# Phase-Shift Analysis of 310-Mev Proton-Proton Scattering Experiments*

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The results of a phase-shift analysis of proton-proton cross-section, polarization, and triple-scattering experiments at ~310 Mev are reported. From an extensive search five satisfactory solutions have been found. Three additional solutions that give fair fits to the data are also reported.

## 1. INTRODUCTION

THE difficulties encountered in field-theoretical attempts to understand nucleon-nucleon interactions have led to the less ambitious hope that a phenomenological potential model for these interactions could be found. Although there has been considerable effort along these lines,[1] no appreciable success has been obtained except in restricted energy regions.[2] Because of this failure to correlate the experimental facts by means of potential models, the value of information that may be obtained from a direct analysis of the data has become increasingly important. The importance derives both from the insight it provides for the construction of particular models and from the possibility of using the information in phenomenological treatments of more complex problems.

A standard method of extracting information from results of scattering experiments is to find sets of phase shifts that reproduce the experimental data. This approach, which has been valuable in the study of pion-nucleon interactions,[3] has also been used in the analysis of high-energy proton-proton experiments.[4] In the hundred-Mev region these efforts have been impaired, however, by limitations in the amount of experimental data that was available. These limitations imposed severe and unrealistic restrictions upon the number of phase shifts that could be considered.[5] Now that *p-p* triple-scattering experiments have been performed the situation is considerably more favorable. Whereas previously the phase shifts were effectively limited to $S$, $P$, and $D$ waves,[6] it now is feasible to include also the $F$, $G$, and $H$ waves. Such an analysis has been carried out with the aid of electronic computers, principally the MANIAC located at Los Alamos. The results are reported in this paper.

The discussion begins in Sec. 2 with an account of the method by which the experimental data were treated. In Sec. 3 a discussion of equations that express the observed quantities in terms of phase shifts is given. Section 4 contains a description of the method by which solutions were found, and a discussion of the extent to which the search for solutions can be considered exhaustive. The accuracy to which the phase shifts are determined is also discussed in this section. The final section contains a discussion of the results, and comments concerning their interpretation.

## 2. TREATMENT OF EXPERIMENTAL DATA

The general theoretical foundation of the analysis is provided by the work of Wolfenstein,[7] and the experimental details may be found in the accompanying experimental papers.[8,9] The purpose of this section is to discuss the manner in which these data were used.

The general policy in the treatment of the data was to leave them in the form in which they were originally provided by the experiments. In keeping with this approach absolute values for the total cross section and the 90° (c.m.) differential cross section were used as input data, whereas at other angles the ratio of the differential cross section, $I_0$, to its 90° value was used. The observed polarization and triple-scattering parameters $P$, $D$, $R$, and $A$ at particular scattering angles were taken directly as input data, as opposed to previous analyses[4] in which, for example, $I_0P$ rather than $P$ was used. Another way in which our treatment differs is that the differential cross section and polariza-

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[1] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950); R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950); R. Christian and P. Noyes, Phys. Rev. **79**, 85 (1950); K. M. Case and A. Pais, Phys. Rev. **80**, 203 (1950); L. J. B. Goldfarb and D. Feldman, Phys. Rev. **88**, 1099 (1952); and Don Swanson, Phys. Rev. **89**, 740 (1953).

[2] H. H. Hall and J. L. Powell, Phys. Rev. **90**, 912 (1953); Yovits, Smith, Hall, Benston, and Breit, Phys. Rev. **85**, 540 (1952); J. Shapiro and M. A. Preston, Can. J. Phys. **34**, 451 (1956).

[3] Fermi, Metropolis, and Alei, Phys. Rev. **95**, 1581 (1954).

[4] R. M. Thaler and J. Bengston, Phys. Rev. **94**, 679 (1954); A. Garren, Phys. Rev. **101**, 419 (1956); C. A. Klein, Nuovo cimento **2**, 581 (1955); E. Clementel and C. Villi, Nuovo cimento **2**, 1165 (1955); and S. Ohnuma and D. Feldman, Phys. Rev. **102**, 1641 (1956).

[5] For a discussion the restrictions imposed by the requirement that the number of phase shifts be less than, or at worst equal to, the number of independent experimental quantities (see the introduction in reference 8).

[6] Thaler and Benston, Klein, and also Garren use $S$ and $P$ phase shifts. Ohnuma and Feldman include $^1D$ and $^3F_2$ and Villi and Clementel include these and also the $^3P_2-^3F_2$ mixing parameter. The inclusion of the $^3F_2$, without $^3F_3$ and $^3F_4$, can be justified by arguing that the $^3P_2-^3F_2$ mixing causes the $^3F_2$ to behave like a $P$ wave at low energies. Recently the effects of $F$ waves and to some extent $G$ and $H$ waves has been investigated by Hull, Ehrman, Hatcher, and Durand, Phys. Rev. **103**, 1047 (1956).

[7] Lincoln Wolfenstein, Phys. Rev. **96**, 1654 (1954).

[8] Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis, preceding paper [Phys. Rev. **105**, 288 (1957)].

[9] J. E. Simmons, Phys. Rev. **104**, 416 (1956).

tion data are not first reduced to coefficients of a power series expansion in $\cos\theta$ but the original measurements at particular scattering angles are used directly. One advantage of this is that, in general, better fits to the original data are obtained. In addition, the analysis of the errors in the phase shifts, arising from errors in the experimental data, is simplified because all experimental errors may be treated as independent.

The values of the experimental quantities are collected in Table I. The experimental errors quoted there contain both the statistical and estimated systematic contributions. It will be noticed that four of the differential cross-section points at small angles are marked by asterisks. These data, obtained by Fischer and Goldhaber,[10] extend into the region where Coulomb effects and higher partial waves are expected to become important and they were not used in the first part of the search program, but were introduced only at a later stage, as is discussed in Sec. 4. In addition the three $A$ measurements were also introduced at a later stage, for this experiment had not yet been performed when the analysis was begun.

It should be mentioned that the laboratory-system energy of the $p$-$p$ collision in the polarization and triple-scattering experiments was about 310 Mev, whereas the cross-section measurements were made at the full beam energy of 340 Mev. In our analysis this difference was completely ignored and the cross-section data were treated as if the measurements has been made at 310 Mev. This procedure is at least partially justified by the observed insensitivity of the proton-proton differential cross section to variations of energy in this region.

TABLE I. The 36 experimental measurements used in this analysis are recorded. The asterisk denotes the Fischer-Goldhaber Coulomb interference data (see Sec. 2). The values predicted by a representative solution are also shown for comparison.

| Designation[a] | Experimental value | Calculated value (Solution 4) |
|---|---|---|
| $\sigma_{\text{total}}\geq 20°$ | 22.24 ±0.70 mb | 22.13 mb |
| $I_0(90°)$ | 3.72 ±0.19 mb | 3.72 mb |
| $r(80.2°)$ | 1.045±0.039 | 1.005 |
| $r(71.4°)$ | 0.971±0.032 | 1.004 |
| $r(64.0°)$ | 0.958±0.032 | 0.991 |
| $r(60.8°)$ | 1.013±0.041 | 0.985 |
| $r(52.4°)$ | 0.997±0.035 | 0.980 |
| $r(44.8°)$ | 1.008±0.026 | 1.000 |
| $r(36.0°)$ | 1.074±0.040 | 1.042 |
| $r(31.9°)$ | 1.031±0.031 | 1.061 |
| $r(23.4°)$ | 1.098±0.033 | 1.083 |
| $*r(18.6°)$ | 1.024±0.078 | 1.072 |
| $*r(14.8°)$ | 1.038±0.086 | 1.041 |
| $*r(11.3°)$ | 0.935±0.108 | 1.010 |
| $*r(9.1°)$ | 1.078±0.091 | 1.065 |
| $s(76.2°)$ | 0.613±0.108 | 0.527 |
| $s(63.9°)$ | 0.635±0.068 | 0.568 |
| $s(53.4°)$ | 0.633±0.052 | 0.649 |
| $s(42.9°)$ | 0.760±0.040 | 0.748 |
| $s(32.3°)$ | 0.837±0.060 | 0.848 |
| $s(21.6°)$ | 0.891±0.067 | 0.955 |
| $t(80.5°)$ | 0.528±0.063 | 0.529 |
| $t(65.2°)$ | 0.497±0.048 | 0.475 |
| $t(52.0°)$ | 0.467±0.060 | 0.488 |
| $t(36.5°)$ | 0.544±0.081 | 0.574 |
| $t(25.8°)$ | 0.701±0.055 | 0.723 |
| $t(23.0°)$ | 0.755±0.079 | 0.779 |
| $u(80.1°)$ | 0.752±0.114 | 0.579 |
| $u(70.9°)$ | 0.381±0.088 | 0.517 |
| $u(54.1°)$ | 0.322±0.058 | 0.251 |
| $u(41.8°)$ | 0.111±0.076 | 0.060 |
| $u(34.4°)$ | −0.175±0.084 | −0.004 |
| $u(22.3°)$ | −0.330±0.142 | −0.068 |
| $v(76.3°)$ | 0.382±0.078 | 0.386 |
| $v(51.4°)$ | 0.016±0.088 | −0.004 |
| $v(25.4°)$ | −1.542±0.363 | −1.414 |

[a] $r(x)\equiv I_0(x)/I_0(90°)$; $s(x)\equiv P(x)/\sin x\cos x$; $t(x)\equiv 1-D(x)$; $u(x)\equiv R(x)/\cos(x/2)$; $v(x)\equiv A(x)/\sin(x/2)$.

## 3. BASIC EQUATIONS

The theory of polarization and triple-scattering experiments for the two-nucleon system has been developed by Wolfenstein[7] and by Wolfenstein and Ashkin.[11] In their treatment the scattering is described by a matrix $M$, in spin space, defined by the equation

$$f_i(\theta,\phi)=\sum M_{ij}a_j. \tag{3.1}$$

The $a_j$ are the amplitudes of the various spin states in the incident plane wave and the $f_i(\theta,\phi)$ are the scattering amplitudes for these states. Summation signs will be over repeated indices unless otherwise stated. The matrix elements $M_{ij}$ are functions of the center-of-mass scattering angles $(\theta,\phi)$, and they completely describe the scattering. Using the formalism of Wolfenstein and Ashkin, the quantities measured in the polarization and triple-scattering experiments may be expressed in terms of the $M_{ij}$. These formulas are given in Table II.[12–14]

The formulas for the observables in terms of phase shifts may now be obtained by expressing the $M_{ij}$ in terms of the phase shifts. Since the phase shifts are related to the $S$ matrix elements, one needs the relation between the $S$ matrix and the $M$ matrix. The $S$ matrix may be expressed as the sum of the unit matrix and the $R$ matrix, where the $R$ matrix satisfies an equation which in the $lsm_lm_s$ representation is

$$f'(lsm_lm_s)=\sum R(lsm_lm_s;l's'm_l'm_s')g(l's'm_l'm_s'). \tag{3.2}$$

[10] D. Fischer and G. Goldhaber, Phys. Rev. **95**, 1350 (1954). These four data were selected from their more numerous data on the basis of estimated systematic errors and consistency with the independent measurements of Chamberlain, Pettengill, Segrè, and Wiegand, Phys. Rev. **95**, 1348 (1954).

[11] L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).

[12] The equations given in Table II were derived from the nonrelativistic formalism of Wolfenstein and Ashkin. The two Pauli amplitudes appearing in this formalism may, however, be reinterpreted as the two amplitudes that appear in the relativistic treatment of the problem. When triple-scattering experiments are considered, some modifications of the nonrelativistic formulas are introduced by this reinterpretation. For a discussion of this point see Henry P. Stapp, Phys. Rev. **103**, 425 (1956).

[13] Henry P. Stapp, University of California Radiation Laboratory Report, UCRL-3098, August, 1955 (unpublished).

[14] L. Wolfenstein, in Annual Reviews of Nuclear Science (Annual Reviews, Inc., Stanford), Vol. 6. We thank Professor Wolfenstein for having made available to us the manuscript of this paper.

TABLE II. The expressions for the various experimental parameters as functions of the $M$ matrix elements are given. The subscripts $1, 0, -1, s$ on the matrix elements refer to the three triplet states $S_z=+1, 0, -1$ and to the singlet state respectively. The $z$ axis is taken along the incident beam direction. The $M_{ij}$ are functions of the usual polar and azimuthal center-of-mass scattering angles $\theta$ and $\phi$. Since the left-hand sides of the above equations are independent of $\phi$, the expressions on the right have, for simplicity, been evaluated at $\phi=0$. For a detailed derivation of these equations see reference 13. The dependence of the polarization correlation parameters $C_{nn}$ and $C_{KP}$ upon the $M$ matrix elements are also included for completeness. A discussion of these parameters may be found in references 13 and 14.

$$I_0=\tfrac{1}{2}|M_{11}|^2+\tfrac{1}{4}|M_{00}|^2+\tfrac{1}{4}|M_{ss}|^2+\tfrac{1}{2}|M_{10}|^2+\tfrac{1}{2}|M_{01}|^2+\tfrac{1}{2}|M_{1-1}|^2$$

$$I_0P=(\sqrt{2}/4)\,\mathrm{Re}[i(M_{10}-M_{01})(M_{11}-M_{1-1}+M_{00})^*]$$

$$I_0(1-D)=\tfrac{1}{4}|M_{11}+M_{1-1}-M_{ss}|^2+\tfrac{1}{4}|M_{11}-M_{1-1}-M_{00}|^2+\tfrac{1}{2}|M_{10}+M_{01}|^2$$

$$\frac{I_0R}{\cos(\theta/2)}=\tfrac{1}{2}\,\mathrm{Re}\left\{\left(M_{00}+(\cos\theta-1)\frac{\sqrt{2}M_{10}}{\sin\theta}\right)(M_{11}+M_{1-1}+M_{ss})^*+\left(\frac{\sqrt{2}M_{10}}{\sin\theta}+\frac{\sqrt{2}M_{01}}{\sin\theta}\right)M_{ss}^*\right\}$$

$$\frac{I_0A}{\sin(\theta/2)}=-\tfrac{1}{2}\,\mathrm{Re}\left\{\left(M_{00}+(\cos\theta+1)\frac{\sqrt{2}M_{10}}{\sin\theta}\right)\times(M_{11}+M_{1-1}+M_{ss})^*-\left(\frac{\sqrt{2}M_{10}}{\sin\theta}+\frac{\sqrt{2}M_{01}}{\sin\theta}\right)(M_{11}+M_{1-1})^*\right\}$$

$$I_0C_{KP}=\{|M_{01}|^2-|M_{10}|^2\}/2\sin\theta$$

$$I_0(1-C_{nn})=\tfrac{1}{2}(|M_{ss}|^2+|M_{11}+M_{1-1}|^2).$$

Here $f'(lsm_lm_s)$ and $g(lsm_lm_s)$ are defined by the following equations:[15,16]

$$\psi^{\mathrm{inc}}(\theta,\phi,r)\simeq -r^{-1}\sum \exp[-i(kr-\tfrac{1}{2}\pi l)]\times g(lsm_lm_s)Y_l^{m_l}(\theta,\phi)\chi_s^{m_s},$$

$$\psi^{\mathrm{sc}}(\theta,\phi,r)\simeq r^{-1}\sum \exp[+i(kr-\tfrac{1}{2}\pi l)]\times f'(lsm_lm_s)Y_l^{m_l}(\theta,\phi)\chi_s^{m_s}. \quad (3.3)$$

The $f'(lsm_lm_s)$ in these equations are connected to the $f_i(\theta,\phi)$ appearing in Eq. (3.1). The latter are defined by the equation

$$\psi^{\mathrm{sc}}(\theta,\phi,r)\simeq r^{-1}\exp(ikr)f_s^{m_s}(\theta,\phi)\chi_s^{m_s}, \quad (3.4)$$

where the $f_i(\theta,\phi)$ in the singlet-triplet representation have been written $f_s^{m_s}(\theta,\phi)$. From the definitions in Eqs. (3.1) to (3.4) and the familiar expansions of the incident plane wave into spherical harmonics,[16] one obtains the relation[13]

$$M_{ij}(\theta,\phi)\equiv M_{sm_s,\,s'm_s'}(\theta,\phi)=\sum Y_l^{m_l}(\theta,\phi)M(lsm_lm_s;s'm_s'), \quad (3.5)$$

where

$$M(lsm_lm_s;s'm_s')=(ik)^{-1}\exp[-\tfrac{1}{2}i\pi l]\sum R(lsm_lm_s;l's'0m_s')\times\exp[\tfrac{1}{2}i\pi l'][\pi(2l'+1)]^{\frac{1}{2}}. \quad (3.6)$$

The most convenient phase shifts are those related to the $R$ matrix elements in the $lsjm_j$ representation. These matrix elements are related to the matrix elements appearing in Eq. (3.6) by means of the Clebsch-Gordan coefficients[16] $C_{ls}(jm_j;m_lm_s)$ according to the equation

$$R(lsm_lm_s;l's'm_l'm_s')=\sum{}' C_{ls}(jm_j;m_lm_s)\times R(lsjm_j;l's'j'm_j')C_{l's'}(j'm_j';m_l'm_s'), \quad (3.7)$$

where the prime on the summation symbol indicates no sum on $l$, $s$, $l'$, and $s'$. The convenience of the $l$, $s$, $j$, $m_j$ representation derives from the fact that the total momentum $j$, its $z$ component $m_j$, and the spin angular momentum $s$ are constants of the motion,[17] and the $R$ matrix therefore contains no off-diagonal elements in these indices. Furthermore the invariance of $R$ with respect to spatial rotations implies that the $R$ matrix elements are independent of $m_j$. The nonzero $R$ matrix elements $R(lsjm_j;l's'j'm_j')$ may therefore be abbreviated as

$$R(l0lm_j;l0lm_j)\equiv R_l,\quad R(l1jm_j;l1jm_j)\equiv R_{lj},\quad R(j\pm1,1,j,m_j;j\mp1,1,j,m_j)\equiv R_\pm^{\,j}=R^j, \quad (3.8)$$

where the equality of $R_+^{\,j}$ and $R_-^{\,j}$ is a consequence of the symmetry of the $S$ matrix. Using some properties of the Clebsch-Gordan coefficients Eqs. (3.6) to (3.8) combine to give

$$M(l000,00)=(ik)^{-1}[\pi(2l+1)]^{\frac{1}{2}}R_l,$$

$$M(l,1,m_s'-m_s,m_s;1,m_s')=(ik)^{-1}\{\sum_{j=l-1}^{l+1}[\pi(2l+1)]^{\frac{1}{2}}C_{l1}(j,m_s';m_s'-m_s,m_s)\times C_{l1}(j,m_s';0m_s')R_{lj}-\sum_{j=l\pm1>1}[\pi(2l'+1)]^{\frac{1}{2}}\times C_{l1}(j,m_s';m_s'-m_s,m_s)C_{l'1}(j,m_s';0m_s')R^j\}, \quad (3.9)$$

where $l'\equiv 2j-l=l\pm2=j\pm1$. The $M(lsm_lm_s,s'm_s')$ that are not of the forms given in Eq. (3.9) are identically zero.

The matrix elements given in Eq. (3.9) refer specifically to the case of two distinguishable nucleons. When the nucleons are identical the antisymmetrized $M$ matrix, $M^a=(1-TS)M$, should be used in place of $M$.[13] Here $T$ and $S$ are the spin- and space-exchange operators. This replacement takes into account the antisymmetry of the wave function and also the fact that the

[15] The symbol $\simeq$ represents equality in the limit $r\to\infty$. The $\psi^{\mathrm{inc}}(\theta,\phi,r)$ is the asymptotic incoming part of the wave function and $\psi^{\mathrm{sc}}(\theta,\phi,r)$ is the asymptotic scattered wave. The $Y_l^m(\theta,\phi)$ are the spherical harmonics as defined by Blatt and Weisskopf (see reference 16) and $\chi_s^{m_s}$ is the spin-state basis vector. The range of the potential is assumed finite; Coulomb effects will be included later.

[16] J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

[17] The spin is constant in the $p$-$p$ system because of conservation of parity and the antisymmetry of the wave function. For the $n$-$p$ system it will be constant if isotopic spin is conserved, which is assumed here.

particles are indistinguishable. Explicit formulas for the $M$ matrix elements may be obtained by evaluating the Clebsch-Gordan coefficients in Eq. (3.9). These formulas, specialized to the proton-proton system, are given in Table III. The Coulomb effects are also included there.

The Coulomb part of the interaction has been treated in the nonrelativistic approximation. In this approximation the equations above continue to be valid provided the quantity $(kr)$ appearing in Eqs. (3.3) and (3.4) is replaced by $(kr-n\ln 2kr)$. Here $n=e^2/\hbar v \cong(137\beta)^{-1}$, where $\beta$ is the laboratory velocity of the incident proton divided by the velocity of light. The $S$ matrix defined in this way gives the scattering due to the combined nuclear and Coulomb effects, and in the limit of no nuclear potential it becomes the pure Coulomb-scattering matrix $S_c\equiv R_c+1$.

Owing to the singular nature of the Coulomb contributions, it is convenient to express the $R$ matrix in the form

$$\begin{aligned} R&=S-1\\ &=(S-S_c)+(S_c-1)\\ &=\alpha+R_c. \end{aligned} \tag{3.10}$$

The matrix $\alpha\equiv S-S_c$ may presumably be analyzed in terms of partial waves, since $S$ differs from $S_c$ only by nuclear effects, and these are expected to vanish for large $l$. The contribution of $R_c$, on the other hand, is not analyzed into partial waves but is treated exactly and contributes the Coulomb scattering amplitude,

$$f_C(\theta)=\frac{-n}{k(1-\cos\theta)}\{\exp[-in\ln\tfrac{1}{2}(1-\cos\theta)]\}. \tag{3.11}$$

The expressions for the $M_{ij}$ that are given in Table III are functions of $f_C(\theta)$ and the matrix elements of $\alpha$.

The matrix elements of $\alpha$ can be expressed in terms of phase shifts. Owing to the unitarity condition the diagonal elements $S_l\equiv R_l+1$ and $S_{lj}\equiv R_{lj}+1$ for $l=j$ are pure phase factors. Consequently the corresponding matrix elements of $\alpha$ may be expressed

$$\begin{aligned} &\alpha_l=e^{2i\delta_l}-e^{2i\Phi_l},\\ &\alpha_{lj}=e^{2i\delta_{lj}}-e^{2i\Phi_l}\quad\text{for } l=j, \end{aligned} \tag{3.12}$$

where

$$\Phi_l\equiv\eta_l-\eta_0=\sum_{x=1}^{l}\arctan(n/x). \tag{3.13}$$

The matrix elements of $\alpha$ between the states with $l=j\pm1$ may be expressed, following Blatt and Biedenharn,[18] by

$$\begin{aligned} &\alpha_{j\pm1,\,j}=\cos^2\epsilon_j e^{2i\delta_{j\pm1,i}}+\sin^2\epsilon_j e^{2i\delta_{j\mp1,j}}-e^{2i\Phi_{j\pm1}},\\ &\alpha^j=\tfrac{1}{2}\sin 2\epsilon_j(e^{2i\delta_{j\mp1,j}}-e^{2i\delta_{j\pm1,i}}). \end{aligned} \tag{3.14}$$

[18] J. M. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (1952) and Revs. Modern Phys. **24**, 258 (1952).

TABLE III. The antisymmetrized $M$ matrix elements for the $p$-$p$ system are expressed as functions of the Coulomb scattering amplitude $f_C(\theta)$ and the matrix elements of $\alpha$. The amplitude $f_C(\theta)$ is defined in Eq. (3.11), and the expressions for the matrix elements of $\alpha$ in terms of phase shifts are given in Eqs. (3.12) through (3.15). The $\alpha_{lj}$ for $j<0$ and the $\alpha^j$ for $j\leq 0$ are defined to be zero. The $P_l^m(\theta)$ are the associated Legendre polynomials, $p=\hbar k$ is the center-of-mass momentum.

$$M_{ss}(\theta,\phi)=f_C(\theta)+f_C(\pi-\theta)+2(ik)^{-1}\sum_{\text{even}\,l}P_l(\theta)\left(\frac{2l+1}{2}\right)\alpha_l,$$

$$\begin{aligned} M_{11}(\theta,\phi)=&f_C(\theta)-f_C(\pi-\theta)+2(ik)^{-1}\sum_{\text{odd}\,l}P_l(\theta)\\ &\times\left\{\left(\frac{l+2}{4}\right)\alpha_{l,\,l+1}+\left(\frac{2l+1}{4}\right)\alpha_{l,\,l}+\left(\frac{l-1}{4}\right)\alpha_{l,\,l-1}\right.\\ &\left.-\tfrac{1}{4}[(l+1)(l+2)]^{\frac{1}{2}}\alpha^{l+1}-\tfrac{1}{4}[(l-1)l]^{\frac{1}{2}}\alpha^{l-1}\right\}, \end{aligned}$$

$$\begin{aligned} M_{00}(\theta,\phi)=&f_C(\theta)-f_C(\pi-\theta)+2(ik)^{-1}\sum_{\text{odd}\,l}P_l(\theta)\\ &\times\left\{\left(\frac{l+1}{2}\right)\alpha_{l,\,l+1}+\left(\frac{l}{2}\right)\alpha_{l,\,l-1}+\tfrac{1}{2}[(l+1)(l+2)]^{\frac{1}{2}}\alpha^{l+1}\right.\\ &\left.+\tfrac{1}{2}[(l-1)(l)]^{\frac{1}{2}}\alpha^{l-1}\right\}, \end{aligned}$$

$$\begin{aligned} M_{01}(\theta,\phi)=2(ik)^{-1}e^{i\phi}\sum_{\text{odd}\,l}P_l^1(\theta)&\left\{-\frac{\sqrt{2}}{4}\left(\frac{l+2}{l+1}\right)\alpha_{l,\,l+1}\right.\\ &+\frac{\sqrt{2}}{4}\left(\frac{2l+1}{l(l+1)}\right)\alpha_{l,\,l}+\frac{\sqrt{2}}{4}\left(\frac{l-1}{l}\right)\alpha_{l,\,l-1}\\ &\left.+\frac{\sqrt{2}}{4}\left(\frac{l+2}{l+1}\right)^{\frac{1}{2}}\alpha^{l+1}-\frac{\sqrt{2}}{4}\left(\frac{l-1}{l}\right)^{\frac{1}{2}}\alpha^{l-1}\right\}, \end{aligned}$$

$$\begin{aligned} M_{10}(\theta,\phi)=2(ik)^{-1}e^{-i\phi}\sum_{\text{odd}\,l}P_l^1(\theta)&\left\{\left(\frac{\sqrt{2}}{4}\right)\alpha_{l,\,l+1}-\left(\frac{\sqrt{2}}{4}\right)\alpha_{l,\,l-1}\right.\\ &\left.+\frac{\sqrt{2}}{4}\left(\frac{l+2}{l+1}\right)^{\frac{1}{2}}\alpha^{l+1}-\frac{\sqrt{2}}{4}\left(\frac{l-1}{l}\right)^{\frac{1}{2}}\alpha^{l-1}\right\}, \end{aligned}$$

$$\begin{aligned} M_{1-1}(\theta,\phi)=2(ik)^{-1}e^{-2i\phi}\sum_{\text{odd}\,l}P_l^2(\theta)&\left\{\left(\frac{1}{4(l+1)}\right)\alpha_{l,\,l+1}\right.\\ &-\left(\frac{2l+1}{4l(l+1)}\right)\alpha_{l,\,l}+\left(\frac{1}{4l}\right)\alpha_{l,\,l-1}\\ &\left.-\tfrac{1}{4}[(l+1)(l+2)]^{-\frac{1}{2}}\alpha^{l+1}-\tfrac{1}{4}[(l-1)l]^{-\frac{1}{2}}\alpha^{l-1}\right\} \end{aligned}$$

$$M_{-1-1}(\theta,\phi)=M_{11}(\theta,-\phi),\qquad M_{01}(\theta,\phi)=-M_{0-1}(\theta,-\phi),$$
$$M_{-11}(\theta,\phi)=M_{1-1}(\theta,-\phi),\qquad M_{10}(\theta,\phi)=-M_{-10}(\theta,-\phi).$$

An alternative expression for these matrix elements, which is useful in the analysis of Coulomb contributions and the interpretation of the results, is

$$\begin{aligned} &\alpha_{j\pm1,\,j}=\cos 2\epsilon_j\exp(2i\bar{\delta}_{j\pm1,\,j})-\exp(2i\Phi_{j\pm1}),\\ &\alpha^j=i\sin 2\epsilon_j\exp[i(\bar{\delta}_{j+1,\,j}+\bar{\delta}_{j-1,\,j})]. \end{aligned} \tag{3.15}$$

The phase shifts appearing in this second definition, Eq. (3.15), will be called bar phase shifts in distinction to

the Blatt and Biedenharn (BB) phase shifts defined by Eq. (3.14). The bar phase shifts for the states that do not involve mixing are the same as the BB phase shifts defined in Eq. (3.12).

The phase shifts defined by Eqs. (3.13) to (3.15) include both Coulomb and nuclear contributions. If there were no nuclear effects these phase shifts would become just the pure Coulomb phase shifts $\Phi_l$. When both nuclear and Coulomb forces are present, it is useful for some purposes to remove the Coulomb contributions from the phase shifts and consider only the nuclear part. Such a separation is possible, however, only if special assumptions are used. If, for instance, the Coulomb force can be assumed to act only outside the region about the origin in which the nuclear effects occur, and if the WKB approximation is valid in this outside region, then the bar phase shifts that would be obtained if the Coulomb potential were removed (leaving the nuclear potential unchanged) are given by the equations

$$\bar{\delta}_l{}^N \equiv \bar{\delta}_l - \Phi_l, \quad \bar{\delta}_{lj}{}^N \equiv \bar{\delta}_{lj} - \Phi_l, \quad \bar{\epsilon}^N \equiv \bar{\epsilon}. \quad (3.16)$$

The phase shifts $\bar{\delta}^N$ and $\bar{\epsilon}^N$ *defined* by these equations will be called the nuclear bar phase shifts. The equations for the nuclear Blatt and Biedenharn phase shifts derived from the same set of assumptions are considerably more complicated than Eqs. (3.16). They may be derived from Eq. (3.16), and the equations that relate the BB phase shifts to the bar phase shifts. These relations are given in the appendix, where it is also shown that for weak potentials the bar phase shifts are proportional to the matrix elements of the potential. Consequently they become small in the limit of weak potentials, and will—for special types of potentials—obey corresponding interval rules. These properties are not shared by the BB parameters. For convenience both the nuclear bar phase shifts and the original (nonnuclear) BB phase shifts are reported in the table of results.

One point concerning the relativistic Coulomb contributions should be mentioned. As stated above, the treatment of the Coulomb effects is nonrelativistic. However, the relativistic corrections are not completely neglected. Although the exact relativistic form of the Coulomb interaction is not known, Garren[4] has calculated the lowest-order field-theoretical relativistic corrections and finds that they do not contain the singular factor $(1-\cos\theta)^{-1}$ which characterizes the nonrelativistic term. If the relativistic effects are indeed nonsingular they need not be separated for special treatment, as was the nonrelativistic term, but may be combined with the nuclear effects in the term that is expanded in partial waves. The nuclear phase shifts would then contain not only nuclear effects, and the remnants of the nonrelativistic Coulomb effects that arise from the approximations made when the Coulomb contributions were subtracted, but also the contributions from the relativistic Coulomb corrections.

## 4. SEARCH PROGRAM

The formulas collected in Tables II and III allow the various observables to be expressed in terms of the phase shifts. These functions will be denoted by $y_n(\delta)$, where $n$ denotes the particular observable. The corresponding experimental values will be denoted by $y_n$. Following the method used by Fermi,[3] a search was made for phase shifts which minimized the expression

$$\mathfrak{M}(\delta) = \sum_n \left( \frac{y_n(\delta) - y_n}{\epsilon_n} \right)^2,$$

where $\epsilon_n$ is the experimental error in the measurement of $y_n$. The procedures were quite similar to those used by Fermi. Both the grid method (in which only one phase shift is changed at a time) and the gradient method[3] were used. The gradient method was replaced in the later stages by a new search method devised by Davidon.[19] The search program consists of taking a large number of random sets of phase shifts and using various combinations of the above searching procedures to obtain from each random set of phase shifts a corresponding solution. These solutions are sets of phase shifts such that an increment of $\pm(1/64)°$ in any phase shift gives a larger value of $\mathfrak{M}(\delta)$.

The work was divided into three stages. In the first stage the results of the $A$ experiments and the Coulomb interference measurements were not included in the data, and the theoretical forms included neither Coulomb effects nor the contributions of $G$ and $H$ waves. The number of random starting sets of phase shifts used in this stage was 360. The corresponding solutions were grouped into 34 tight clusters whose members were the same in all phase shifts to within approximately 0.2°. The various clusters evidently correspond to different relative minima. Of the 34 minima all but two were obtained three or more times. Of the two, one was obtained twice and the other only once. The $A$ data, which were not available at the beginning of the analysis, were then incorporated and the 34 minima were used as the starting points for the second phase of the search program. Two independent search procedures (grid and Davidon) were used independently to obtain solutions from the 34 starting points. The solutions obtained from a particular starting point by means of the two procedures were not the same in all cases, but from each procedure alone we obtained from the 34 starting points just 19 different final solutions, and these 19 solutions were the same for the two procedures. Each of these 19 solutions was obtained from at least five of the original 360 random starting points (even if only the grid search or only the Davidon search from the 34 intermediate minima is considered). Of these 19 solutions the best seven are significantly superior to the remaining twelve. The search was there-

[19] William C. Davidon, Bull. Am. Phys. Soc. Ser. II, **1**, 51 (1956).

TABLE IV. The eight best solutions obtained after the introduction of the Coulomb effects, the $G$ and $H$ wave contributions, and the Fischer-Goldhaber Coulomb interference data. The values of $2\delta$ are recorded. The column headed "BB" are the Blatt and Biedenharn phase shifts defined by Eqs. (3.10) through (3.14) of the text. They are the total phase shifts and include both Coulomb and nuclear effects. In the absence of nuclear forces they reduce to the Coulomb phase shifts $\Phi_l$. In the column headed "Nuclear bar" are the values of twice the nuclear bar phase shifts defined in Eqs. (3.15) and (3.16). In the absence of nuclear forces these phase shifts reduce to zero.

| | Solution 1 ($\mathfrak{M}=17.9$) | | Solution 2 ($\mathfrak{M}=21.7$) | | Solution 3 ($\mathfrak{M}=23.8$) | | Solution 4 ($\mathfrak{M}=24.5$) | |
|---|---|---|---|---|---|---|---|---|
| Type | Nuclear bar | BB | Nuclear bar | BB | Nuclear bar | BB | Nuclear bar | BB |
| $^1S_0$ | −20.2±5.0° | −20.2±5.0° | −39.0±3.9° | −39.0±3.9° | −21.9±4.9° | −21.9±4.9° | −53.9±3.9° | −53.9±3.9° |
| $^1D_2$ | 25.7±1.4° | 27.6±1.4° | 8.7±1.2° | 10.6±1.2° | 26.6±1.5° | 28.5±1.5° | 9.7±1.2° | 11.6±1.2° |
| $^1G_4$ | 2.0° | 4.6° | 2.6° | 5.2° | 2.2° | 4.8° | 2.1° | 4.7° |
| $^3P_0$ | −28.6±4.3° | −27.4±4.3° | −72.1±3.8° | −70.8±3.8° | −8.1±2.7° | −6.9±2.7° | −50.7±3.8° | −49.4±3.8° |
| $^3P_1$ | −53.3±2.6° | −52.0±2.6° | −23.4±2.0° | −22.2±2.0° | −39.5±1.6° | −38.2±1.6° | −14.5±2.0° | −13.2±2.0° |
| $^3F_3$ | −8.8±2.1° | −6.5±2.1° | 0.5±0.7° | 2.8±0.7° | −5.1±1.1° | −2.8±1.1° | 3.1±0.7° | 5.4±0.7° |
| $^3H_5$ | 0.2° | 3.0° | −2.7° | 0.1° | 1.8° | 4.7° | −1.7° | 1.2° |
| $^3H_6$ | 2.6° | 5.5° | 2.7° | 5.5° | −1.2° | 1.7° | −1.6° | 1.3° |
| $^3P_2$ | 32.2±1.9° | 33.6±1.9° | 37.6±2.1° | 46.3±1.5° | 45.1±1.2° | 46.7±1.3° | 46.2±2.1° | 51.5±1.5° |
| $^3F_2$ | 1.6±2.0° | 3.8±2.0° | −1.0±2.3° | −6.0±1.8° | −4.0±1.1° | −2.0±1.1° | −2.7±2.3° | −4.5±1.8° |
| $\epsilon_2$ | −1.9±1.1° | −7.5±4.1° | −18.6±1.7° | −46.2±3.1° | 3.5±0.8° | 8.6±2.0° | −15.0±1.3° | −33.4±3.1° |
| $^3F_4$ | 6.3±1.0° | 9.9±1.0° | 5.0±1.5° | 10.2±1.5° | 1.0±0.9° | 4.2±0.9° | 5.2±1.5° | 8.0±1.5° |
| $^3H_4$ | 2.9° | 4.5° | 4.2° | 4.2° | −2.2° | −0.2° | −1.4° | 1.0° |
| $\epsilon_4$ | −2.3° | −57.5° | −3.0° | −87.5° | −1.8° | −53.5° | −1.8° | −31.1° |

| | Solution 5 ($\mathfrak{M}=34.2$) | | Solution 6 ($\mathfrak{M}=34.6$) | | Solution 7 ($\mathfrak{M}=41.3$) | | Solution 8 ($\mathfrak{M}=52.3$) | |
|---|---|---|---|---|---|---|---|---|
| Type | Nuclear bar | BB | Nuclear bar | BB | Nuclear bar | BB | Nuclear bar | BB |
| $^1S_0$ | 94.4±3.7° | 94.4±3.7° | −0.5±4.6° | −0.5±4.6° | 23.7±4.1° | 23.7±4.1° | 57.2±5.1° | 57.2±5.1° |
| $^1D_2$ | 1.8±0.9° | 3.6±0.9° | 25.7±1.2° | 27.7±1.2° | −0.6±2.4° | 1.2±1.7° | 9.5±1.6° | 11.4±1.6° |
| $^1G_4$ | 5.8° | 8.4° | −2.1° | 0.6° | 3.1° | 5.8° | −0.6° | 2.0° |
| $^3P_0$ | 75.0±3.7° | 76.2±3.7° | −129.4±3.8° | −128.2±3.8° | 7.4±4.6° | 8.7±4.6° | 134.0±6.7° | 135.2±6.7° |
| $^3P_1$ | 9.9±1.8° | 11.2±1.8° | −26.8±1.8° | −25.5±1.8° | 71.0±1.9° | 72.3±1.9° | 18.9±2.2° | 20.1±2.2° |
| $^3F_3$ | −7.5±1.5° | −5.2±1.5° | 6.1±2.2° | 8.4±2.2° | −11.5±1.7° | −9.1±1.4° | −14.9±1.5° | −12.6±1.5° |
| $^3H_5$ | −0.9° | 2.0° | −3.9° | −1.0° | −1.1° | 1.8° | 0.5° | 3.4° |
| $^3H_6$ | 1.2° | 4.0° | 0.6° | 3.5° | 1.2° | 4.1° | 1.0° | 3.9° |
| $^3P_2$ | 13.4±1.0° | 15.6±0.7° | 16.3±1.0° | 17.5±1.0° | 8.0±0.7° | 9.2±0.7° | 4.3±1.1° | 5.6±1.1° |
| $^3F_2$ | −29.8±2.0° | −28.5±2.1° | −4.2±1.4° | −1.9±1.3° | −26.0±1.7° | −23.7±1.7° | −17.5±1.5° | −15.3±1.5° |
| $\epsilon_2$ | 6.7±0.6° | 18.1±3.8° | −0.4±1.2° | −2.1±6.2° | −0.9±1.4° | −3.1±4.9° | −1.0±1.6° | −5.6±7.1° |
| $^3F_4$ | 2.8±0.5° | 8.7±0.5° | 6.5±0.5° | 10.8±0.5° | 3.2±0.5° | 7.3±0.5° | 4.5±0.5° | 7.3±0.5° |
| $^3H_4$ | 2.3° | 1.5° | 4.4° | 5.4° | 0.0° | 1.2° | 1.6° | 4.0° |
| $\epsilon_4$ | −3.6° | −90.8° | 2.6° | 73.8° | −2.7° | −64.3° | −1.2° | −46.9° |

fore continued until these seven had been obtained five additional times from new random points, the $A$ data now being included. All the solutions obtained from the 60 new random points were included among the 19 solutions previously found.

For the third and final stage of the search program, the Coulomb effects and the contribution of $G$ and $H$ waves were introduced into the theoretical forms and the data were augmented by including measurements for which $\theta$ was less than 20°. For these angles the Coulomb effects and higher partial waves would be expected to become important. The work was shifted from the MANIAC to the IBM 704 to accommodate the increased complexity of the problem. No large-scale search was attempted. The 19 MANIAC solutions, suitably adjusted to account for the inclusion of Coulomb effects, were used as starting points for both the grid and the Davidon search procedures. The two procedures led to the same final solutions in all but four cases. Except in these four cases the final solution did not differ significantly from the corresponding starting points, the differences being of the order of 3° or less in all phase shifts. From the seven best MANIAC solutions eight final solutions were obtained (in one case the grid and Davidon procedures led to different solutions). These eight solutions, which are significantly better than the other solutions obtained, are recorded in Table IV. The errors quoted in the results were derived from the error matrix. The treatment of errors is the same as was used by Anderson, Davidon, Glicksman, and Kruse,[20] and the method is adequately described in their paper. The error matrices were computed for the MANIAC solutions and have not been recalculated for the final "704" solutions. For this reason errors for the $G$ and $H$ phase shifts are not given.

## 5. DISCUSSION

The comparison of the values of the experimental quantities predicted by the five best solutions to the measured values are shown in Figs. 1 through 5. Except for the two small-angle points in the $R$ experiment, the fit of all these solutions is good.

The solutions may also be evaluated by comparing the values of $\mathfrak{M}$ with the value expected from statistical considerations. If the true phase shifts for partial waves higher than $H$ waves were really zero, and if the errors are statistical, then the most probable value of $\mathfrak{M}$ at that relative minimum which lies in the neighborhood

[20] Anderson, Davidon, Glicksman, and Kruse, Phys. Rev. **100**, 279 (1955).

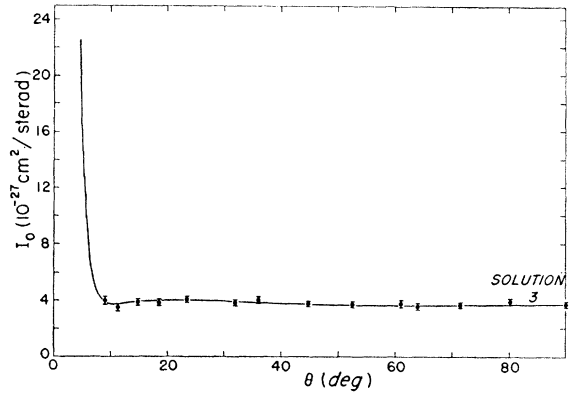


FIG. 1. Differential scattering cross section as a function of center-of-mass scattering angle, $\theta$, as predicted by phase shift solution 3. Nearly identical curves are obtained for solutions 1, 2, 4, and 6. Experimental values are shown for comparison.

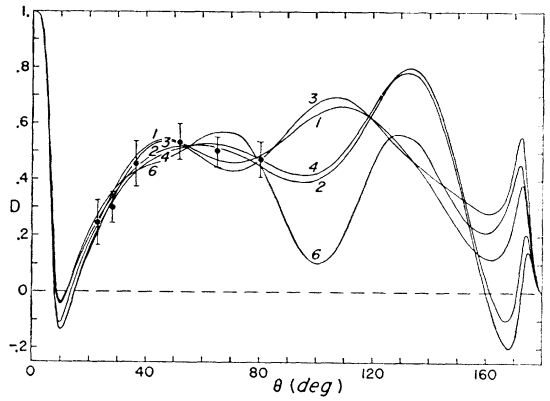


FIG. 3. Plot of $D$ *vs* $\theta$ for solutions 1, 2, 3, 4, and 6. Experimental values are shown for comparison.

of the true solution is the difference between the number of observables and the number of phase shifts. This is $36-14=22$. There is a 90% chance that $\mathfrak{M}$ will be larger than $\sim 14$ and smaller than $\sim 34$. The probability that $\mathfrak{M}>34$ is $\sim 5\%$ and the probability that it is greater than 40 is $\sim 1\%$. The values of $\mathfrak{M}$ for the solutions listed are given in Table IV. The four best solutions are seen to have $\mathfrak{M}$ in the range $17<\mathfrak{M}<24$, and the four fair solutions have $\mathfrak{M}$ in the range $34<\mathfrak{M}<53$. The solutions not listed have $\mathfrak{M}>62$.

It is natural to see to what extent the phase shifts corresponding to the same $l$ value obey the interval rules for $\mathbf{L}\cdot\mathbf{S}$ and the tensor forces. If the potential were due to a sum of central and $\mathbf{L}\cdot\mathbf{S}$ forces, then in the Born approximation we would have

$$\rho_1\equiv\frac{\bar{\delta}_{10}-\bar{\delta}_{11}}{\bar{\delta}_{11}-\bar{\delta}_{12}}=\tfrac{1}{2},\quad \rho_3\equiv\frac{\bar{\delta}_{32}-\bar{\delta}_{33}}{\bar{\delta}_{33}-\bar{\delta}_{34}}=\tfrac{3}{4}. \tag{5.1}$$

Only Solutions 2 and 4 satisfy these relations even approximately. Solution 4 gives $\rho_1=0.60\pm0.08$ and $\rho_3=0.70\pm0.32$, in close agreement with the interval rule; and Solution 2 gives $\rho_1=0.80\pm0.08$ and $\rho_3=0.33\pm0.53$ in fair agreement. However, the mixing parameters, which would be zero if only central and $\mathbf{L}\cdot\mathbf{S}$ forces were present, are rather large in the $j=2$ states for both Solutions 2 and 4. A large mixing parameter would indicate the presence of a tensor force. The interval rules that would obtain if only tensor and central forces were present are

$$\rho_1=-5/2,\quad \rho_3=-27/20, \tag{5.2}$$

but neither of these relations is satisfied even approximately by any of the solutions.

The failure of the simple interval rules seems to indicate that either the spin-dependent forces must contain important contributions of more than one type or that the forces are sufficiently strong to invalidate the Born approximation results even in sign and order of magnitude. The latter possibility is being studied by Gammel and Thaler, who are investigating the possibility that the tensor force is strong enough to cause the $^3P_0$ phase shift to change sign. The 310-Mev data can be satisfactorily explained in this way, but it is not known whether an energy- and charge-independent potential can be found.

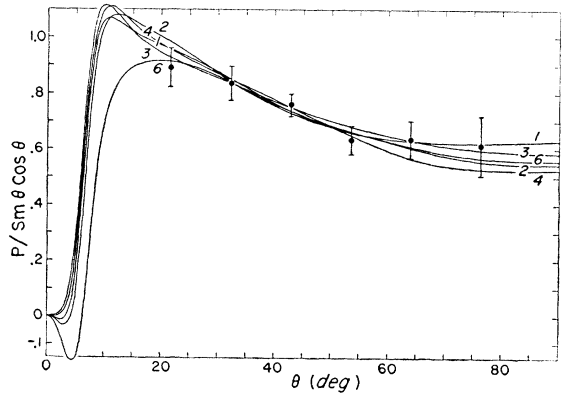


FIG. 2. Plot of $P/\sin\theta\cos\theta$ *vs* $\theta$ for solutions 1, 2, 3, 4, and 6. Experimental values are shown for comparison.

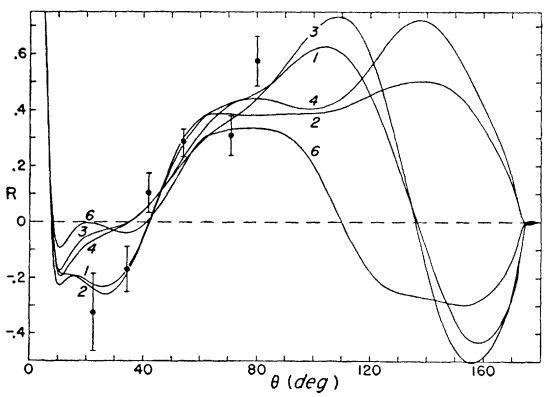


FIG. 4. Plot of $R$ *vs* $\theta$ for solutions 1, 2, 3, 4, and 6. Experimental values are shown for comparison.

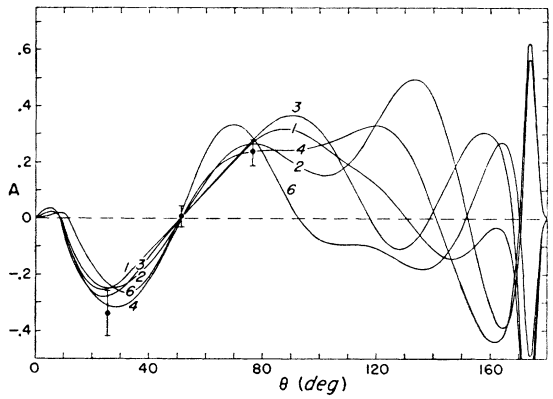


FIG. 5. Plot of $A$ *vs* $\theta$ for solutions 1, 2, 3, 4, and 6. Experimental values are shown for comparison.

Additional restrictions on the $p$-$p$ phase shifts may be obtained from a study of the reaction $p+p \rightarrow \pi+d$.[21] The analysis of this reaction, which is given in reference 8, shows Solutions 5, 7, and 8 to be unsatisfactory. The remaining five solutions are therefore preferred. There are several characteristics common to the five good solutions. All have negative $^1S_0$ phase shifts and positive $^1D_2$ and $^1G_4$ phase shifts. This suggests that in the singlet state there is a repulsive hard core surrounded by an attractive potential. In the triplet states the $^3P_0$ and $^3P_1$ phase shifts are negative, whereas the $^3P_2$ phase shifts are large and positive. This suggests that $\mathbf{L}\cdot\mathbf{S}$ forces are more important than the tensor forces unless the tensor forces are very strong. Indeed it had been concluded previously by Gammel and Thaler, on the basis of an extensive machine analysis of the solutions reported in an earlier (unpublished) version of the present work, that combinations of central and tensor forces alone could not produce any of the reported sets of phase shifts unless potentials strong enough to produce resonances were used. Similar conclusions have also been reached by Wolfenstein[22] from a direct analysis of the experiments near 90°.

The present analysis has been restricted to the Berkeley $p$-$p$ experiments near 300 Mev. It is to be expected that the extension to lower energies and, with the assumption of charge independence, to the $n$-$p$ system would provide a means of selecting from among the solutions that have been obtained. The use of dispersion relations to select from among the possible solutions—a method which has been very useful in the case of pion-nucleon phase shifts—has not been possible because the extension of dispersion relations to $p$-$p$ interactions has, as yet, not been achieved.

## ACKNOWLEDGMENTS

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## APPENDIX. SUBTRACTION OF COULOMB EFFECTS

For those partial waves in which the angular momentum is a constant, nuclear phase shifts may be defined by

$$\delta_l{}^N=\delta_l-\Phi_l, \quad \delta_{ll}{}^N=\delta_{ll}-\Phi_l, \tag{A.1}$$

where the $\Phi_l$ are defined in Eq. (3.13). Under the condition that the Coulomb interaction can be considered confined to a region outside the nuclear region and that the WKB approximation is valid in this outside region, the $\delta^N$ defined by Eq. (A.1) are the phase shifts that would be obtained from the pure nuclear interaction alone. The same result may also be obtained for the case where mixing occurs if the nuclear phase shifts are defined in terms of a nuclear $S$ matrix $S_N$, which is related to the $S$ matrix by the matrix equation

$$S_N=[\exp(-i\Phi)]S[\exp(-i\Phi)], \tag{A.2}$$

where $\Phi$ is the diagonal matrix whose elements are $\Phi_l$. The nuclear $S$ matrix, $S_N$, may be expressed in terms of nuclear phase shifts by means of the same equations as were used to express the $S$ matrix in terms of the original (total) phase shifts. By using Eq. (A.2) the relations between the nuclear and total phase shifts may then be derived. The expressions for the Blatt and Biedenharn nuclear phase shifts in terms of the Blatt and Biedenharn total phase shifts are quite complicated for those phase shifts which involve mixing. An alternative method of defining phase shifts in the coupled case is to write the $S$ matrix in the form

$$S=(\exp i\bar{\delta})(\exp 2i\bar{\epsilon})(\exp i\bar{\delta}), \tag{A.3}$$

where $\bar{\delta}$ is the diagonal matrix with elements $\bar{\delta}_l$, and $\bar{\epsilon}$ is a symmetric matrix with zeros on the diagonal. For the two-by-two case this gives

$$(S)_j=\begin{pmatrix}\exp(i\bar{\delta}_{j-1,j}) & 0\\ 0 & \exp(i\bar{\delta}_{j+1,j})\end{pmatrix}\begin{pmatrix}\cos 2\bar{\epsilon}_j & i\sin 2\bar{\epsilon}_j\\ i\sin 2\bar{\epsilon}_j & \cos 2\bar{\epsilon}_j\end{pmatrix}\times\begin{pmatrix}\exp(i\bar{\delta}_{j-1,j}) & 0\\ 0 & \exp(i\bar{\delta}_{j+1,1})\end{pmatrix}. \tag{A.4}$$

The conversion between nuclear phase shifts and total phase shifts is simple when the bar phase shifts are used. One finds

$$\bar{\delta}_{j\pm1}{}^N=\bar{\delta}_{j\pm1,j}-\Phi_{j\pm1}, \quad \bar{\epsilon}_j{}^N=\bar{\epsilon}_j. \tag{A.5}$$

[21] Robert Tripp, Phys. Rev. **102**, 862 (1956).

[22] Lincoln Wolfenstein, Bull. Am. Phys. Soc. Ser. II, **1**, 36 (1956).

To obtain the nuclear Blatt and Biedenharn phase shifts, one may use Eqs. (A.5) together with the equations connecting the Blatt and Biedenharn phase shifts to the barred phase shifts. These equations are:

$$\begin{aligned}&\delta_{j+1,j}+\delta_{j-1,j}=\bar{\delta}_{j+1,j}+\bar{\delta}_{j-1,j},\\&\sin(\bar{\delta}_{j-1,j}-\bar{\delta}_{j+1,j})=\tan 2\bar{\epsilon}_j/\tan 2\epsilon_j,\\&\sin(\delta_{j-1,j}-\delta_{j+1,j})=\sin 2\bar{\epsilon}_j/\sin 2\epsilon_j.\end{aligned}\tag{A.6}$$

The mixing parameter $\bar{\epsilon}_j$ has a simple interpretation. It gives the proportions into which an incoming beam in one channel (partial wave) divides between the two outgoing channels. It measures therefore the degree to which $l$ is not conserved. Equations (A.6) show that $\epsilon_j$ does not give a good indication of the degree to which $l$ is conserved, since it becomes large when the phase shifts $\delta_{lj}$ become small in comparison to $\bar{\epsilon}$. Furthermore, the parameter $\epsilon_j$ depends upon the phases of the basis vectors in terms of which the $S$ matrix is defined.[13] Its value is, therefore, a reflection more of the mathematical conventions than of the actual physics.

Another set of parameters that have been used to describe the scattering in the coupled states are the real parameters $x_{j\pm1,j}$, $y_j$, defined by writing the asymptotic forms of two independent solutions to the coupled equations in the form[23]

$$\begin{aligned}&u_{j-1,j}(r)\simeq F_{j-1}(r)+x_{j-1,j}G_{j-1}(r),\\&u_{j+1,j}(r)\simeq y_jG_{j+1}(r),\\&u_{j-1,j}'(r)\simeq y_jG_{j-1}(r),\\&u_{j+1,j}'(r)\simeq F_{j+1}(r)+x_{j+1,j}G_{j+1}(r).\end{aligned}\tag{A.7}$$

Here $r^{-1}F_l(r)$ and $r^{-1}G_l(r)$ are the regular and irregular solutions of the Coulomb radial partial-wave equations[16] and the $r^{-1}u_{lj}(r)$ are the radial wave functions in the $lj$ channel. The fact that $y_j$ is the same for the two solutions is a consequence of the Wronskian condition. A chief advantage of these parameters is that in the Born approximation they have the simple forms[24]

$$\begin{aligned}x_{j\pm1,j}&=-(j\pm1,\,j|V|j\pm1,\,j),\\y_j&=-(j-1,\,j|V|j+1,\,j)\\&=-(j+1,\,j|V|j-1,\,j),\end{aligned}\tag{A.8}$$

where $V$ is the potential in units of the center-of-mass energy of the system. The matrix elements are defined as

$$(lj|V|l'j')\equiv\int_0^\infty d(kr)\int d\Omega F_l(r)Y_{jls}^{m_j}(\theta\phi)V\times Y_{j'l's}^{m_j}(\theta\phi)F_{l'}(r).\tag{A.9}$$

They are independent of $m_j$ if $V$ is invariant under rotations. These parameters are related to the Blatt and Biedenharn phases shifts by the equations[25]

$$\begin{aligned}x_{j\mp1}&=\cos^2\epsilon\,\tan\delta_{j\mp1,j}+\sin^2\epsilon\,\tan\delta_{j\pm1,j},\\y_j&=\tfrac{1}{2}\sin 2\epsilon(\tan\delta_{j-1}-\tan\delta_{j+1}).\end{aligned}\tag{A.10}$$

The reciprocal equations are

$$\begin{aligned}&\tan 2\epsilon_j=2y_j/(x_{j-1,j}-x_{j+1,j}),\\&\tan\delta_{j\pm1,j}=\tfrac{1}{2}\{x_{j-1,j}+x_{j+1,j}\\&\qquad\mp[(x_{j-1,j}-x_{j+1,j})^2+(2y_j)^2]^{\frac{1}{2}}\}.\end{aligned}\tag{A.11}$$

For small phases shifts, where the sines and tangents of $\delta_{j\pm1,j}$ may be replaced by their arguments, one finds from a comparison of Eqs. (A.6) and (A.11) the correspondence

$$y_j\to\epsilon_j,\quad x_{j\pm1,j}\to\bar{\delta}_{j\pm1,j}.\tag{A.12}$$

The barred phase shifts, like the $x$ and $y$, are therefore proportional to the matrix elements for weak interactions.

[23] J. L. McHale and R. L. Thaler, Phys. Rev. **98**, 273 (1955).

[24] Roy Thaler (private communication).

[25] John Gammel (private communication).