Microwave Conductivity of an Ionized Decaying Plasma at Low Pressures*

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Margenau's analysis for the microwave conductivity of an infinite decaying plasma in a uniform field is extended to the case of a bounded plasma in a slightly nonuniform field. It is shown that, if we assume a power expansion for the electron-collision frequency as a function of energy, the conductivity at low pressures can be computed as a function of time and position when the spatial and time variations of the density and energy moments of the electron-distribution function are known. An approximate method, based on a convenient integration of Boltzmann equation is given to compute these quantities, when inelastic collisions can be neglected. The steady-state conductivity in the late afterglow of a diffusion-controlled decaying plasma is thus explicitly determined for two experimental conditions: a plasma filling a cubic quartz bottle centered in a parallelepiped microwave cavity and a plasma filling a quartz tube of square cross section in a wave guide. The limit for the validity of the theory set by the appearance of inelastic collisions at high electric fields is investigated.

INTRODUCTION

IN this paper we shall extend Margenau's analysis[1] for the microwave conductivity of an infinite decaying plasma in a uniform electric field to the case of a bounded plasma in a slightly nonuniform field. These are the experimental conditions we meet in measuring the conductivity of the plasma in a microwave cavity or in a wave guide. We shall limit the discussion to the case of a decaying plasma at low pressures, so that the electron-collision frequency is much smaller than the angular frequency of the applied field; this case is the most important one when measurements of microwave conductivity are used to determine the collision frequency in a gas at low electron energies.[2,3]

DISTRIBUTION FUNCTION AND THE BOLTZMANN EQUATION

To determine the conductivity of a plasma we must know the distribution $F(\mathbf{r},\mathbf{v},t)$ of the free electrons as a function of position $\mathbf{r}$, velocity $\mathbf{v}$, and time t. This distribution is determined by the Boltzmann transport equation in phase space. We assume that the following experimental conditions are satisfied: (a) the microwave electric field $\mathbf{E}(\mathbf{r})\exp(j\omega t)$ used to measure the conductivity is the only applied field; (b) the angular frequency ω is sufficiently high so that between cycles the electrons undergo no appreciable loss of energy or significant change in density; (c) over the significant part of the distribution function $\nu_m \gg \nu_x$, ν_i, where $\nu_m(v)$, $\nu_x(v)$, and $\nu_i(v)$ are the collision frequencies of an electron with a molecule for momentum transfer, excitation, and ionization, respectively; (d) the electron and ion densities are low enough so that phenomena like electron-ion collisions,[4] electron-electron interactions,[5] and plasma resonance[6] can be neglected; (e) the amplitude of the electron oscillations, which is proportional to $E/[\omega(\nu_m^2+\omega^2)^{\frac{1}{2}}]$, is smaller than the dimensions of the container, so that the electrons do not travel completely across and collide with the walls in every half-cycle; (f) any dimension of the container is signicantly larger than the electron mean free path.

With these assumptions the distribution function can be written[7]:

$$F = F_0{}^0 + \mathbf{v}\cdot[\mathbf{F}_0{}^1 + \mathbf{F}_1{}^1\exp(j\omega t)]/v, \tag{1}$$

where $F_0{}^0$, $\mathbf{F}_0{}^1$, and $\mathbf{F}_1{}^1$ are functions of $\mathbf{r}$, v, and t (the variation with t is slow). By substituting Eq. (1) in the Boltzmann equation, and by separating and eliminating the various harmonic terms, we obtain (see Margenau,[1] Allis and Brown,[7] Bernstein and Holstein,[8] and Rose and Brown[9] for particular cases):

$$\begin{aligned}\partial F_0{}^0/\partial t &- (1/3v)(\nabla + \mathbf{a}_s\partial/v\partial v)\cdot(v^3/\nu_m)(\nabla + \mathbf{a}_s\partial/v\partial v)F_0{}^0 \\ &- (a^2/6v^2)\partial[\nu_m(\nu_m{}^2+\omega^2)^{-1}v^2\partial F_0{}^0/\partial v]/\partial v \\ &- (m/Mv^2)\partial(\nu_m v^3 F_0{}^0)/\partial v - (2U/3Mv^2) \\ &\times\partial(\nu_m v^2\partial F_0{}^0/\partial v)\partial v + (\nu_x+\nu_i+\alpha n_+ - q)F_0{}^0 = 0,\end{aligned} \tag{2}$$

$$\nu_m\mathbf{F}_0{}^1 = -(v\nabla + \mathbf{a}_s\partial/\partial v)F_0{}^0, \tag{3}$$

$$(\nu_m + j\omega)\mathbf{F}_1{}^1 = -\mathbf{a}\partial F_0{}^0/\partial v. \tag{4}$$

Here $\mathbf{a}(\mathbf{r}) = -(e/m)\mathbf{E}(\mathbf{r})$, $\mathbf{a}_s(\mathbf{r},t) = -(e/m)\mathbf{E}_s(\mathbf{r},t)$, $\mathbf{E}_s$ being the dc electric field set by the space charge, e and m the electron charge and mass, ∇ the gradient operator in configuration space, M the mass of a molecule and

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[1] H. Margenau, Phys. Rev. **69**, 508 (1946).
[2] L. Gould and S. C. Brown, Phys. Rev. **95**, 897 (1954).
[3] A. L. Gilardini and S. C. Brown, following paper [Phys. Rev. **105**, 31 (1956)].
[4] J. M. Anderson and L. Goldstein, Phys. Rev. **100**, 1037 (1955).
[5] W. P. Allis, *Handbuch der Physik* (Springer-Verlag, Berlin, 1956) Vol. 21, p. 439.
[6] K. S. Champion and S. C. Brown, Phys. Rev. **98**, 559(A) (1955).
[7] W. P. Allis and S. C. Brown, Phys. Rev. **87**, 419 (1952).
[8] I. B. Bernstein and T. Holstein, Phys. Rev. **94**, 1475 (1954).
[9] D. J. Rose and S. C. Brown, Phys. Rev. **98**, 310 (1955).

U its average kinetic energy, α the electron-ion recombination coefficient, $n_+(\mathbf{r},t)$ the positive-ion density, and qF_0^0 the rate of appearance of electrons at low energies as a result of excitation and ionization. The space-charge field is given by Poisson's equation:

$$\nabla\cdot\mathbf{E}_s=e(n_+-n)/\epsilon_0, \qquad (5)$$

where ϵ_0 is the permittivity of free space and $n(\mathbf{r},t)$ is the electron density:

$$n=\int_0^\infty F_0^0 4\pi v^2 dv. \qquad (6)$$

Equations (2), (5), and (6) form an integro-differential system of equations for F_0^0; Eqs. (3) and (4) give $\mathbf{F}_0^1$ $\mathbf{F}_1^1$ from F_0^0.

MICROWAVE CONDUCTIVITY

Before investigating further the solution of the Boltzmann equation, we must see how to use this solution to compute the microwave conductivity of a plasma. The complex ac conductivity of a plasma $\sigma(\mathbf{r},t)$ is defined as the ratio between the current density and the electric field that produces the current. The ac current density is given by the equation[7]:

$$\sigma\mathbf{E}=-e\int_0^\infty \mathbf{F}_1^1(4\pi v^3/3)dv. \qquad (7)$$

This, with (4), gives

$$\sigma=-\frac{4\pi e^2}{3m\omega}\int_0^\infty \frac{(\nu_m/\omega)-j}{1+(\nu_m/\omega)^2}\frac{\partial F_0^0}{\partial v}v^3dv. \qquad (8)$$

Henceforth we limit the discussion to the low-pressure case $\nu_m^2\ll\omega^2$; with the aid of a partial integration and of Eq. (6), from Eq. (8) we obtain

$$\sigma=(e^2n/m\omega)(\rho p_0-j), \qquad (9)$$

where p_0 is the normalized pressure and

$$\rho=-(4\pi/3n)\int_0^\infty(\nu_m/\omega p_0)(\partial F_0^0/\partial v)v^3dv \qquad (10)$$

is a function of time and position, called the conductivity ratio.

Let us expand ν_m in powers of electron energy (the convenience of this expansion will be apparent in the rest of the paper):

$$\nu_m/\omega p_0=\sum_{l=0}^\infty b_l(mv^2/2)^l \qquad (11)$$

and substitute in (10); this gives[10]:

$$\rho=\sum_{l=0}^\infty[1+(2l/3)]b_lw_l, \qquad (12)$$

where

$$nw_l=4\pi(m/2)^l\int_0^\infty F_0^0v^{2(l+1)}dv \qquad (13)$$

are the energy moments of F_0^0. From Eq. (6) we have $w_0=1$.

Equations (9) and (12) determine the conductivity as a function of time and position, when the density (6) and moments (13) are known as functions of these variables. The next section is devoted to the determination of these density and moments functions for the case of a decaying plasma. We limit the discussion to this case because it is easy and useful when the conductivity measurements are used to determine the collision frequency for electrons in a gas.

DENSITY AND MOMENTS VARIATION IN A DECAYING PLASMA

We define the decaying plasma as a plasma in which there is no production of new ions; more generally, we assume no inelastic losses or $\nu_x=\nu_i=q=0$. These assumptions are usually satisfied in the late afterglow of a pulsed discharge. In these conditions also the usual assumptions of ambipolar diffusion can be made: $n_+\approx n$ and $\mathbf{\Gamma}_+\approx\mathbf{\Gamma}$, where $\mathbf{\Gamma}_+$ and $\mathbf{\Gamma}$ are the dc positive-ion and electron flow, respectively; the limits of electron density, temperature, and size of the container, in which the ambipolar diffusion theory is valid, have been discussed by Allis and Rose.[11]

When the density of electrons and ions are set equal, the space-charge field can no longer be determined from Poisson's equation and the following procedure must be adopted.[12] From Eq. (3), we have

$$\mathbf{\Gamma}=\int_0^\infty \mathbf{F}_0^1(4\pi v^3/3)dv=-\nabla(Dn)-\mu n\mathbf{E}_s, \qquad (14)$$

where D and μ are the electron diffusion coefficient and dc mobility, defined by

$$Dn=(4\pi/3)\int_0^\infty F_0^0(v^4/\nu_m)dv, \qquad (15)$$

$$\mu n=-(4\pi e/3m)\int_0^\infty(\partial F_0^0/\partial v)(v^3/\nu_m)dv. \qquad (16)$$

For the positive ion flow we have the analogous equation:

$$\mathbf{\Gamma}_+=-\nabla(D_+n_+)+\mu_+n_+\mathbf{E}_s, \qquad (17)$$

where D_+ and μ_+ are the positive ion diffusion coefficient and dc mobility, generally much smaller than the corresponding quantities for electrons. Consequently the flow $\mathbf{\Gamma}_+$ is approximately equal to $\mathbf{\Gamma}$, but it is much smaller than any one of the two terms on the right of Eq. (14); these terms must therefore balance

[10] Phelps, Fundingsland, and Brown, Phys. Rev. 84, 559 (1951).

[11] W. P. Allis and D. J. Rose, Phys. Rev. 93, 84 (1954).

[12] W. Schottky, Physik. Z. 25, 342 (1924).

each other, and hence we obtain

$$\mathbf{E}_s \approx -(1/\mu n)\nabla(Dn). \tag{18}$$

Equation (18) now replaces Eq. (5). The flow is obtained by substituting (18) in (17). Assuming that D_+ and μ_+ are independent of position and that $D_+/\mu_+ = 2U/3e$, the distribution for positive ions being Maxwellian, we obtain

$$\mathbf{\Gamma} \approx -\mu_+[(2U/3e)\nabla n + (1/\mu)\nabla(Dn)]. \tag{19}$$

We are now in a position to calculate the differential equations for the density $n(\mathbf{r},t)$ and the moments $w_l(\mathbf{r},t)$. We obtain the first multiplying Eq. (2) by $4\pi v^2 dv$, and integrating over velocity space. The result, using Eqs. (3) and (14), is

$$\partial n/\partial t + \nabla\cdot\mathbf{\Gamma} + \alpha n^2 = 0. \tag{20}$$

The equation for the moments is obtained analogously, multiplying Eq. (2) by $4\pi(m/2)^l v^{2(l+1)}dv/\nu_m$ and integrating over velocity space. In the integration, the collision frequency ν_m is assumed proportional to v^h, h being a constant, which is a cruder approximation than (11); but we shall see that this approximation affects only the correction terms for the nonuniformity in the applied field, and is, therefore, adequate. The final result is

$$w_l = \tfrac{1}{3}(1+2l)(U+Ma^2/4\omega^2)w_{l-1} + (M/m)I_l, \quad \text{for } l \geq 1, \tag{21}$$

where

$$I_l = (4\pi/n)(m/2)^l(2l-h)^{-1}\int_0^\infty (v^{2(l+1)}/\nu_m) \times\{(1/3v)(\nabla+\mathbf{a}_s\partial/v\partial v)\cdot(v^3/\nu_m)(\nabla+\mathbf{a}_s\partial/v\partial v)F_0^0 + [(\nabla\cdot\mathbf{\Gamma})-n\partial/\partial t](F_0^0/n)\}dv. \tag{22}$$

For an infinite plasma in a uniform field and steady-state conditions for electron energies, diffusion and space charge field do not exist and $\mathbf{\Gamma}=I_l=0$; in this case we find Margenau's result—that the electron distribution function is Maxwellian with an average energy $U+Ma^2/4\omega^2$. This result can be directly derived from (21) or, in a more general way independent of the approximation we made for ν_m, solving Eq. (2). This suggests that, if nonuniformities are small, if electron energies are near to steady-state values and diffusion cooling effects can be neglected,[13] the terms $(\nabla\cdot\mathbf{\Gamma})$ in Eq. (20) and I_l in Eq. (21) can be computed assuming a Maxwellian distribution in velocity for F_0^0; for the average electron energy of this distribution we assume the correct value w_1. In this way, with the approximation $D_+ \ll D$, we obtain from Eqs. (19) and (22) [see reference 2 for Eq. (23)]:

$$-(3e/2\mu_+)\nabla\cdot\mathbf{\Gamma} = (U+w_1)\nabla^2 n + (2-\tfrac{1}{2}h)\nabla n\cdot\nabla w_1 + (1-\tfrac{1}{2}h)n\nabla^2 w_1, \tag{23}$$

$$I_l = (1/\pi^{\frac{1}{2}}\nu_m)(2w_1/3)^l\{[\Gamma(\tfrac{3}{2}+l-h)/9m\nu_m] \times[A_l(\nabla n\cdot\nabla w_1)/n + B_l\nabla^2 w_1 + C_l(\nabla w_1)^2/w_1] - \Gamma(\tfrac{3}{2}+l-\tfrac{1}{2}h)\partial w_1/w_1\partial t\}, \tag{24}$$

where

$$A_l = 2(5-h),\quad B_l = 2(3+2l-2h),\quad C_l = 4+2l+4l^2-6(1+l)h+3h^2, \tag{25}$$

and ν_m has to be taken at the velocity $(4w_1/3m)^{\frac{1}{2}}$. These formulas are correct only for $h<5/2$; outside of this range some of the integrals used in the derivation diverge. Also, in this range they can be used only as far as the resulting I_l are small compared to the other terms of Eq. (21).

Equations (20) and (21) for $l=1$, with the aid of (23), (24), and (25), form a system of two equations in the unknowns $n(\mathbf{r},t)$ and $w_1(\mathbf{r},t)$, that can be solved when the boundary conditions of a particular experimental case are given. When n and w_1 are known, Eq. (21) gives the space and time variation of higher order moments. Finally, from Eqs. (9) and (12) we can compute $\sigma(\mathbf{r},t)$. The problem of determining the conductivity is thus solved in principle and in the next section we shall apply this theory to a specific experimental case.

From Eqs. (21) and (24) we see that, when $\nabla n/n$ is independent of time, a steady-state solution for electron energies exists for which $\partial w_1/\partial t=0$, and all the moments w_l become independent of time. This is the case in the experimental conditions we shall discuss in the next section.

QUARTZ BOTTLE IN A MICROWAVE CAVITY

We shall determine the steady-state conductivity of a decaying plasma contained in a cubic quartz bottle of side d centered in a parallelepiped microwave cavity with sides L_x, L_y, L_z along the directions x, y, z. The applied electric field corresponds to the fundamental mode TM_{011} and is directed along the x axis; $\mathfrak{U}_c$ is the quantity $M(eE/2m\omega)^2$ at the center of the cavity. We assume that the field is not modified by the presence of the electrons, an acceptable hypothesis at the low electron densities where the present theory is correct, according to assumption (*d*) of the first section. The boundary condition for the density is the usual one: $n=0$ at the walls of the bottle. We assume no recombination and no appreciable presence of higher-order diffusion modes; consequently the decay of n is exponential in time. In this case $\nabla n/n$ is independent of time, and as stated in the last section, the moments w_l in energy steady-state conditions are independent of time. We solve the equations of the last section using

[13] The experimental work of Biondi [M. A. Biondi, Phys. Rev. **93**, 1136 (1954)] on rare gases shows that diffusion cooling is appreciable only at very low pressures, about in the range where condition (f) stated in the beginning of this paper is no longer satisfied.

trigonometric expansions in space for n and w_l, limiting these expansions to the first significant term after the ones for uniform field. This is usually a good approximation, particularly at low pressures. Therefore, we write

$$n=n_0\cos(\pi x/d)[\cos(\pi y/d)+c_y\cos(3\pi y/d)]\times[\cos(\pi z/d)+c_z\cos(3\pi z/d)]\exp(-\gamma t), \quad (26)$$

$$w_l=\bar{w}_l+\Delta_y w_l\cos(2\pi y/d)+\Delta_z w_l\cos(2\pi z/d), \quad (27)$$

and assume that c_y, $c_z\ll 1$, and $\Delta_y w_l$, $\Delta_z w_l\ll\bar{w}_l$. We give final results only for the average and y components; the expressions for the z components are obtained by interchanging the variables. For the energy w_1, we obtain

$$\bar{w}_1=U+\Phi(d/L_y)\Phi(d/L_z)\mathfrak{U}_c+(5-h)\Gamma(5/2-h)\beta(\Delta_y w_1+\Delta_z w_1)/\Gamma(5/2), \quad (28)$$

$$\Delta_y w_1=\Psi(d/L_y)\Phi(d/L_z)[1+5(3-h)\Gamma(5/2-h)\beta/\Gamma(5/2)]^{-1}\mathfrak{U}_c, \quad (29)$$

where

$$\beta=(M/6m)(\pi\mathfrak{L}/d)^2, \quad (30)$$

$\mathfrak{L}$ being the mean free path at the velocity $(4\bar{w}_1/3m)^{\frac{1}{2}}$ and

$$\Phi(\xi)=\tfrac{1}{2}[1+\sin(\pi\xi)/\pi\xi], \quad (31)$$

$$\Psi(\xi)=\sin(\pi\xi)/\pi(\xi^{-1}-\xi). \quad (32)$$

The parameters for the density are

$$c_y=(3h-11)\Delta_y w_1/16(U+\bar{w}_1), \quad (33)$$

$$\gamma=(\pi/d)^2(2\mu_+/e)[U+\bar{w}_1+\tfrac{1}{6}(3-h)(\Delta_y w_1+\Delta_z w_1)]. \quad (34)$$

Higher-order moments can be computed and substituted in Eq. (12); then we obtain the conductivity ratio:

$$\rho=\sum_{l=0}^{\infty}[\Gamma(5/2+l)/\Gamma(5/2)](1+R_l)b_l(2\bar{w}_1/3)^l. \quad (35)$$

R_l is a correction factor for the field nonuniformity, given by

$$R_l\bar{w}_1=(5-h)\delta_l\beta(\Delta_y w_1+\Delta_z w_1)+(l-\eta_l\beta)[\cos(2\pi y/d)\Delta_y w_1+\cos(2\pi z/d)\Delta_z w_1], \quad (36)$$

where $\delta_0=\delta_1=\eta_0=\eta_1=0$, and, for $l\geqslant 2$,

$$\delta_l=\sum_{p=2}^{l}[\Gamma(3/2+p-h)/\Gamma(3/2+p)]-(l-1)\Gamma(5/2-h)/\Gamma(5/2), \quad (37)$$

$$\eta_l=\sum_{p=2}^{l}[(11+4p-5h)\Gamma(3/2+p-h)/\Gamma(3/2+p)]-5(l-1)(3-h)\Gamma(5/2-h)/\Gamma(5/2). \quad (38)$$

As a particular case we have the formulas for ρ in a uniform field: evaluating the limit for d going to zero, we obtain Eq. (35) with $R_l=0$ and $\bar{w}_1=U+\mathfrak{U}_c$. Equations (9), (26), and (35) give the final result: the conductivity of the plasma as a function of time and position.

The same formulas with a few modifications are valid for a plasma contained in a square quartz tube of side d centered in a wave guide of sides L_x and L_y, when the wave propagates along the z axis in the fundamental mode TM_{01} and the electric field is parallel to the x axis. The modifications required are: (a) put $c_z=\Delta_z w_l=0$; and (b) replace Eqs. (33) and (34) with

$$c_y=(3h-10)\Delta_y w_1/16(U+\bar{w}_1), \quad (39)$$

$$\gamma=(\pi/d)^2(4\mu_+/3e)[U+\bar{w}_1+\tfrac{1}{4}(2-h)\Delta_y w_1]. \quad (40)$$

When recombination is present, the solutions are more complicated; if the loss of electrons for recombination predominates over diffusion losses, we can assume n uniform and the formulas for this case can be easily derived.

We have now solved the problem that we have been discussing, namely, the determination of the steady-state microwave conductivity of a low-pressure decaying plasma in a nonuniform field, as a function of position and time, for the most common geometries. We shall devote the next section to relating this conductivity to the integral parameters that are actually measured in our experiments with resonant cavities or wave guides. We shall then discuss the limit imposed on the theory by the appearance of inelastic collisions when the electric field is sufficiently strong.

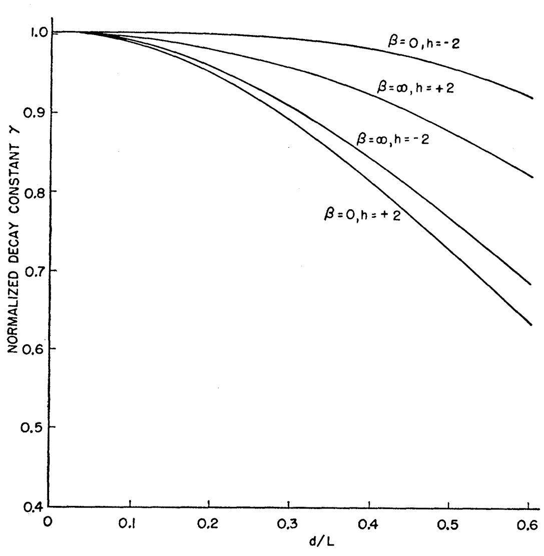

FIG. 1. Normalized decay constant γ as a function of the ratio d/L between the sides of the quartz bottle and the cavity.

AVERAGE CONDUCTIVITY

The conductivity we measure in a resonant cavity or in a wave guide is a spatial average conductivity $\langle\sigma\rangle$, given by[10,14]

$$\langle\sigma\rangle=\int_\tau \sigma E^2 d\tau \bigg/ \int_\tau E^2 d\tau, \qquad (41)$$

where τ is the volume of the cavity or the area of the wave guide.

For the case of the cubic quartz bottle discussed in the last section, we obtain:

$$\langle\sigma\rangle=(e^2n_0/m\omega)(2d/\pi L_x)\Theta_0(d/L_y)\Theta_0(d/L_z) \times[1+c_y\Theta_1(d/L_y)/\Theta_0(d/L_y) +c_z\Theta_1(d/L_z)/\Theta_0(d/L_z)](\langle\rho\rangle p_0-j)\exp(-\gamma t), \qquad (42)$$

where

$$\Theta_n(\xi)=\frac{(-1)^n 2\xi}{(2n+1)\pi}\left\{1+\frac{\cos(\pi\xi)}{1-[2\xi/(2n+1)]^2}\right\}, \qquad (43)$$

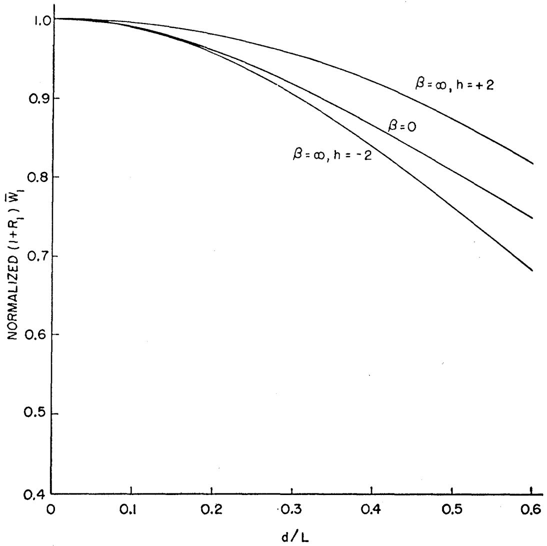

FIG. 2. Normalized first coefficient in the expansion for the average conductivity ratio $(1+R_1)\bar{w}_1$ as a function of d/L.

and $\langle\rho\rangle$ is the average conductivity ratio given by the same Eq. (35), in which R_l is

$$R_l\bar{w}_1=\{\tfrac{1}{2}l+[(5-h)\delta_l-\tfrac{1}{2}\eta_l]\beta\}(\Delta_y w_1+\Delta_z w_1) +\tfrac{1}{2}(l-\eta_l\beta)[\Theta_1(d/L_y)\Delta_y w_1/\Theta_0(d/L_y) +\Theta_1(d/L_z)\Delta_z w_1/\Theta_0(d/L_z)]. \qquad (44)$$

For the case of the square quartz tube the formulas are analogous with the modifications already discussed and $\Theta_n(d/L_z)=1$.

Formula (42) shows that the average conductivity, as compared with the conductivity in a uniform field, is a function of the additional parameters d/L_x, d/L_y, d/L_z, h, and β. To see the effects of these parameters on the conductivity, we plotted in Figs. 1, 2, and 3 the decay constant γ, and the first two coefficients in the expansion for the average conductivity ratio: $(1+R_1)\bar{w}_1$,

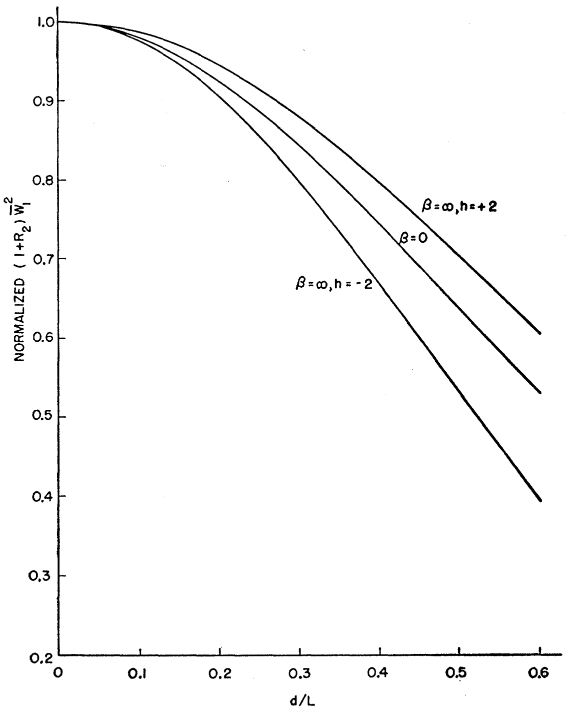

FIG. 3. Normalized second coefficient in the expansion for the average conductivity ratio $(1+R_2)\bar{w}_1^2$ as a function of d/L.

$(1+R_2)\bar{w}_1^2$, respectively, as a function of the ratio d/L for a cubic cavity and for the case $\mathfrak{U}_c\gg U$. The values of the ordinates are normalized to unity for $d/L\to 0$, which represent the uniform field limit. The highest and lowest curve of any graph delimit a region in which fall all the curves for the range $-2<h<2$ and any values of β.

Finally we recall the relation between $\langle\sigma\rangle$ and the parameters which are generally measured at microwave frequencies. If we have a microwave cavity, we measure $\Delta\omega_0$, the change in the resonant frequency ω_0, and $\Delta(1/Q_L)$, the change in the reciprocal of the loaded Q, from the condition of no plasma. We have, for small perturbations,

$$\langle\sigma\rangle/\epsilon_0=\omega_0\Delta(1/Q_L)-2j\Delta\omega_0. \qquad (45)$$

Obviously the measure of Q_L with the plasma present must be performed without changing the amplitude of the field in the cavity. In a wave guide we measure the complex propagation constant with the plasma γ_p and without the plasma γ_0; there we have

$$j\omega\mu_0\langle\sigma\rangle=\gamma_p^2-\gamma_0^2, \qquad (46)$$

where μ_0 is the free-space permeability.

ELECTRON-ENERGY RANGE IN WHICH THE THEORY IS VALID

In the previous sections we neglected the effects of inelastic collisions; this is correct until the average electron energy w_1 is less than a certain value W. In

[14] D. J. Rose and S. C. Brown, J. Appl. Phys. 23, 1028 (1952).

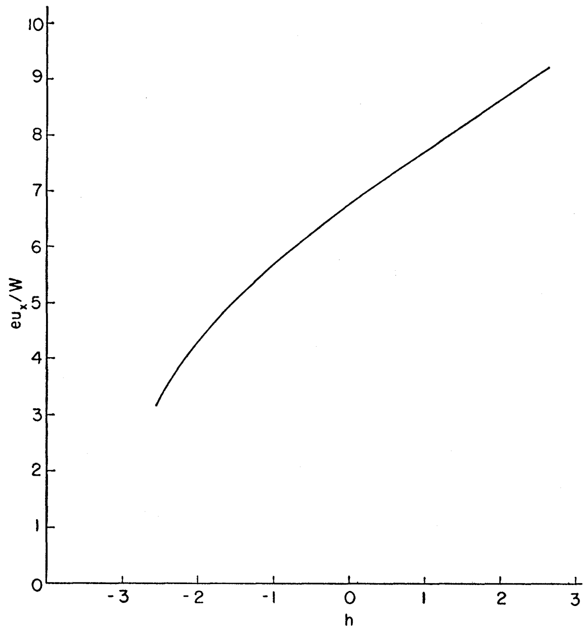

FIG. 4. Ratio of the first excitation potential u_x and the maximum electron energy W as a function of h.

this section we shall derive W as the value of w_1 at which the fraction of the absorbed power going into excitation and ionization becomes appreciable; let us say, one percent.

If the average energy loss in ev per inelastic collision is u_j, the power per electron which goes into excitation and ionization will be $u_j(\langle\nu_x\rangle+\langle\nu_i\rangle)$ where the angular brackets indicate averages over the distribution function. To determine $\langle\nu_x\rangle$ and $\langle\nu_i\rangle$, we have to know the distribution function in the inelastic region; in this region the inelastic collisions dominate all other collision processes, so that the differential equation for the steady-state distribution function can be derived from Eq. (2) neglecting diffusion, recoil, and thermal energies terms, as well as q and α. Then we obtain

$$(\nu_x+\nu_i)F_0{}^0=(a^2/6\omega^2v^2)\partial[\nu_m v^2\partial F_0{}^0/\partial v]/\partial v. \quad (47)$$

Multiplying both terms by $4\pi v^2dv$ and integrating from v_x (the electron velocity corresponding to the potential u_x where excitation starts to take place) to infinity, we obtain

$$\langle\nu_x\rangle+\langle\nu_i\rangle=-[2\pi a^2\nu_m(v_x)v_x^2/3\omega^2n](\partial F_0{}^0/\partial v)_{v=v_x}. \quad (48)$$

In this equation we must know only the derivative at $v=v_x$ of $F_0{}^0$ and, since at this point the distribution function for the inelastic region has to join that for the elastic one, we can substitute the derivative of the latter in Eq. (48). From the discussion given in the previous sections and because we are discussing a case in which the effect of inelastic losses is still very small, we are justified in assuming a Maxwellian velocity distribution in the elastic region.

To determine the energy W, we equate the power per electron going into inelastic losses, computed in the way just mentioned, to one hundredth of the power absorbed per electron, which is $ma^2\langle\nu_m\rangle/2e\omega^2$, when $\nu_m{}^2\ll\omega^2$. When the usual assumption that ν_m varies as v^h is introduced and the average $\langle\nu_m\rangle$ is computed using a Maxwellian distribution in velocity, we end with the following equation for W:

$$(eu_j/W)(eu_x/W)^{(3+h)/2}\exp(-3eu_x/2W) = (2/3)^{(3+h)/2}\Gamma(3/2+h/2)/100. \quad (49)$$

In this formula u_j is not exactly known, but it is not very much larger than u_x, and the final result is rather insensitive to it. In Fig. 4 the quantity eu_x/W is plotted *versus* h, for the case $u_j=u_x$. We can conclude that the theory discussed in this paper is correct for electric-fields less than approximately $2\omega m(W-U)^{\frac{1}{2}}/eM^{\frac{1}{2}}$.

CONCLUSION

In this paper we have given the formulas for computing the steady-state conductivity of a low-pressure decaying plasma contained in a quartz bottle, centered in a microwave cavity or in a wave guide. Knowing the field at the center of the cavity and the electron-collision frequency as a function of the velocity, we compute the average electron energy $\bar{w}_1$ and the first-harmonic terms Δw_1 for the spatial distribution of the electron energy, using formulas (28) and (29) for the particular geometry that we have discussed. The average conductivity, which we measure experimentally, can be computed by means of the formulas (42), (35), and (44). These same formulas can be used reciprocally to determine the collision frequency for slow electrons in a gas when the conductivity is measured as a function of the applied field. We shall discuss this application in the following paper.[3]