} \frac{(p_x^4 + p_y^4 + p_z^4 - \frac{2}{3}p^4)}{p^4} \right]^{\frac{1}{2}} - 1 \right\}.$$

The energy-momentum relation has been artificially written in this form, to reduce the problem to the general form we have considered, with $g(\mathbf{p}/p)$ as small as possible. From these expressions, it is clear that the $(+)$ and $(-)$ sign refer, respectively, to the light and heavy holes.

On the basis of experimental results,[7,23] we may expand the square root in Eq. (9) to first order, with a maximum error of approximately 0.9% for germanium and 7% for silicon. This maximum error occurs only over very small regions on a constant-energy surface, and produces a much smaller effect on the averages appearing in the Boltzmann theory. Then, $g(\mathbf{p}/p)$ in the cubic frame of reference may be reduced to

$$g(\mathbf{p}/p) = \kappa[(p_x/p)^4 + (p_y/p)^4 + (p_z/p)^4 - \tfrac{2}{3}], \quad (10)$$

where

$$\kappa = \frac{\mp C^2}{4(B^2 + \frac{1}{6}C^2)^{\frac{1}{2}}[A \pm (B^2 + \frac{1}{6}C^2)^{\frac{1}{2}}]}.$$

Our task in analyzing the cyclotron resonance of holes is now to evaluate the terms appearing in the expressions for the cyclotron resonance current in spherical coordinates, with the polar axis along $\mathbf{H}$, making use of Eq. (10).

The situation for the light holes is rather simple in

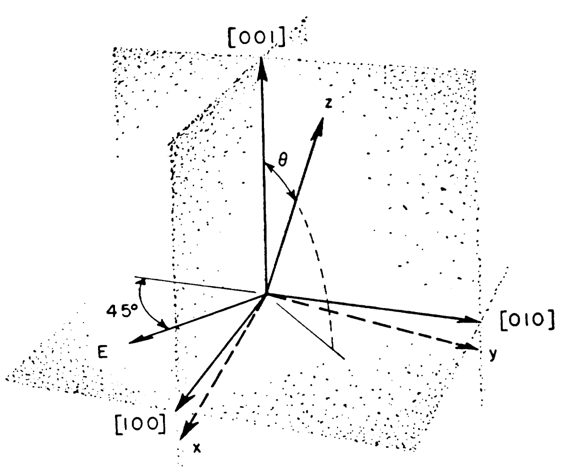


FIG. 1. Coordinate systems used in analyzing cyclotron resonance in germanium and silicon. The crystal axes are labeled [100], [010], and [001]. $(\hat{x}, \hat{y}, \hat{z})$ are "magnetic" coordinate axes, with x and y respectively symmetrical to [100] and [010].

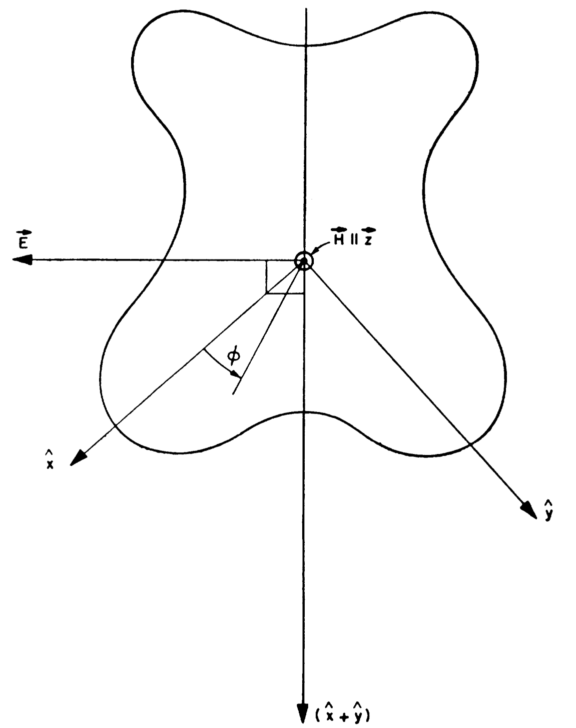


FIG. 2. Schematic, showing the intersection of a plane normal to $\mathbf{H}$, with the warped constant energy surfaces in germanium or silicon. $(\hat{x}, \hat{y}$, and $\hat{z})$, with $\hat{z}$ out of the paper, are magnetic coordinates.

both germanium and silicon. In both these materials, κ is small for the light holes ($\sim$0.16 for germanium, 0.32 for silicon)[23] so that there is a very slight anisotropy of the cyclotron resonant frequency and line shape. Harmonic cyclotron resonance is much too small to observe experimentally.

We shall consider two cases of cyclotron resonance of heavy holes in germanium and silicon: Case (1) *transverse cyclotron resonance:* $\mathbf{E}$ along the $[1\bar{1}0]$ direction, and $\mathbf{H}$ in the plane normal to $\mathbf{E}$ (see Fig. 1); Case (2) *longitudinal cyclotron resonance:* $\mathbf{E}$ in the $(1\bar{1}0)$ plane and $\mathbf{H}$ parallel to $\mathbf{E}$. We shall discuss line shapes of the fundamental high-mass-hole resonance, for Case (1), and the relative intensities of fundamental and harmonic cyclotron resonance for both Cases (1) and (2).

(b) Fundamental Cyclotron Resonance Line Shapes

The magnetic frame of reference $(\hat{x}\hat{y}\hat{z})$ is chosen, as shown in Fig. 1, with the unit vectors $\hat{x}$ and $\hat{y}$ symmetrical with respect to the $(1\bar{1}0)$ plane. In this frame of reference, $\mathbf{E} = (E/\sqrt{2})(\hat{x} - \hat{y})$. (See Fig. 2.) Using Eq. (6) for the cyclotron power absorption and making use of simple relations which follow from mirror plane symmetry with respect to the $(1\bar{1}0)$ plane, expressions for the power absorbed in transverse cyclotron resonance have been obtained. For the fundamental and first 2 harmonics,

$$P_1=-\frac{3}{8}\frac{Ne^2}{m}\tau E^2 \times \mathrm{Re}\int\frac{(1+j\omega\tau)(u_{x1}-u_{y1})(S_{x1}-S_{y1})\sin\Theta d\Theta}{(1+j\omega\tau)^2+[\omega^0(1+R_0)]^2\tau^2},$$

$$P_2=-\frac{3}{8}\frac{Ne^2}{m}\tau E^2\,\mathrm{Re}\int\frac{2(1+j\omega\tau)(u_{x2}S_{x2})\sin\Theta d\Theta}{(1-j\omega\tau)^2+[2\omega^0(1+R_0)]^2\tau^2}, \quad (11)$$

$$P_3=-\frac{3}{8}\frac{Ne^2}{m}\tau E^2 \times \mathrm{Re}\int\frac{(1+j\omega\tau)(u_{x3}-u_{y3})(S_{x3}-S_{y3})\sin\Theta d\Theta}{(1+j\omega\tau)^2+[3\omega^0(1+R_0)]^2\tau^2}.$$

The expressions for longitudinal cyclotron resonance absorption are

$$P_1=-\frac{3}{8}\frac{Ne^2}{m}\tau E^2\,\mathrm{Re}\int\frac{2(1+j\omega\tau)(u_{z1}S_{z1})\sin\Theta d\Theta}{(1+j\omega\tau)^2+[\omega^0(1+R_0)]^2\tau^2},$$

$$P_2=-\frac{3}{8}\frac{Ne^2}{m}\tau E^2\,\mathrm{Re}\int\frac{(1+j\omega\tau)(u_{z-2}S_{z-2})\sin\Theta d\Theta}{(1+j\omega\tau)^2+[2\omega^0(1+R_0)]^2\tau^2}, \quad (12)$$

$$P_3=-\frac{3}{8}\frac{Ne^2}{m}\tau E^2\,\mathrm{Re}\int\frac{2(1+j\omega\tau)(u_{z3}S_{z3})\sin\Theta d\Theta}{(1+j\omega\tau)^2+[3\omega^0(1+2R_0)]^2\tau^2}.$$

Higher harmonics also appear, but are of very low intensity.

In the appendix, we discuss the approximate evaluation of the quantities appearing in the integrals of Eq. (11) and Eq. (12), as a function of the angle Θ, and of the angle θ in the $(1\bar{1}0)$ plane, made by the magnetic field with respect to the [001] direction. The expression for the fundamental cyclotron resonant frequency, to first order in the warping, is

$$\omega\cong\omega^0(1+R_0)=(eH/mc)\{1+\tfrac{1}{16}\kappa[3(-5/9-2\alpha^2+3\alpha^4)+7(1+10\alpha^2-15\alpha^4)A^4]\}, \quad (13)$$

where $\alpha=\cos\theta$, $A=\cos\Theta$.

Dresselhaus, Kip, and Kittel,[7] show curves of m^* *versus* k_H, for **H** along [100], [111], and [110]. Their curves seem consistent with Eq. (13).

It is interesting to note that, for $\theta\sim 29.5°$, the first order cyclotron frequency is independent of Θ, so that the resonance line should be most nearly Lorentzian in this case.

It should be pointed out that the assumption of slight warping, made in developing the present cyclotron expressions, does not hold too well for regions on a constant energy surface far from the center contour for heavy holes in germanium and silicon. However, the integration over Θ favors the region of the constant-energy surface near the center, where the approximations are more valid. McClure[22] and Luttinger and Goodman[21] have developed more general theories of conductivity for warped constant-energy surfaces. Luttinger and Goodman have been carrying out accurate calculations of cyclotron resonance line shapes for holes in germanium, and our results seem to be in reasonable agreement with theirs.

By using the results of the appendix, the relative line intensities of the fundamental cyclotron resonance absorption of heavy holes as a function of magnetic field have been calculated for **H** along the [001], [111], and [110]. The parameters chosen for the calculation were $\omega\tau=5.5$, and $\kappa=0.854$. The resulting curves are shown in Fig. 3. Lorentzian lines with $\omega\tau=5.5$, for the center contour frequency ($\cos\Theta=0$) are shown for purposes of comparison.

These computations were begun before the best possible experimental determination of κ had been made for either germanium or silicon. However, the experimental values of κ for germanium and silicon are not very much different from 0.854, ($\sim$1.15 for germanium, $\sim$0.95 for silicon) so that an approximate comparison of line shapes should be possible. Experimental curves are shown in a previous paper.[23]

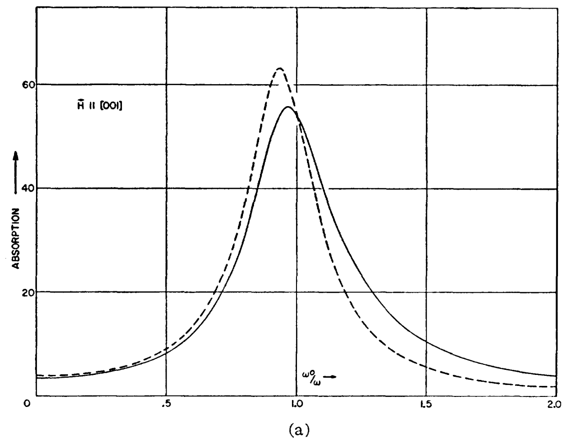
In the limit of large $\omega\tau$, an approximate expression can be obtained for the fractional shift, $\Delta H/H$, of the absorption peak from the center slice peak, due to the warping. The result for **H** along the [001] direction is a fractional shift of $\sim 0.22/\omega\tau$, toward higher fields. For **H** along the [110] and [111] directions, the fractional shift is $\sim 0.22/\omega\tau$ toward lower fields. This result is consistent with the curves of Fig. 3, and is useful in determining the constants A, B, and C from the experimental data.[23]

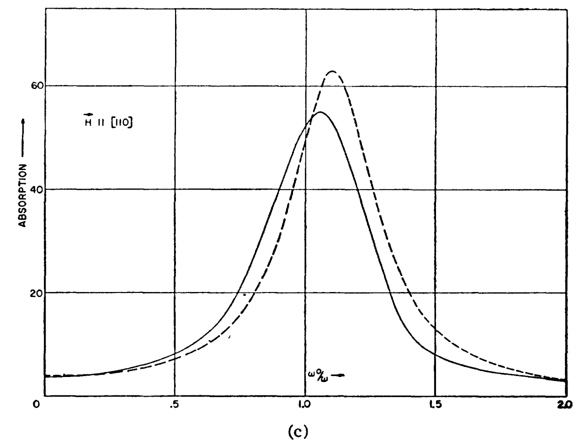
The value of $\omega\tau$ can be determined from the heavy-mass-hole resonances, for $\omega\tau\gg 1$. The method is applicable for **H** along the [001], the [111], and the [110] directions. Let H' be the value of the magnetic field on the steeper side of the resonance such that $(H'-H_0)/H_0=1/\omega\tau$, where H_0 is the value of the field at the peak. Then, an approximate expression can be obtained for the ratio of the power absorption, P', at H', to the power absorption, P_0, at H_0. The result is $P'/P_0\cong 0.55$ for **H** along the [001] direction, and $P'/P_0\cong 0.51$ for **H** along the [111] and [110] directions.

(c) Harmonic Intensities

Integrated harmonic intensities have been calculated for second and third harmonic cyclotron resonance lines, for Case (1) and Case (2), using the first-order expressions of the appendix. Detailed line shape calculations were not made for the harmonics, since experimentally they are of low intensity, and are obscured by the presence of other, larger, resonances. The integrated intensity is an approximate measure of peak intensity, if the anisotropy spread in resonant frequency is less than the line width, $2/\tau$.

Certain simple relations hold for the cyclotron resonance harmonic intensities. For transverse cyclotron





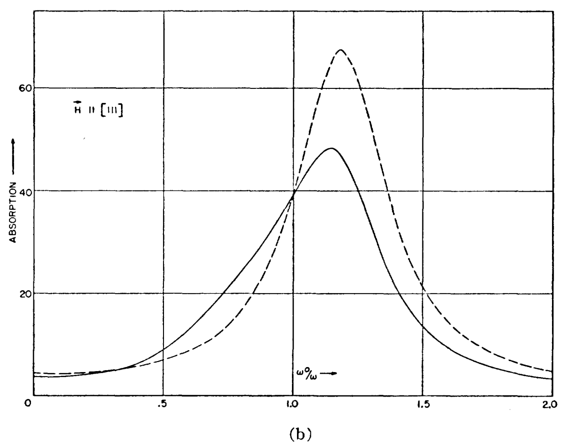


FIG. 3. Cyclotron resonance of heavy holes for H along (a) [001], (b) [111], and (c) [110]. Curves show power absorption (arbitrary scale) *versus* magnetic field, in units of $\omega^0/\omega = eH/mc\omega$. Parameters chosen were $\omega\tau = 5.5$, $K = 0.854$. Curves of the center contour Lorentz lines are shown for comparison.

resonance [Case (1)] the second harmonic intensity should be identically zero for **H** along the [001] and the [110] directions; and the third harmonic intensity should be zero for **H** along the [111] direction. The contribution of the center contour to the transverse second harmonic cyclotron resonance intensity is zero for **H** anywhere in the $(1\bar{1}0)$ plane; however, the contribution of the remainder of the constant energy surface is considerable.

For longitudinal cyclotron resonance [Case (2)], the intensity of the fundamental should be zero for **H** along the [001], [111], and [110] directions. The second harmonic intensity should be zero for **H** along [001] and [111]. The third harmonic intensity should be zero for **H** along [001] and [110].

Figure 4 shows the integrated intensity of second and third harmonic cyclotron resonance, both transverse and longitudinal, as a function of the angle θ. The experimental line intensities of harmonics[23] show only partial agreement between theory and experiment.

IV. DISCUSSION

The agreement between cyclotron resonance observations, and the classical Boltzmann models with τ independent of energy, discussed in this paper, are on the whole fairly good. Both resonant frequencies and general features of the line shapes are reproduced by the theories. The occurrence of harmonics of the cyclotron resonance lines for heavy holes in Ge and Si is explained, although the observed intensities are not completely consistent with theory. However, in experimental configurations where the rf electric field is presumed to be perpendicular to the dc magnetic field, there may also be some parallel rf electric field present. This would make the comparison of observed harmonic intensities with the theory very inaccurate.

Cyclotron resonance may also be considered from the quantum-mechanical point of view.[28–30] Carriers exist in Ge and Si, in quantized levels (Landau levels) in the magnetic field. The rf electric field produces electric dipole transitions between levels, and the resonant frequencies are given by the same expressions as the classical ones. For the case of warped surfaces, the warping produces a mixing of Landau levels and a violation of the usual selection rules, giving rise to

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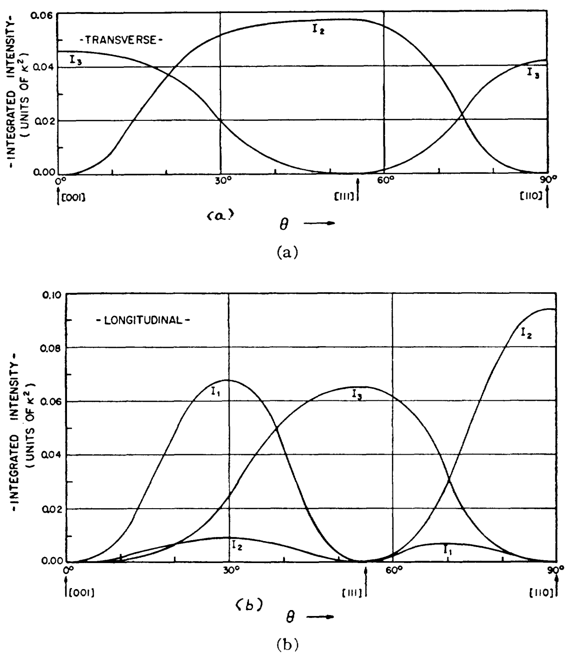

FIG. 4. Integrated harmonic intensities of heavy-mass-hole resonances, as a function of the angle θ, between the [001] and **H**. The intensities are given in units of κ^2, relative to the fundamental transverse cyclotron resonance of heavy holes. (a) Shows the intensities for second and third harmonic transverse cyclotron resonance; (b) shows the intensities for fundamental second and third harmonic longitudinal cyclotron resonance.

harmonic cyclotron resonance absorption. All of the quantum results are the same as those of the classical theory, except at very low quantum numbers for the case of bands degenerate at $k=0$ (see reference 28). The theory then predicts a change in resonant frequencies. Observations by Fletcher *et al.*[31] of cyclotron resonance of holes in germanium at 4.2°K to 1.3°K, and very low power levels, probably indicate the occurrence of such low quantum number effects.

Fletcher *et al.*[31] have also observed line narrowing at very low power levels, (10^{-6} mw). This perhaps indicates that, at higher power levels (10^{-3} mw) and liquid helium temperatures, the electrons and holes are at temperatures higher than the lattice because the lattice-electron relaxation, through the mechanism of phonon creation and annihilation, is less effective at such low temperatures.[32]

The assumption of energy-independent relaxation time τ, for the calculations in this paper, was mainly for the purpose of simplifying the results. The fact that the main features of the line shapes can be explained by an energy-independent τ should not be considered as very strong confirmation of this particular assumption. The general results are probably rather insensitive to the energy dependence of τ. The theory of cyclotron resonance for warped energy surfaces can be modified to take into account the energy dependence of τ, by using the more general theory of Luttinger and Goodman[21] and McClure.[22] A careful measurement of the temperature dependence of cyclotron resonance line widths at low rf powers, as a function of such variables as impurity concentration, crystal prefection, and dislocation density, could help toward a better understanding of scattering mechanisms and the energy dependence of τ.

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APPENDIX. EVALUATION OF CYCLOTRON RESONANCE EXPRESSIONS

To evaluate the quantities appearing in the expressions for the cyclotron resonance of holes in germanium and silicon, it is necessary to write the quantity $(g/\kappa) = (p_x^4+p_y^4+p_z^4-\frac{2}{3}p^4)/p^4$ in terms of spherical coordinates (p,Θ,ϕ) in the "magnetic" frame of references $(\hat{x},\hat{y},\hat{z})$. (See Fig. 2.) The expression is

$$(g/\kappa)=g_0+g_1(\cos\phi+\sin\phi)+g_{-2}\sin2\phi +g_3(\cos3\phi-\sin3\phi)+g_4(\cos4\phi),$$

where

$$g_0=\left(-\frac{5}{48}-\frac{3}{8}\alpha^2+\frac{9}{16}\alpha^4\right)+A^2\left(\frac{3}{8}+\frac{15}{4}\alpha^2-\frac{45}{8}\alpha^4\right) +A^4\left(-\frac{7}{16}-\frac{35}{8}\alpha^2+\frac{105}{16}\alpha^4\right),$$

$$\begin{aligned}
g_1&=AB\frac{\alpha\beta}{2\sqrt{2}}(3\alpha^2-1)(3-7A^2),\\
g_{-2}&=\tfrac{1}{4}B^2(-1+4\alpha^2-3\alpha^4)(7A^2-1),\\
g_3&=\frac{AB^3\alpha\beta}{2\sqrt{2}}(5-3\alpha^2),\\
g_4&=\tfrac{1}{16}B^4(-3+10\alpha^2-3\alpha^4),
\end{aligned}\tag{14}$$

$A=\cos\Theta$, $\alpha=\cos\theta$, and θ is the angle made by the magnetic field direction $B=\sin\Theta$, $\beta=\sin\theta$, with respect to the [001] axis.

The vectors **u** and **S** are given by

$$\mathbf{u}=[(1+g)\hat{p}+\tfrac{1}{2}\nabla g]$$
$$=\hat{x}\bigg[(1+g)\sin\Theta\cos\phi-\frac{1}{2\sin\Theta}\times\left(\frac{\partial g}{\partial\phi}\right)\sin\phi+\frac{1}{2}\left(\frac{\partial g}{\partial\Theta}\right)\cos\Theta\cos\phi\bigg]$$
$$+\hat{y}\bigg[(1+g)\sin\Theta\sin\phi+\frac{1}{2\sin\Theta}\times\left(\frac{\partial g}{\partial\phi}\right)\cos\phi+\frac{1}{2}\left(\frac{\partial g}{\partial\Theta}\right)\cos\Theta\sin\phi\bigg]$$
$$+\hat{z}\bigg[(1+g)\cos\Theta-\frac{1}{2}\left(\frac{\partial g}{\partial\Theta}\right)\sin\Theta\bigg],$$
$$\mathbf{S}=\left[\frac{1}{4\pi}\int(1+g)^{-5/2}\sin\Theta d\Theta d\phi\right]^{-1}(1+g)^{-5/2}\mathbf{u}\equiv F(1+g)^{5/2}\mathbf{u}. \quad (15)$$

From these relations, the quantities appearing in Eqs. (11) and (12) for cyclotron resonance power absorption have been obtained, to first order in the warping. The resulting expressions are

$$R_0=\tfrac{1}{16}\kappa[3(-5/9-2\alpha^2+3\alpha^4)+7(1+10\alpha^2-15\alpha^4)A^4],$$
$$(u_{x1}-u_{y1})(S_{x1}-S_{y1})=FB^2\bigg[1+\frac{\kappa}{16}\left(-\frac{13}{6}+15\alpha^2-\frac{27}{2}\alpha^4\right)+\frac{3}{8}\kappa\left(-\frac{9}{2}-17\alpha^2+\frac{63}{2}\right)\alpha^4A^2+\frac{63}{16}\kappa\left(\frac{3}{2}+\alpha^2-\frac{9}{2}\alpha^4\right)A^2\bigg],$$
$$(u_{x2})=\frac{\kappa\alpha\beta AB^2}{2\sqrt{2}}[(5-3\alpha^2)-(1-9\alpha^2)A^2],$$
$$(S_{x2})=\frac{F\kappa\alpha\beta AB^2}{4\sqrt{2}}[(S-21\alpha^2)-7(1-9\alpha^2)A^2], \quad (16)$$

$$(u_{x3}-u_{y3})=\frac{\kappa B^3}{2048}[3(-3+10\alpha^2-3\alpha^4)+(25-102\alpha^2+81\alpha^4)A^2],$$
$$(S_{x3}-S_{y3})=\frac{F\kappa B^3}{4096}[(-23+90\alpha^2-63\alpha^4)+7(25-102\alpha^2+81\alpha^4)A^2],$$
$$u_{z1}=\frac{\kappa\alpha\beta(3\alpha^2-1)B}{4\sqrt{2}}(3-21A^2+14A^4),$$
$$S_{z1}=\frac{F\kappa\alpha\beta(3\alpha^2-1)B}{4\sqrt{2}}(3-36A^2+49A^4),$$
$$u_{z-2}=\frac{7\kappa AB^4}{4}(-1+4\alpha^2-3\alpha^4), \quad (17)$$
$$S_{z-2}=\frac{F\kappa AB^2}{8}(-1+4\alpha^2-3\alpha^4)(19-49A^2),$$
$$u_{z3}=\frac{\kappa\alpha\beta(5-3\alpha^2)}{4\sqrt{2}}B^3(1-2A^2),$$
$$S_{z3}=\frac{F\kappa\alpha\beta(5-3\alpha^2)}{4\sqrt{2}}B^3(1-7A^2).$$

The power absorption for the fundamental or any of the harmonics, may be written in the form

$$P=\mathrm{Re}(1+j\omega\tau)\int_{-1}^{1}\frac{Q(A^2)dA}{(1+j\omega\tau)^2+H^2(a+bA^4)},$$

where $Q(A^2)$ is a polynomial in A^2, and a and b are functions of the magnetic field direction. These integrals may be evaluated exactly. However, the final evaluation of the power absorption as a function of H still requires a numerical calculation, since the result contains a term which is the real part of the logarithm of a complex argument.