# Particle Aspect of the Electromagnetic Field Equations

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In this paper it is shown that Maxwell's theory of the electromagnetic field in vacuum can be stated in a form closely parallel to Dirac's theory of the electron. The electromagnetic field is described by a three-by-one column matrix whose elements are linear combinations of the three independent spinor components of the field. The Maxwell equations take a form similar to the Dirac equation for a free electron but involving three-by-three matrices. In terms of a wave function normalized so that the integral-square is the number of photons, the classical expressions for the energy, momentum, and angular momentum in the field are related to expected values of the Hamiltonian, momentum, and angular momentum operators. The angular momentum operator consists of an orbital part plus a three-by-three spin-one matrix part. As in Dirac's theory, the contributions from the states of negative Hamiltonian eigenvalue must be subtracted—these are states of opposite circular polarization.

## I. INTRODUCTION

SINCE any quantity which transforms linearly with respect to Lorentz transformations continuous with the identity is a spinor,[1] spinor quantities are regarded as fundamental in general considerations of field theories. The basic equations of such theories can often be written in spinor form, but this is not convenient for discussing the solutions because of the complicated properties of the spinor operator that corresponds to the gradient. A compromise is to describe the field by the independent spinor components (or a linear combination of them) but otherwise write the field equations in terms of the ordinary differentiation operators and matrices. This approach permits the fundamental description of the field to be used without introducing the spinor operations. The situation is illustrated by Dirac's equations for an electron; although, as was originally shown by van der Waerden,[2] Dirac's theory can be expressed completely in spinor form,[3] discussions about solutions of the equations are made more easily if the equations are written in terms of ordinary differentiation operations applied to the spinor components. In this paper Maxwell's equations for the electromagnetic field in vacuum are studied from this point of view.

The spinor formulation of Maxwell's equations was investigated by Laporte and Uhlenbeck.[4] They found that the electromagnetic field can be described by a spinor of the second rank,

$$g_{\dot{1}\dot{1}}=-2i(\psi_2+i\psi_1),$$
$$g_{\dot{2}\dot{2}}=-2i(\psi_2-i\psi_1),$$
$$g_{\dot{1}\dot{2}}=g_{\dot{2}\dot{1}}=2\psi_3,$$

where $\psi_j$ is the complex three-vector,

$$\psi_j=E_j+iB_j, \tag{1}$$

and $E_j$, $B_j$ are the electric and magnetic fields. In the following section Maxwell's equations are restated in terms of the three quantities $\psi_j$ formed into a column matrix $\psi$.

The usefulness of a complex vector such as $\psi_j$ is well known in classical electromagnetic theory.[5] The equation for $\psi$ discussed below was studied by Oppenheimer[6] without identifying the real and imaginary parts of $\psi$ with the electric and magnetic fields. Recently Archibald[7] pointed out that Maxwell's equations could be cast into this form.

The fact that the field equations become similar to the Dirac equations permits the field to be discussed using standard quantum-mechanical methods. The transformation properties of the $\psi$ matrix do not lead to the classical conservation theorems. However, as shown below, a wave function $\phi$ can be defined in such a way that the classical formulas for the energy, momentum, and angular momentum in the field are found from expected values of the usual type of quantum-mechanical operator.

## II. BASIC EQUATIONS

Maxwell's equations for the electromagnetic field in vacuum, in Gaussian unrationalized units, are[8]

$$\epsilon_{jkl}\partial E_l/\partial x_k+c^{-1}\partial B_j/\partial t=0, \tag{2}$$

$$\epsilon_{jkl}\partial B_l/\partial x_k-c^{-1}\partial E_j/\partial t=0, \tag{3}$$

$$\partial E_j/\partial x_j=0, \tag{4}$$

$$\partial B_j/\partial x_j=0. \tag{5}$$

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[1] See, for example, B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, 1932), Sec. 20.

[2] B. L. van der Waerden, Nachr. Ges. Wiss. Göttingen 100 (1929).

[3] This subject has recently been reviewed by W. L. Bade and H. Jehle, Revs. Modern Phys. 25, 714 (1953).

[4] O. Laporte and G. E. Uhlenbeck, Phys. Rev. 37, 1380 (1931).

[5] See, for example, J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), Sec. 1.12.

[6] J. R. Oppenheimer, Phys. Rev. 38, 725 (1931).

[7] W. J. Archibald, Can. J. Phys. 33, 565 (1955).

[8] The conventions used are that Latin indices run from one to three, Greek from one to four, and that a sum is understood to be made on indices repeated within a term.

In terms of the $\psi_j$ of Eq. (1), the equations become

$$\epsilon_{jkl}\partial\psi_l/\partial x_k - ic^{-1}\partial\psi_j/\partial t = 0, \tag{6}$$

$$\partial\psi_j/\partial x_j = 0. \tag{7}$$

In the usual way, one can conclude from Eq. (6) that $\partial\psi_j/\partial x_j$ is constant in time. Therefore one only needs to consider Eq. (6); the solutions of Eqs. (6) and (7) are the solutions of Eq. (6) which have zero divergence at the start. Equation (7) does not even need to be considered in the initial conditions if the solutions of a definite nonzero frequency are to be found, for if

$$\psi_j(\mathbf{x},t) = w_j(\mathbf{x})e^{-i\omega t},$$

then Eq. (7) is a consequence of Eq. (6).

To write Eq. (6) in matrix form one may introduce three matrices $s_j$,

$$(s_j)_{kl} = i\epsilon_{jkl}, \tag{8}$$

so that Eq. (6) becomes

$$-cs_kp_k\psi = i\hbar\partial\psi/\partial t, \tag{9}$$

where $p_k$ is written for $-i\hbar\partial/\partial x_k$. The operator $-cs_kp_k$ may be called the Hamiltonian $H$. It is not to be identified with the energy. The $s_j$ satisfy the angular momentum commutation rules,

$$[s_j,s_k] = -i\epsilon_{jkl}s_l,$$

and are a representation of spin one.

Next the plane wave solutions of Eqs. (7) and (9) will be written down—they are of interest in themselves and are needed for later reference. The substitution

$$\psi = u\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x} - Wt)]$$

[the symbol $p_k$ is used for the eigenvalue as well as for the operator] reduces Eq. (9) to the matrix eigenvalue problem

$$-c\begin{pmatrix} 0 & ip_3 & -ip_2 \\ -ip_3 & 0 & ip_1 \\ ip_2 & -ip_1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = W\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

The secular equation has the solutions

$$W = \pm cp,\ 0$$

where $p$ is the positive root of $p_kp_k$. A solution for the eigenvectors, which in form prefers the 3-axis, is

$$u_\pm = [2p^2(p_1^2+p_2^2)]^{-\frac{1}{2}}\begin{pmatrix} \pm ipp_2 - p_1p_3 \\ \mp ipp_1 - p_2p_3 \\ p_1^2+p_2^2 \end{pmatrix}, \tag{10}$$

$$u_0 = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

One sees that the zero-eigenvalue solution is longitudinal and, by dotting with the wave vector, that the other solutions are transverse. Only the first two solutions satisfy Eq. (7)—the longitudinal solution is to be disregarded. The matrices $u_\pm$ have the properties[9]

$$u_\pm^H u_\mp = 0, \tag{11}$$

$$u_\pm^H u_\pm = 1, \tag{12}$$

$$u_\pm = u_\mp^C. \tag{13}$$

Therefore a complete set of functions for the electromagnetic field are

$$\bar{\psi}_\pm(\mathbf{p};\mathbf{x},t) = (2\pi\hbar)^{-\frac{3}{2}}u_\pm\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x}\mp cpt)], \tag{14}$$

with the properties

$$H\bar{\psi}_\pm = \pm cp\bar{\psi}_\pm, \tag{15}$$

$$\int d\mathbf{x}\bar{\psi}_\pm^H(\mathbf{p})\bar{\psi}_\mp(\mathbf{p}') = 0, \tag{16}$$

$$\int d\mathbf{x}\bar{\psi}_\pm^H(\mathbf{p})\bar{\psi}_\pm(\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}'), \tag{17}$$

where $\delta$ is the three-dimensional Dirac delta function. The physical features of the solutions $\bar{\psi}_\pm$ can easily be seen by specializing to coordinates with the $x_3$ axis in the $\mathbf{p}$ direction. If one sets $p_1$ equal to zero, takes the limit as $p_2$ approaches zero, and replaces $p_3$ by $p$, the result is

$$\bar{\psi}_\pm = (2\pi\hbar)^{-\frac{3}{2}}2^{-\frac{1}{2}}\begin{pmatrix} \pm i \\ -1 \\ 0 \end{pmatrix}\exp[i\hbar^{-1}p(x_3\mp ct)].$$

Finally, from

$$\bar{\psi}_{\pm;j} = \bar{E}_{\pm;j} + i\bar{B}_{\pm;j},$$

the electric and magnetic fields are found:

$$\begin{aligned} \bar{E}_{\pm;1} &= \mp C\sin[\hbar^{-1}p(x_3\mp ct)], \\ \bar{E}_{\pm;2} &= -C\cos[\hbar^{-1}p(x_3\mp ct)], \\ \bar{B}_{\pm;1} &= \pm C\cos[\hbar^{-1}p(x_3\mp ct)], \\ \bar{B}_{\pm;2} &= -C\sin[\hbar^{-1}p(x_3\mp ct)], \\ \bar{E}_{\pm;3} &= \bar{B}_{\pm;3} = 0, \end{aligned} \tag{18}$$

where $C$ is the constant $(2\pi\hbar)^{-\frac{3}{2}}2^{-\frac{1}{2}}$. It is seen that a positive/negative eigenvalue solution corresponds to a wave which propagates in the positive/negative $\mathbf{p}$ direction and which is right/left hand circularly polarized with the propagation direction.

### III. TRANSFORMATION PROPERTIES AND CONSERVATION THEOREMS

The purpose of this section is to write down the conservation equations which arise in consequence of the covariance of Eqs. (7) and (9) with respect to transformations to new origins of the space and time coordinates and with respect to Lorentz transformations. Most of these theorems do not correspond to the classical conservation laws for the field but they lead to new considerations so that in Sec. IV a complete

[9] The superscripts $^C$, $^T$, and $^H$ denote the complex conjugate transpose, and Hermitian conjugate matrices.

correspondence between the classical conserved quantities and expected values of transformation operators can be made.

The basic theorem is that, if the transformation

$$x_\mu'=x_\mu'(x), \qquad \psi'=\psi'(\psi)$$

[$x_4$ is $ict$ and $x_\mu'(\mathbf{x},x_4)$ is abbreviated to $x_\mu'(x)$] carries the equation

$$cs_k\partial\psi(x)/\partial x_k=\partial\psi(x)/\partial t$$

into the equation

$$cs_k\partial\psi'(x')/\partial x_k'=\partial\psi'(x')/\partial t',$$

then the operator $\mathcal{O}$ defined by

$$\psi'(x)=\mathcal{O}\psi(x)$$

gives a conservation equation

$$\begin{aligned}\partial(\psi^H\mathcal{O}\psi)/\partial t&=\partial(\psi^H\psi')/\partial t\\&=(cs_k\partial\psi/\partial x_k)^H\psi'+\psi^H(cs_k\partial\psi'/\partial x_k)\\&=\partial(c\psi^H s_k\mathcal{O}\psi)/\partial x_k.\end{aligned} \tag{19}$$

The density of the conserved quantity is $\psi^H\mathcal{O}\psi$ and its flux is $-c\psi^H s_k\mathcal{O}\psi$. If $\mathcal{O}$ is chosen Hermitian, the real part of $\psi^H\mathcal{O}\psi$ is significant because only the real part contributes to $\int d\mathbf{x}\psi^H\mathcal{O}\psi$ when the integration extends over the whole volume where $\psi$ is not zero.

The transformations to be considered are the identity, infinitesimal space and time displacements, infinitesimal space rotations, infinitesimal pure Lorentz transformations, space reflection, and time reflection:

$$x_\mu'=x_\mu, \tag{20a}$$

$$x_\mu'=x_\mu+\alpha_\mu, \tag{20b}$$

$$x_j'=x_j-\epsilon_{jkl}x_k\theta_l, \tag{20c}$$

$$x_j'=x_j-v_jt,\ t'=t-c^{-2}v_kx_k, \tag{20d}$$

$$x_j'=-x_j,\ x_4'=x_4, \tag{20e}$$

$$x_j'=x_j,\ x_4'=-x_4, \tag{20f}$$

where $\alpha_\mu, \theta_l, v_j$ are smallness parameters. The corresponding $\psi$-transformations are

$$\psi'(x')=\psi(x), \tag{21a,b}$$

$$\psi'(x')=\psi(x)+is_l\theta_l\psi(x), \tag{21c}$$

$$\psi'(x')=\psi(x)+c^{-1}s_kv_k\psi(x), \tag{21d}$$

$$\psi'(x')=-\psi^C(x), \tag{21e}$$

$$\psi'(x')=\psi^C(x). \tag{21f}$$

These are found directly from the transformation properties of the electromagnetic field. Møller[10] gives the pure Lorentz transformation in a convenient form and Watanabe's[11] assignments for the time reflection are used.

[10] C. Møller, *The Theory of Relativity* (Oxford University Press, London, 1952), p. 110.
[11] S. Watanabe, Revs. Modern Phys. **27**, 26 (1955).

In consequence of Eqs. (21) the operators $\mathcal{O}$ are

$$\mathcal{O}_a=1, \tag{22a}$$

$$\mathcal{O}_b=-i\partial/\partial x_\mu, \tag{22b}$$

$$\mathcal{O}_c=-i\epsilon_{lmn}x_m(\partial/\partial x_n)-s_l, \tag{22c}$$

$$\mathcal{O}_d=x_kc^{-1}(\partial/\partial t)+ct(\partial/\partial x_k)+s_k, \tag{22d}$$

$$\mathcal{O}_e=-KP, \tag{22e}$$

$$\mathcal{O}_f=KT. \tag{22f}$$

Here $K$ is the operator that changes a function into its complex conjugate, $P$ changes the sign of each space-coordinate, and $T$ changes the sign of the time-coordinate. Unit terms in $\mathcal{O}_b$, $\mathcal{O}_c$, $\mathcal{O}_d$ are disregarded. The densities of some of the conserved quantities are

$$\psi^H\psi=E_jE_j+B_jB_j, \tag{23a}$$

$$\mathrm{Re}[\psi^H(-i\partial/\partial x_\mu)\psi]=E_j\partial B_j/\partial x_\mu-B_j\partial E_j/\partial x_\mu, \tag{23b}$$

$$\begin{aligned}\mathrm{Re}[\psi^H\mathcal{O}_c\psi]=\epsilon_{lmn}x_m[E_p\partial B_p/\partial x_n-B_p\partial E_p/\partial x_n]\\+2\epsilon_{lmn}E_mB_n.\end{aligned} \tag{23c}$$

The identity gives the Poynting theorem but the other conservation rules are unrelated to the classical rules.

## IV. CONNECTION WITH THE CLASSICAL CONSERVATION THEOREMS

The conservation theorems of Sec. III cannot be identified with the classical theorems for momentum, angular momentum, and center-of-energy of the electromagnetic field. However, there is a direct connection with the classical theorems if $\psi$ is replaced by $\phi$, defined by

$$\phi=|8\pi H|^{-\frac{1}{2}}\psi. \tag{24}$$

The operator $H$ has zero eigenvalues and so in general has no inverse. However, as long as the discussion is confined to the radiation field, as it is below, the functions $\psi$ can be expanded in terms of the $\bar{\psi}_\pm$ of Eq. (14) and $|H|^{-\frac{1}{2}}\bar{\psi}_\pm$ can be defined to be $(cp)^{-\frac{1}{2}}\bar{\psi}_\pm$. The value $p=0$ does not arise in the expansion because it corresponds to fields constant in space and time.

The expansions may be written in the form

$$\psi=\int d\mathbf{p}(8\pi cp)^{\frac{1}{2}}A_+(\mathbf{p})\bar{\psi}_++\int d\mathbf{p}(8\pi cp)^{\frac{1}{2}}A_-(\mathbf{p})\bar{\psi}_-, \tag{25}$$

where the factors $(8\pi cp)^{\frac{1}{2}}$ are included so that $A_\pm(\mathbf{p})$ are the expansion coefficients for $\phi$:

$$\phi=\int d\mathbf{p}A_+(\mathbf{p})\bar{\psi}_++\int d\mathbf{p}A_-(\mathbf{p})\bar{\psi}_-. \tag{26}$$

It is seen that

$$\int d\mathbf{x}\phi^H\phi=\int d\mathbf{p}A_+{}^CA_++\int d\mathbf{p}A_-{}^CA_-. \tag{27}$$

Also, according to Eqs. (31) and (32) below, $d\mathbf{p}A_\pm{}^C A_\pm$ is to be interpreted as the number of right or left hand circularly polarized photons in $d\mathbf{p}$. Therefore $\phi$ is normalized so that $\int d\mathbf{x}\phi^H\phi$ is the total number of photons in the field.

It is clear that if $\psi$ satisfies Eqs. (7) and (9), then so also does $\phi$. Therefore, the process of Sec. III for finding conservation theorems applies to $\phi$ as well as to $\psi$. The $\phi$ operator is found from the $\psi$ operator $\mathcal{O}$ as follows:

$$\begin{aligned}\phi'(x)&=|8\pi H|^{-\frac{1}{2}}\psi'(x)\\&=|8\pi H|^{-\frac{1}{2}}\mathcal{O}\psi(x)\\&=|H|^{-\frac{1}{2}}\mathcal{O}|H|^{\frac{1}{2}}\phi(x).\end{aligned}\tag{28}$$

The $\psi$ and $\phi$ operators are identical if $\mathcal{O}$ commutes with $H$.

The $\phi$ operators to be considered are 1, $H$, $\mathbf{p}$,

$$J_j=\epsilon_{jkl}x_kp_l-\hbar s_j,\tag{29}$$

$$\begin{aligned}G_j&=|H|^{-\frac{1}{2}}[x_j(i\hbar\partial/\partial t)-c^2t(-i\hbar\partial/\partial x_j)+i\hbar s_j]|H|^{\frac{1}{2}}\\&=|H|^{-\frac{1}{2}}Hx_j|H|^{\frac{1}{2}}-c^2tp_j.\end{aligned}\tag{30}$$

These expressions are found from Eqs. (22) by including appropriate factors and replacing $i\hbar\partial/\partial t$ by $H$ as may be done if the operators are to be applied to solutions of the field equations. The identity gives the conservation of the number of photons. The classical expressions for energy, momentum, and angular momentum in the field are related to $H$, $\mathbf{p}$, $\mathbf{J}$ as follows:

$$\begin{aligned}\int d\mathbf{x}(E^2+B^2)/8\pi&=\int d\mathbf{p}cpA_+{}^CA_++\int d\mathbf{p}cpA_-{}^CA_-\\&=\int d\mathbf{x}\phi_+{}^HH\phi_+-\int d\mathbf{x}\phi_-{}^HH\phi_-,\end{aligned}\tag{31}$$

$$\begin{aligned}\int d\mathbf{x}(\mathbf{E}\times\mathbf{B})_j/4\pi c&=\int d\mathbf{p}p_jA_+{}^CA_+-\int d\mathbf{p}p_jA_-{}^CA_-\\&=\int d\mathbf{x}\phi_+{}^Hp_j\phi_+-\int d\mathbf{x}\phi_-{}^Hp_j\phi_-,\end{aligned}\tag{32}$$

$$\int d\mathbf{x}[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]_j/4\pi c=\int d\mathbf{x}\phi_+{}^HJ_j\phi_+-\int d\mathbf{x}\phi_-{}^HJ_j\phi_-,\tag{33}$$

where $\phi_+$ is the first term in Eq. (26) and $\phi_-$ the second. These equalities can be verified by expressing the classical integrals and the $\phi$ integrals separately in terms of $A_\pm$ and comparing the results. The proof for the momentum relation for example is as follows:

$$\begin{aligned}\int d\mathbf{x}(\mathbf{E}\times\mathbf{B})_j/4\pi c&=-(8\pi c)^{-1}\int d\mathbf{x}\psi^Hs_j\psi\\&=-\int d\mathbf{p}pA_+{}^CA_+u_+{}^Hs_ju_+-\int d\mathbf{p}pA_-{}^CA_-u_-{}^Hs_ju_-,\end{aligned}$$

where Eq. (25) is used for $\psi$, Eq. (14) for $\bar{\psi}_\pm$, and the $x$-integration is performed first. The cross terms drop out since, from Eq. (13),

$$\begin{aligned}u_\pm{}^Hs_ju_\mp&=u_\mp{}^Ts_ju_\mp\\&=i\epsilon_{jkl}u_{\mp;k}u_{\mp;l}\\&=0.\end{aligned}$$

Also it is easily seen that

$$u_\pm{}^Hs_ju_\pm=\mp p_j/p;\tag{34}$$

one of these results can be found by direct calculation from Eq. (10) and the two results are complex conjugates. Therefore the momentum integral becomes

$$\int d\mathbf{x}(\mathbf{E}\times\mathbf{B})_j/4\pi c=\int d\mathbf{p}p_jA_+{}^CA_+-\int d\mathbf{p}p_jA_-{}^CA_-.$$

When this is combined with

$$\int d\mathbf{x}\phi_\pm{}^Hp_j\phi_\pm=\int d\mathbf{p}p_jA_\pm{}^CA_\pm,$$

the result is Eq. (32) as required. The proof of a component of Eq. (33) can be made in a similar way, by using the intermediate results

$$\begin{aligned}\epsilon_{3kl}u_\pm{}^H(\partial/\partial p_k)s_lu_\pm&=0,\\\epsilon_{3kl}u_\pm{}^H(\partial/\partial p_k)s_lu_\mp&=0,\\\epsilon_{3kl}u_\pm{}^H(\partial u_\pm/\partial p_k)p_l&=\pm ip_3/p.\end{aligned}$$

An analogous connection between the angular momentum in the field and the angular displacement operator $\mathbf{J}$ was found by Franz.[12]

The connection between the classical center-of-energy theorem and the operator $\mathbf{G}$ can be shown without making expansions. A statement of the classical theorem for the free Maxwell field is that the following quantity is conserved[13]:

$$\begin{aligned}\int d\mathbf{x}[x_j(E^2+B^2)/8\pi-c^2t(\mathbf{E}\times\mathbf{B})_j/4\pi c]&=(8\pi)^{-1}\int d\mathbf{x}\psi^Hx_j\psi-c^2t\int d\mathbf{x}(\mathbf{E}\times\mathbf{B})_j/4\pi c\\&=\int d\mathbf{x}\phi^H[|H|^{\frac{1}{2}}x_j|H|^{\frac{1}{2}}-c^2t(H/|H|)p_j]\phi\\&=\int d\mathbf{x}\phi^H(H/|H|)G_j\phi.\end{aligned}\tag{35}$$

[12] W. Franz, Z. Physik **127**, 363 (1950).
[13] See, for example, reference 10, p. 170.

The justification for replacing the momentum by

$$\int d\mathbf{x}\phi^H(H/|H|)p_j\phi$$

in the second term is that

$$\int d\mathbf{x}\phi_\pm{}^H\,\mathcal{O}\phi_\mp = \pm\frac{1}{2}\int d\mathbf{x}\phi_\pm{}^H[(H/|H|)\,\mathcal{O}-\mathcal{O}(H/|H|)]\phi_\mp = 0$$

for any operator $\mathcal{O}$ that commutes with $H$.

In Eqs. (31), (32), and (33) the energy, momentum, and angular momentum in the field are related to expected values of the operators $H$, $\mathbf{p}$, $\mathbf{J}$. In each case the contribution from the expected value of the left-hand polarized part of the wave is subtracted from the right-hand contribution. A parallel situation arises in Dirac's theory of the electron where, to obtain the physically observable energy, momentum, or angular momentum, the expected value of the appropriate displacement operator in the part of the wave function with negative Hamiltonian eigenvalues is subtracted from the contribution of the positive eigenvalue part. In his theory the Hamiltonian can be assumed to be the energy operator, the vacuum to consist of filled negative energy states, and the subtraction to correspond to the removal of electrons from the vacuum because the electrons follow Fermi-Dirac statistics: the Pauli principle prevents an electron from cascading indefinitely down into the negative energy states. This interpretation cannot be made for the photon equations above because the photons will follow Bose-Einstein statistics.

The factor of $(H/|H|)$ that appears in Eq. (35) suggests that a different assignment of the operators should be made. Since $H$, $\mathbf{p}$, $\mathbf{J}$ commute with $H$, Eqs. (31), (32), (33) can be rewritten as

$$\int d\mathbf{x}(E^2+B^2)/8\pi = \int d\mathbf{x}\phi^H(H/|H|)H\phi, \qquad (36)$$

$$\int d\mathbf{x}(\mathbf{E}\times\mathbf{B})_j/4\pi c = \int d\mathbf{x}\phi^H(H/|H|)p_j\phi, \qquad (37)$$

$$\int d\mathbf{x}[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]_j/4\pi c = \int d\mathbf{x}\phi^H(H/|H|)J_j\phi, \qquad (38)$$

so that uniformly the classical integrals are equal to the expected values of $(H/|H|)$ times the displacement operators. The operators to be identified with the physically observable quantities are then $(H/|H|)$ times the displacement operators.

The theorem of Sec. III, relating conservation equations to transformation properties, can be revised to take account of the $(H/|H|)$ factor. If the transformation

$$x_\mu' = x_\mu'(x), \quad \phi' = \phi'(\phi)$$

carries the equation

$$cs_k\partial\phi(x)/\partial x_k = \partial\phi(x)/\partial t$$

into the equation

$$cs_k\partial\phi'(x')/\partial x_k' = \partial\phi'(x')/\partial t',$$

then, defining the operator $\mathcal{O}$ by

$$\phi'(x) = \mathcal{O}\phi(x),$$

one finds that

$$\partial[\phi^H(H/|H|)\,\mathcal{O}\phi]/\partial t = c(\partial\phi/\partial x_k)^H s_k(H/|H|)\phi' + c\phi^H(H/|H|)s_k(\partial\phi'/\partial x_k).$$

In the second term, the order of the operators $(H/|H|)$ and $s_k$ can be reversed since $s_k\partial/\partial x_k$ is proportional to $H$; one finds

$$\partial[\phi^H(H/|H|)\,\mathcal{O}\phi]/\partial t = \partial[c\phi^H s_k(H/|H|)\,\mathcal{O}\phi]/\partial x_k,$$

and this seems to be the appropriate theorem to relate the classical integrals to the transformation operators. The flux of the quantity whose density is $\phi^H(H/|H|)\,\mathcal{O}\phi$ is $-c\phi^H s_k(H/|H|)\,\mathcal{O}\phi$. However, for the number of photons and the unit operator the theorem without the $(H/|H|)$ factor is to be used so that $\phi^H\phi$ is the expected photon density and $-c\phi^H s_k\phi$ is the expected photon flux.

When these assignments are applied to the plane wave solutions of Eq. (14), the following results for the density of photons, energy, and momentum are obtained [if $\phi$ is $\bar\psi_\pm$, the fields are the real and imaginary parts of $(8\pi cp)^{\frac{1}{2}}\bar\psi_\pm$]:

$$\bar\psi_\pm{}^H\bar\psi_\pm = (2\pi\hbar)^{-3},$$
$$\bar\psi_\pm{}^H|H|\bar\psi_\pm = cp(2\pi\hbar)^{-3},$$
$$\bar\psi_\pm{}^H(H/|H|)p_j\bar\psi_\pm = \pm p_j(2\pi\hbar)^{-3},$$

where $u_\pm{}^H u_\pm$ is given by Eq. (12). The corresponding results for the fluxes are

$$-c\bar\psi_\pm{}^H s_k\bar\psi_\pm = c(\pm p_k/p)(2\pi\hbar)^{-3},$$
$$-c\bar\psi_\pm{}^H s_k|H|\bar\psi_\pm = c(\pm p_k/p)cp(2\pi\hbar)^{-3},$$
$$-c\bar\psi_\pm{}^H s_k(H/|H|)p_j\bar\psi_\pm = c(\pm p_k/p)(\pm p_j)(2\pi\hbar)^{-3},$$

where $u_\pm{}^H s_k u_\pm$ is given by Eq. (34). These values of the densities and fluxes are consistent with the idea of a stream of particles with energy $cp$, momentum $\pm p_j$, speed $c$, moving in the $\pm p_k/p$ direction. As argued below Eq. (18), $\pm p_j/p$ is also the wave propagation direction. Also, from Eq. (18) it is seen that the wave frequency is $cp/h$ and the wavelength is $h/p$; therefore the operator assignments of the preceding paragraphs contain implicitly the Planck relation between energy and frequency and the de Broglie relation between momentum and wavelength. For a right/left hand circularly polarized wave, the spin density is

$$\bar{\psi}_\pm{}^H(H/|H|)(-\hbar s_k)\bar{\psi}_\pm=\pm\hbar(\pm p_k/p)(2\pi\hbar)^{-3},$$

so that the spin is parallel/antiparallel to the propagation direction.

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# Commutation Relations of Interacting Spinor Fields*

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It is shown that the requirement that the Hamiltonian density commute with itself on a spacelike surface precludes the possibility that three or more different spinor fields, coupled to one another in Yukawa-type interactions, commute with each other. If the Hamiltonian contains only two such fields, however, they may be assumed either to commute or to anticommute without violating this requirement.

## I. INTRODUCTION

THE form of the commutation relations between field operators that represent physically different Fermi-Dirac particles has been investigated recently by Kinoshita.[1] He has shown that if the Lagrangian contains interaction terms that are bilinear in spinor fields, these fields must anticommute[2] in order that unique equations of motion be obtained from Schwinger's variational principle. However, if the equations of motion are obtained from the canonical commutation laws

$$-i\partial\psi^i/\partial t=[H,\psi^i],\quad -i\partial\bar{\psi}^i/\partial t=[H,\bar{\psi}^i],\qquad (1.1)$$

the results are unique regardless of whether the spinor fields commute or anticommute. Since self-consistent results are obtained from the canonical formalism, it is not clear whether the inconsistency obtained by Kinoshita reflects the impropriety of the commutation relations or the inapplicability of the variational principle in this case. It is of interest, therefore, to determine whether Kinoshita's conclusions can be obtained without recourse to the variation formalism.

The question of whether different spinor fields commute or anticommute is of no practical importance when the Hamiltonian contains only two such fields, since the physical observables obtained using either choice of commutation relations are the same. On the other hand, the transition amplitude for a particular process involving three different spinor fields is calculated in Sec. 2 by the formal application of the Dyson expansion of the $S$ matrix,[3] and the result is found to depend on the choice of commutation relations. However, it is shown in Sec. 4 that if three or more spinor fields interact with each other via Yukawa-type interactions,[4] the assumption that they commute with one another is inconsistent with the requirement that the Hamiltonian density commute with itself at two points on a spacelike surface.[5] If the Hamiltonian contains only two different spinor fields, they may be assumed to either commute or anticommute without violating the above requirement, which will henceforth be referred to as Postulate II.

The case of three or more interacting spinor fields is thus fundamentally different from that of only two such fields in that Postulate II places a restriction on the commutation relations in the former case but not in the latter. Section 5 contains some speculations concerning the apparent distinction between these two cases.

## II. TRANSITION MATRIX ELEMENTS

In this section the transition matrix for a simple process is evaluated by the formal application of Dyson's $S$-matrix expansion. This example illustrates a difference between the cases in which the different spinor fields are assumed to commute or anticommute.

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[1] T. Kinoshita, Phys. Rev. **96**, 199 (1954).

[2] As used in this paper, the expression "commuting spinor fields" will always refer to different spinor fields. For a single spinor field the usual anticommutation relations are assumed.

[3] F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[4] By "Yukawa-type interactions" we merely mean that an interaction term in the Hamiltonian contains the spinor fields bilinearly and the boson field linearly.

[5] See, for example, W. Pauli, Progr. Theoret. Phys. (Japan) **5**, 526 (1950). This is a special case of what Pauli refers to as Postulate II: "Physical quantities (observables) commute with each other in two space-time points with a space-like distance." Strictly speaking, only the Hamiltonian density integrated over a finite volume is an observable. For this reason, in order to deal with physical quantities at two different points of space-time, $x_1$ and $x_2$, one may integrate the densities over suitable regions of space $R_1$ and $R_2$, so that all points in $(R_1,t_1)$ are spacelike with respect to all points in $(R_2,t_2)$. The requirement that the Hamiltonian density commute with itself on a spacelike surface is also an integrability condition on the Tomonaga-Schwinger equation. In connection with this see K. Nishijima, Progr. Theoret. Phys. (Japan) **5**, 187 (1950).