Proton Bremsstrahlung*

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The differential cross section for the production of gamma rays from the bremsstrahlung of protons of 30- to 140-Mev kinetic energy in the center-of-mass system is calculated. An energy-dependent complex square well is used to represent the nuclear interaction. The complexity of the potential enhances gamma-ray production. Furthermore, the theory predicts a low intensity for high-energy gamma rays. It is also found that the continuous spectrum of radiation is closed between very low-frequency and high-energy gamma rays of energy just below the incident proton kinetic energy. By using the experimental information, a qualitative result about the energy dependence of the imaginary part of the potential is obtained in comparable form with those used by other people.

I. INTRODUCTION

HIGH energy nucleon-nucleus collisions are, usually, accompanied by a continuous gamma-ray emission. An interesting gamma-ray source arises from the bombardment of nuclei by high-energy protons with energies below the threshold π_0-production energy. In this paper we are interested in the gamma-ray production through the nuclear interaction only. However, for low-energy protons, about 2 Mev, the Coulomb field is, effectively, the only force available to cause a proton bremsstrahlung and the cross section for this process has been calculated by Drell and Huang.[1] In an exact calculation the Coulomb interaction must also be included. But with energies much higher than $\frac{1}{2}$ Mev and for low Z the Coulomb effect is negligible. The Coulomb effect for incident proton energies as low as 30 Mev can be disregarded since at this energy the distance of closest approach, $r_0=(Ze^2/mc^2)(mc^2/E)=(Ze^2/mc^2)\times(1/60)$, is much less than the range of nuclear forces. However, the Coulomb interaction, in this problem, can be an important effect in the case of high-energy gamma rays.

The available data on gamma rays from high-energy inelastic proton scattering consist of the experimental results of Wilson[2] and Cohen.[3] Besides these two experiments, the Coulomb excitation experiment by Stelson and McGowan[4] yielded gamma rays of discrete energy from excitation of nuclear states superimposed over a continuum of bremsstrahlung gamma rays. The characteristic of the latter radiation lies in its high yield of very low-energy gamma rays and in an increase of the yield with energy from 2- to 5-Mev incident proton energy. In this energy range the experiment on Bi, Th, and Sn targets did not show a particular Z dependence of the radiation, but only a difference in the intensity of the radiation.

For proton energies above 15 Mev, however, Cohen's experiment contains quite different results than that of Stelson and McGowan. In the neighborhood of 8 Mev there is a sudden increase of yield of gamma rays. It was also observed that the yield for gamma rays produced from 100-Mev proton bombardment on Cu, Al, C, and Be increases with A. At 15-Mev gamma-ray energy there is a distinct divergence of the spectral curves corresponding to the gamma rays arising from the bombardment of Cu, Al, C, and Be. The spectral curves come together just above the high end of the energy axis, beyond 35 Mev. The primary energy dependence of the spectrum from Be^9 bombarded by 38-, 100-, and 140-Mev protons show a similar behavior of divergence near 15-Mev photons and a convergence at the high end of the energy axis. The yield increases with decreasing proton energy.

No data were shown below about 8 Mev. The present theory does, however, predict the continuation of the spectrum down to zero-energy radiation. The latter part of the spectrum may, *inter alia*, arise from the multiple scattering of high-energy protons in the nuclear matter. By multiple scattering the protons will lose energy and eventually Coulomb interaction will set in to produce the low-energy bremsstrahlung gamma-rays. This mechanism is not, of course, contained in our theory where the Coulomb interaction is ignored from the start. At any rate, the spectrum below 8 Mev can be looked upon as the nuclear part of the Coulomb bremsstrahlung, i.e., as a correction to it plus purely nuclear bremsstrahlung gamma rays.

In this problem we shall use an energy-dependent complex square-well potential to describe the proton-nucleus interaction. Because of the energy dependence of the interaction we actually have two complex fields corresponding to initial and final states. A complex well replaces all the possible reaction channels by an average one and inelastic processes like (p,d) reactions are already included in a complex representation. The main inelastic process of interest is the interaction of the

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[1] S. D. Drell and K. Huang, Phys. Rev. **99**, 686 (1955).

[2] R. Wilson, Phys. Rev. **85**, 563 (1950).

[3] D. Cohen, Phys. Rev. **95**, 664 (1954) and University of California Radiation Laboratory Report UCRL-3230, 1955 (unpublished).

[4] P. H. Stelson and F. K. McGowan, Phys. Rev. **99**, 112 (1955).

incident protons with the radiation field after being processed by the complex nucleus.

An undesirable feature of a complex well lies in its sharp discontinuity at the nuclear surface which may give rise to too much reflection and to less penetration (very small mean free path in nuclear matter) for all angular momentum states of the protons. This would greatly inhibit the yield of gamma radiation from the inelastic proton scattering. But the energy dependence of the well can partly remove this difficulty.

The above-mentioned drawback, among others, of a complex well should be disregarded in the light of the many successes obtained by the use of an optical model of the nucleus. One of the latest of these is the "cloudy crystal ball model"[5] used for the description of the low-energy neuton elastic scattering. The motivation behind the present attempt comes from considering an optical model of the nucleus in the light of a different problem[6] from that of mere elastic scattering of nuclear particles. In this paper we would like to study the optical model by calculating transition probabilities with it, using the existing experimental data on photon production from high-energy proton scattering. In particular, an investigation of the effect of an energy-dependent complex potential on the gamma-ray production is another interesting aspect of the problem.

II. KINEMATICS OF THE GAMMA-RAY PRODUCTION

The maximum energy E_g of the photon produced in the collision of two particles with masses M_P and M_A, in the center-of-mass system, is given by

$$E_g = M_P M_A (q-1)c^2/(M_P{}^2 + M_A{}^2 + 2M_P M_A q)^{\frac{1}{2}}, \quad \text{(II.1)}$$

where

$$q = 1 \bigg/ \left(1 - \frac{v_p{}^2}{c^2}\right)^{\frac{1}{2}}.$$

If we put

$$E_0 = (q-1)M_P c^2,$$

we obtain

$$E_g = E_0 \bigg/ \left[\left(1 + \frac{M_P}{M_A}\right)^2 + \frac{2E_0}{M_A c^2}\right]^{\frac{1}{2}}. \quad \text{(II.2)}$$

If $M_A c^2$ is the rest energy of the nucleus, we can neglect $2E_0/M_A c^2$ and, neglecting binding energy, we write

$$E_g = \left(\frac{A}{A+1}\right) E_0. \quad \text{(II.3)}$$

For a nucleon-nucleon collision the maximum energy of the gamma ray produced is one half of the incident kinetic energy. We see that the yield is higher in the case of nucleon-nucleus collisions. Thus not all the kinetic energy of the proton can be used for photon production. At the maximum energy of gamma-ray production, with high-energy protons bombarding light nuclei, enough kinetic energy will be left after photon production for the proton to get out of the nucleus. The minimum kinetic energy of the protons in the final state is

$$E_0 - E_g = E_0/(A+1). \quad \text{(II.4)}$$

In order that the nuclear force (of range e^2/mc^2) be large compared to the Coulomb force, we must have

$$E_0 \gg (A+1)mc^2. \quad \text{(II.5)}$$

Thus the Coulomb interaction in the final state for Be^9 will be negligible if the primary energy E_0 is much larger than 5 Mev. For Cu and heavier nuclei the Coulomb interaction is a nonnegligible effect for low incident proton energies. Since the experiment has mainly concentrated on gamma rays from Be^9, our neglect of the Coulomb effect is a good approximation.

The fact that the spectral curves in Cohen's experiment do not touch the energy axis at the high end but come together just above the axis may be taken as a kinematic effect.

III. FORMULATION OF THE PROBLEM

Since the kinetic energy of the protons are small compared to their rest energy, the use of nonrelativistic theory is a sufficiently good approximation. For a first orientation we shall neglect the magnetic moment of the proton. The Hamiltonian of the system is given by

$$H = H_0 + H_r + H_{PN} + H', \quad \text{(III.1)}$$

where H_0 = kinetic energy of the protons, H_r = energy of the radiation field, H_{PN} = interaction energy of protons and the nucleus, H' = interaction energy of protons and radiation field. The interaction H' will be treated only in the first order.

The Schrödinger's equation defining the initial and final states for the Hamiltonian $H_0 + H_r + H_{PN}$, in the center-of-mass system, is

$$\nabla^2\psi + \frac{2\mu}{\hbar^2}(W - V)\psi = 0, \quad \text{(III.2)}$$

where W is E_0 in the initial state and E in the final state of the proton, $E = E_0 - E_g$, $\mu = [A/(A+1)]M$, $E_0 = [A/(A+1)]E_L$, and $E_g = \hbar ck$ = energy of the emitted photon.

The nuclear potential V is defined by

$$\begin{aligned} V_0 &= -[V_R(E_0) + iV_I(E_0)], \quad r < R \\ &= 0, \quad r > R. \end{aligned} \quad \text{(III.3)}$$

The final state potential V is defined by replacing E_0 in (III.3) by the final energy E of the proton. The nuclear radius R is given as

$$R = r_0 A^{\frac{1}{3}} = 1.33 \times 10^{-13} A^{\frac{1}{3}} \text{ cm}. \quad \text{(III.4)}$$

[5] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).
[6] B. Kurşunoğlu, Phys. Rev. **98**, 1156 (1955).

The initial and final states are defined by solving (III.2) in the form

$$\psi_I=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})i^n a_n P_n(\cos\alpha_I)\times[g_n(k_I r)+h_n(k_I r)],\quad r<R$$
$$=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})i^n P_n(\cos\alpha_I)\times[g_n(k_{0I}r)+\eta_I{}^n h_n(k_{0I}r)],\quad r>R \tag{III.5}$$

and

$$\psi_F=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})i^n b_n P_n(\cos\alpha_F)\times[g_n(k_F r)+h_n(k_F r)],\quad r<R$$
$$=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})i^n P_n(\cos\alpha_F)\times[\eta_F{}^n g_n(k_{0F}r)+h_n(k_{0F}r)],\quad r>R \tag{III.6}$$

respectively, where

$$k_{0I}{}^2=\frac{2\mu E_0}{\hbar^2},\quad k_I{}^2=k_{0I}{}^2+\frac{2\mu}{\hbar^2}[V_R(E_0)+iV_I(E_0)],$$
$$k_{0F}{}^2=\frac{2\mu E}{\hbar^2},\quad k_F{}^2=k_{0F}{}^2+\frac{2\mu}{\hbar^2}[V_R(E)+iV_I(E)], \tag{III.7}$$

and where

$$\eta_I{}^n=\exp(2i\delta_I{}^n),\quad \eta_F{}^n=\exp(-2i\delta_F{}^n). \tag{III.8}$$

The functions g_n and h_n are defined by

$$h_n=j_n+in_n,\quad g_n=j_n-in_n,$$

where j_n and n_n are spherical Bessel functions.

The choice of the reaction coefficients $\eta_I{}^n$ and $\eta_F{}^n$ related to the phase shifts in the form (III.8) follows from the fact that the final state in a scattering problem of this kind consists of a plane wave propagating in the positive k_{0F} direction plus a spherical wave which is ingoing rather than outgoing. This fact has recently been pointed out by Breit and Bethe[7] and by Bethe and Maximon[8] in connection with general scattering problems and electron bremsstrahlung.

It is easily seen that the asymptotic form of the final state for $r>R$ is

$$\psi_F=\exp(i\mathbf{k}_{0F}\cdot\mathbf{r})+\frac{1}{r}f_F(\alpha_F)\exp(-ik_{0F}r), \tag{III.9}$$

where

$$f_F(\alpha_F)=\frac{1}{-2ik_{0F}}\sum_{n=0}^{\infty}(2n+1)(-1)^n\times[\exp(-2i\delta_F{}^n)-1]P_n(\cos\alpha_F).$$

[7] G. Breit and H. A. Bethe, Phys. Rev. **93**, 888 (1955).
[8] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1955).

The asymptotic form of the initial state for $r>R$ is

$$\psi_I=\exp(i\mathbf{k}_{0I}\cdot\mathbf{r})+\frac{1}{r}f_I(\alpha_I)\exp(ik_0 r), \tag{III.10}$$

where

$$f_I(\alpha_I)=\frac{1}{2ik_{0I}}\sum_{n=0}^{\infty}(2n+1)[\exp(2i\delta_I{}^n)-1]P_n(\cos\alpha_I).$$

The coefficients a_n, b_n and the complex phase shifts $\delta_I{}^n$, $\delta_F{}^n$ are to be determined by fitting the internal and external wave functions at $r=R$. The finiteness of the wave functions at the origin and their normalization imply that we may conveniently fit the functions $[rk_I R_I{}^n(r),\ rk_{0I}\exp(-i\delta_I{}^n)R_{0I}{}^n(r)]$ and $[rk_F R_F{}^n(r),\ rk_{0F}\exp(i\delta_F{}^n)R_{0F}{}^n(r)]$ and their derivatives at $r=R$. After some manipulations with the various properties of the Bessel functions, we obtain the results

$$\eta_I{}^n=\frac{z_{0I}j_n(z_I)h_{n-1}{}^*(z_{0I})-z_I j_{n-1}(z_I)h_n{}^*(z_{0I})}{z_I j_{n-1}(z_I)h_n(z_{0I})-z_{0I}j_n(z_I)h_{n-1}(z_{0I})}, \tag{III.11}$$

$$\eta_F{}^n=\frac{z_{0F}j_n(z_F)h_{n-1}(z_{0F})-z_F j_{n-1}(z_F)h_n(z_{0F})}{z_F j_{n-1}(z_F)h_n{}^*(z_{0F})-z_{0F}j_n(z_F)h_{n-1}{}^*(z_{0F})}, \tag{III.12}$$

and

$$a_n=\frac{1}{z_I}\frac{i\exp(-i\delta_I{}^n)}{z_{0I}j_n(z_I)h_{n-1}(z_{0I})-z_I j_{n-1}(z_I)h_n(z_{0I})}, \tag{III.13}$$

$$b_n=\frac{1}{z_F}\frac{-i\exp(i\delta_F{}^n)}{z_{0F}j_n(z_F)h_{n-1}{}^*(z_{0F})-z_F j_{n-1}(z_F)h_n{}^*(z_{0F})}. \tag{III.14}$$

In order that the ordinary reaction cross sections be positive, we must have

$$|\eta_I{}^n|^2<1,\quad |\eta_F{}^n|^2<1,$$

which mean that $\delta_I{}^n$ and $\delta_F{}^n$ must have the forms

$$\delta_I{}^n=u_I{}^n+iv_I{}^n,\quad \delta_F{}^n=u_F{}^n-iv_F{}^n, \tag{III.15}$$

where $v_I{}^n$ and $v_F{}^n$ are positive numbers. In the above relations the dimensionless numbers are obtained as $z_{0I}=k_{0I}R$, $z_I=k_I R$, $z_F=k_F R$, $z_{0F}=k_{0F}R$, $z=kR$.

From the above phase shift relations it is easy to deduce that if all the kinetic energy of the proton is transformed into a photon, i.e., if $z_{0F}=0$, then

$$\delta_F{}^n=n\pi,\quad b_n=0. \tag{III.16}$$

In this case the final state reaction and scattering cross sections vanish. This result confirms the kinematic relation that the maximum energy of the photon must be less than the primary energy. From the first relation of (III.16), we get

$$u_F{}^n=n\pi,\quad v_F{}^n=0.$$

The vanishing of $v_F{}^n$ implies that $V_I(E)=0$ for $E_g=E_0$. Although kinematically the emission of a gamma ray of energy equal to E_0 is not possible, it would not be unreasonable if we were to assume the energy dependence of V_I in the final state to have the form

$$V_I = aE^b, \qquad \text{(III.17)}$$

where a and b are numerical constants.

Two other useful relations are obtained by putting $E_g=0$ in (III.12) and (III.14), giving

$$\delta_I{}^n - \delta_F{}^n = n\pi, \quad a_n = b_n. \qquad \text{(III.18)}$$

The last relations show that the final state wave function does not reduce to the initial state wave function by putting $E_g=0$ in the final state, since the production of a zero-energy radiation is also a transition. If, however, the final state wave function in the asymptotic limit was chosen to contain outgoing spherical waves instead of ingoing ones, then $E_g=0$ would not be a transition and the final state wave function would automatically reduce to the initial state wave function.[9]

The matrix element to be calculated consists of internal and external transition amplitudes given by

$$M = M^{(i)} + M^{(e)}, \qquad \text{(III.19)}$$

where

$$M^{(i)} = \int \psi_F{}^* H' \psi_I dV, \quad r<R, \qquad \text{(III.20)}$$

$$H' = -ie\left(\frac{2\pi\hbar^2c^2}{E_g}\right)^{\frac{1}{2}} \frac{\hbar}{\mu_c} \exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\varepsilon}\cdot\boldsymbol{\nabla}. \qquad \text{(III.21)}$$

The external transition amplitude $M^{(e)}$ follows from integrating over the domain $r>R$.

The above formulation of the problem can also be applied to the radiative capture problems but this aspect of the inelastic reaction is not included in our initial conditions.

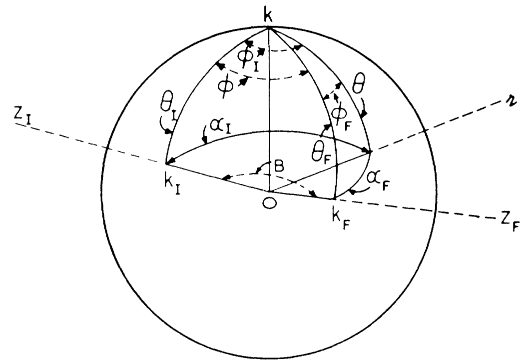

FIG. 1. Angles required in the gamma-ray production. The polarization angle (not shown in the figure) is the angle γ between the planes $(\mathbf{k},\mathbf{k}_I)$ and $(\mathbf{k},\boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon}$ is the unit polarization vector of the photon and lies in a plane perpendicular to $\mathbf{k}$.

IV. DIFFERENTIAL CROSS SECTION

Since quite a number of angles are involved in the calculation, it will be found convenient if we represent them in Fig. 1. The angles α_I and α_F are related to θ_I, θ, and φ_I by

$$\cos\alpha_I = \cos\theta_I \cos\theta + \sin\theta_I \cos\varphi_I \sin\theta,$$
$$\cos\alpha_F = \cos\theta_F \cos\theta + \sin\theta_F \sin\theta \cos(\varphi - \varphi_I).$$

The volume element dV and the angle between $\boldsymbol{\varepsilon}$ and the unit vector along the r-direction are given by

$$dV = r^2 \sin\theta d\varphi_I dr, \quad \boldsymbol{\varepsilon}\cdot\mathbf{r}/r = \sin\theta\cos(\gamma-\varphi_I),$$

respectively, and

$$\boldsymbol{\varepsilon}\cdot\mathbf{k}=0.$$

We can now record the matrix element $M^{(i)}$ as

$$M^{(i)} = -ie\left(\frac{2\pi\hbar^2c^2}{E_g}\right)^{\frac{1}{2}} \frac{\hbar}{\mu c} \sum_{smn=0}^{\infty} \left\{ (2s+1)(2m+1)(2n+1) i^{s+m+n}(-1)^s k_I b_s{}^* a_n \int_0^R drr^2 j_s{}^*(k_F r) j_m(kr) \right.$$
$$\left. \times\left[j_{n-1}(k_I r) - \frac{n+1}{k_I r} j_n(k_I r)\right] \int_0^{2\pi} d\varphi_I \int_0^{\pi} d\theta \sin^2\theta P_s(\cos\alpha_F) P_m(\cos\theta) P_n(\cos\alpha_I) \right\} \cos(\gamma - \varphi_I). \qquad \text{(IV.1)}$$

We use the addition theorem for Legendre polynomials and carry out the φ_I and θ-integrations to get

$$M^{(i)} = -ie\left(\frac{2\pi\hbar^2c^2}{E_g}\right)^{\frac{1}{2}} \left(\frac{4\pi\hbar}{\mu c}\right) \sum_{smn=0}^{\infty} \left\{ (-1)^s k_I i^{s+m+n} b_s{}^* a_n \int_0^R drr^2 j_s{}^*(k_F r) j_m(kr) \right.$$
$$\left. \times\left[j_{n-1}(k_I r) - \frac{n+1}{k_I r} j_n(k_I r)\right] U_{smn} \right\}, \qquad \text{(IV.2)}$$

where

$$U_{smn} = \sum_{m_s=-s}^{s} \{\exp[i(m_s\varphi+\gamma)](\alpha_m A_{smnm_s} - \alpha_{m+2} A_{s\ m+2\ nm_s})\Theta_n{}^{m_s+1}(\cos\theta_I) + \exp[i(m_s\varphi-\gamma)]$$
$$\times(\alpha_m B_{smnm_s} - \alpha_{m+2} B_{s\ m+2\ nm_s})\Theta_n{}^{m_s-1}(\cos\theta_I)\} \Theta_s{}^{m_s}(\cos\theta_F). \qquad \text{(IV.3)}$$

[9] Ingoing wave condition in the final state provides the continuity of the transitions; i.e., it does not exclude any possible energy of the gamma ray.

The functions $\Theta_s^{m_s}$ are defined by[10]

$$\Theta_n^{m_n}(\cos\theta)=(-1)^{m_n}\left(\frac{2n+1}{2}\right)^{\frac{1}{2}}\left[\frac{(n-m_n)!}{(n+m_n)!}\right]^{\frac{1}{2}}P_n^{m_n}(\cos\theta).$$

Other constants are given by

$$\alpha_m=\left[\frac{2m(m-1)}{2m-1}\right]^{\frac{1}{2}},$$

$$A_{s\,m+2\,nm_s}=\int_0^\pi d\theta\,\sin\theta\Theta_{m+1}{}^1(\cos\theta)\times\Theta_s{}^{m_s}(\cos\theta)\Theta_n{}^{m_s-1}(\cos\theta),$$

with the conditions

$$s+m+n=\text{an odd number},$$
$$m_s+n=\text{an odd number},$$
$$|m-s+1|\leqq n\leqq m+s+1,\quad m\geqq 0 \text{ for } A,$$

and

$$|m-n+1|\leqq s\leqq m+n+1,\quad m\geqq 0 \text{ for } B \quad \text{(IV.4)}$$

By noting that A_{smnm_s} and B_{smnm_s} are obtained from $A_{s\,m+2\,nm_s}$ and $B_{s\,m+2\,nm_s}$ by replacing m by $m-2$, respectively, we can further simplify the expression (IV.3) to

$$M^{(i)}=ie\left(\frac{2\pi\hbar^2c^2}{E_g}\right)^{\frac{1}{2}}\frac{4\pi\hbar}{\mu c}\frac{\hbar cR}{E_g}z_I\sum_{smn=0}^{\infty}\Bigg\{(-1)^s\times i^{s+m+n}b_s{}^*a_n(2m+3)\alpha_{m+2}\int_0^1 du u j_s{}^*(z_Fu)\times j_{m+1}(zu)\left[j_{n-1}(z_Iu)-\frac{n+1}{z_Iu}j_n(z_Iu)\right]V_{smn}\Bigg\},\quad \text{(IV.5)}$$

where

$$V_{smn}=\sum_{m_s=-s}^{s}\left[\exp(i(m_s\varphi+\gamma))A_{s\,m+2\,nm_s}\times\Theta_n{}^{m_s+1}(\cos\theta_I)+\exp(i(m_s\varphi-\gamma))B_{s\,m+2\,nm_s}\times\Theta_n{}^{m_s-1}(\cos\theta_I)\right]\Theta_s{}^{m_s}(\cos\theta_F),\quad \text{(IV.6)}$$

and

$$u=r/R.$$

In recording the relation (IV.5), we made use of

$$\frac{2m+3}{x}j_{m+1}(x)=j_m(x)+j_{m+2}(x).$$

The number $A_{s\,m+2\,nm_s}$ is given by

$$A_{s\,m+2\,nm_s}=\frac{(-1)^{h-m-m_s-1}(2h-2s)!h!}{(h-m-1)!(h-s)!(h-n)!(2h+1)!}[(2m+3)(2s+1)(2n+1)]^{\frac{1}{2}}\times\left[\frac{(n-m_s-1)!(m+2)!(m_s+s)!(s-m_s)!}{2(n+m_s+1)!m!}\right]^{\frac{1}{2}}\sum_q(-1)^q\frac{(n+m_s+q+1)!(m+s-m_s-q)!}{(n-m_s-q-1)!(m+m_s-s+q+2)!(s-m_s-q)!q!},\quad \text{(IV.7)}$$

where $2h=s+m+n+1$ and the number B follows from A by replacing s and n by n and s, respectively, and m_s by m_s-1. The summation in (IV.7) is to be understood as q taking on all integral values consistent with the factorial notation, the factorial of a negative number being meaningless.

From the form of $M^{(i)}$ given by (IV.5) and from the definition of the wave functions, we can easily infer the external matrix element $M^{(e)}$ to be

$$M^{(e)}=ie\left(\frac{2\pi\hbar^2c^2}{E_g}\right)^{\frac{1}{2}}\left(\frac{\pi\hbar}{\mu c}\right)\frac{\hbar cR}{E_g}z_{0I}\times\sum_{smn=0}^{\infty}(-1)^s i^{s+m+n}(2m+3)\alpha_{m+2}\int_1^\infty du u\times\Bigg\{[\eta_F{}^{s*}h_s(z_{0F}u)+h_s{}^*(z_{0F}u)]j_{m+1}(zu)\times\left[h_{n-1}{}^*(z_{0I}u)+\eta_I{}^nh_{n-1}(z_{0I}u)-\frac{n+1}{z_{0I}u}(h_n{}^*(z_{0I}u)+\eta_I{}^nh_n(z_{0I}u))\right]\Bigg\}V_{smn},\quad \text{(IV.8)}$$

The cross section for the emission of a gamma ray of polarization $\boldsymbol{\varepsilon}$ in the energy interval (E_g, E_g+dE_g) into a solid angle $d\Omega_k$ with the proton recoiling into the solid angle $d\Omega_F$ is given by

$$d\sigma(\gamma,\varphi,\theta_I,\theta_F)=\frac{2\pi}{\hbar}|M^{(i)}+M^{(e)}|^2\rho_F\frac{\mu}{p_{0I}},\quad \text{(IV.9)}$$

where ρ_F is the number of final states defined by

$$\rho_F dE_{0F}=\frac{\Delta N_g}{V_g}\cdot\frac{\Delta N_p}{V_p}=\left[\left(\frac{1}{2\pi}\right)^3k^2dkd\Omega_k\right]\times\left[\left(\frac{1}{2\pi}\right)^3\frac{p_{0F}{}^2}{\hbar^3}dp_{0F}d\Omega_F\right],$$

or

$$\rho_F=\frac{\mu c^3p_{0F}E_g{}^2dE_gd\Omega_kd\Omega_F}{(2\pi\hbar c)^6},\quad \text{(IV.10)}$$

where

$$d\Omega_F=\sin\theta_Fd\theta_Fd\varphi,\quad E_{0F}=E=E_0-E_g.$$

Integrating over the recoil proton solid angle $d\Omega_F$ and summing over photon polarization, we obtain the differential cross section for gamma-ray emission in the

[10] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951), p. 176.

energy interval (E_g, E_g+dE_g) in the form

$$d\sigma(\theta_I)=\frac{R^2}{8\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{E}{E_0}\right)^{\frac{1}{2}}\frac{dE_g}{E_g}d\Omega_k Q, \qquad (IV.11)$$

where

$$Q=\sum_{smn=0}^{\infty}K_{smn}L_{smn}, \qquad (IV.12)$$

$$\begin{aligned}K_{smn}=&16|z_I|^2|b_s|^2|a_n|^2\alpha_{m+2}^2|I_{smn}|^2(2m+3)^2\\&+z_{0I}^2(2m+3)^2\alpha_{m+2}^2|F_{smn}|^2\\&+8z_{0I}\,\mathrm{Re}[z_Ib_s^*a_n(2m+3)^2F_{smn}^*I_{smn}\alpha_{m+2}^2],\end{aligned} \qquad (IV.13)$$

$$L_{smn}=\sum_{m_s=-s}^{s}[A_{s\,m+2\,nm_s}^2(\Theta_n^{m_s+1}(\cos\theta_I))^2+B_{s\,m+2\,nm_s}^2(\Theta_n^{m_s-1}(\cos\theta_I))^2], \qquad (IV.14)$$

$$I_{smn}=\int_0^1 du\left[uj_s^*(z_Fu)j_{m+1}(zu)\times\left(j_{n-1}(z_Iu)-\frac{n+1}{z_Iu}j_n(z_Iu)\right)\right], \qquad (IV.15)$$

$$F_{smn}=\int_1^\infty du\left\{u[h_s^*(z_{0F}u)+\eta_F{}^{s*}h_s(z_{0F}u)]j_{m+1}(zu)\times\left[h_{n-1}^*(z_{0I}u)+\eta_I{}^nh_{n-1}(z_{0I}u)-\frac{n+1}{z_{0I}u}(h_n^*(z_{0I}u)+\eta_I{}^nh_n(z_{0I}u))\right]\right\}. \qquad (IV.16)$$

The angular distribution of the radiation is contained in the functions L_{smn}. We shall be interested in the experimentally measured quantities

$$\alpha=\frac{d^2\sigma(\theta_I)}{d\Omega_kdE_g}=\frac{R^2}{8\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{E}{E_0}\right)^{\frac{1}{2}}\frac{1}{E_g}Q, \qquad (IV.17)$$

and the total cross section σ_t for the emission of a quantum E_g in the range dy,

$$\sigma_tdy=\frac{R^2}{8\pi}\left(\frac{e^2}{\hbar c}\right)\frac{dy}{y}(1-y)^{\frac{1}{2}}\int Qd\Omega_k,$$

or

$$\sigma_t=\frac{R^2}{4}\left(\frac{e^2}{\hbar c}\right)\frac{1}{y}(1-y)^{\frac{1}{2}}Q_t, \qquad (IV.18)$$

where

$$y=\frac{E_g}{E_0},\quad Q_t=\sum_{smn=0}^{\infty}K_{smn}L_{smn}{}^t$$

and

$$L_{smn}{}^t=\sum_{m_s=-s}^{s}(A_{s\,m+2\,nm_s}{}^2+B_{s\,m+2\,nm_s}{}^2), \qquad (IV.19)$$

A suitable unit in which to measure the cross section for proton bremsstrahlung is to take

$$\sigma_0=R^2(e^2/\hbar c)=1.282\times10^{-28}A^{\frac{2}{3}}\ \mathrm{cm}^2,$$

and write

$$\alpha=\frac{1}{8\pi}\sigma_0\frac{1}{E_g}(1-y)^{\frac{1}{2}}Q, \qquad (IV.20)$$

$$\sigma_t=\frac{1}{4}\sigma_0\frac{1}{y}(1-y)^{\frac{1}{2}}Q_t. \qquad (IV.21)$$

The experimentally measured total cross section for the production of gamma rays with greater than 20-Mev energy by 140-Mev protons on Be^9 is $(1.3\pm0.5)\times10^{-29}$ cm^2, so that neglecting center-of-mass corrections, we must have

$$\int_{1/7}^{9/10}\left[\frac{1}{y}(1-y)^{\frac{1}{2}}Q_t\right]_{E_0=140\ \mathrm{Mev}}dy=0.047\pm0.018. \qquad (IV.22)$$

This result together with the experimentally determined values of $\alpha(90°)$ for various E_g and E_0 can be used in a numerical analysis of the cross section to find the complex potential V.

V. DISCUSSION

A machine analysis of this calculation is needed to define the shape of the spectral curves and the energy dependence of the complex potential. Owing to difficulties in obtaining the services of a machine we were unable to perform the required numerical analysis of the cross section. An analytic approach by expansion in powers of y for low energies is a hopeless task and no such attempt will be made. However, a few qualitative features of the cross section can be seen without much complication.

We first consider the low-frequency radiation. For very small E_g the quantity z is small compared to 1, and in this case the effective dependence on z of I_{smn} defined by (IV.15) has the form z^{m+1} since, for $0\leqq u\leqq1$, we can write

$$j_{m+1}(zu)\xrightarrow[z=0]{}\frac{z^{m+1}u^{m+1}}{(2m+3)!}.$$

When $1<u<\infty$, as in F_{smn} defined by (IV.16), special care is needed to find the behavior of $j_{m+1}(zu)$ for $z=0$. We use the multiplication theorem[11] for spherical Bessel functions:

$$j_{m+1}(zu)=u^{m+1}\sum_{q=0}^{\infty}\frac{(-1)^q(u^2-1)^q}{2^qq!}z^qj_{q+m+1}(z), \qquad (V.1)$$

where u is unrestricted, so that we can determine the

[11] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944), p. 142.

asymptotic form of $j_{m+1}(zu)$ for $z\to 0$ in the form

$$j_{m+1}(zu) \underset{z=0}{\longrightarrow} z^{m+1}u^{m+1}\sum_{q=0}^{\infty}\frac{(-1)^q(u^2-1)^q z^{2q}}{2^q q!(2m+2q+3)!}$$

or

$$j_{m+1}(zu) \underset{z=0}{\longrightarrow} \left(\frac{u^2}{u^2-1}\right)^{\frac{1}{2}(m+2)} j_{m+1}[z(u^2-1)^{\frac{1}{2}}]. \quad \text{(V.2)}$$

Regardless of what u may be, the effective dependence of F_{smn} on z, for small enough z, has again the form z^{m+1}. Thus for the production of very low-energy gamma rays the quantities $\mathcal{a}$ and σ_t vary according to E_g. It is, of course, obvious that the integrands in I_{smn} and F_{smn}, for any value of z, behave well at the limits of integration.

At the high-frequency limit, however, we only need to consider the quantities F_{smn}. If we use the result (III.16), which holds approximately for very small z_{0F}, then we get

$$h_s^*(z_{0F}u)+\eta_{F^{s*}}h_s(z_{0F}u) \underset{z_{0F}=0}{\longrightarrow} 2j_s(z_{0F}u)\to 0, \quad s>0.$$

Hence for high frequencies the quantities $\mathcal{a}$ and σ_t are very small. Actually, at the extreme limit of $E_g=E_0$, $\mathcal{a}$ and σ_t both vanish. In this way it is seen that the result is a closed spectrum of radiation.

In order to see the effect of the complex potential on the gamma-ray production, as a good approximation we consider the upper limit of a typical factor $j_s(z_F u)$ in the cross section. From the well-known Bessel inequality, we can write

$$|j_s(z_F u)|\leqq\frac{\Gamma(\frac{1}{2})}{2^{s+1}}\frac{|z_F{}^s|u^s}{\Gamma(s+\frac{3}{2})}\exp(u|y_F|), \quad \text{(V.3)}$$

where

$$z_F=x_F+iy_F=\left(\frac{\mu R^2}{\hbar^2}\right)^{\frac{1}{2}}\left(q_F+i\frac{V_I(E)}{q_F}\right),$$

$$q_F=[E_0(1-y)+V_R(E)+[(E_0(1-y)+V_R(E))^2+(V_I(E))^2]^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \text{(V.4)}$$

and

$$\left(\frac{\mu R^2}{\hbar}\right)^{\frac{1}{2}}=\left(\frac{A^{5/3}}{A+1}\right)^{\frac{1}{2}}\times\frac{1}{4.8\ \text{Mev}^{-\frac{1}{2}}}.$$

Let us now consider the contribution to I_{smn} arising from the initial state S and final state P waves:

$$I_{100}=\int_0^1 du\,u\,j_1^*(z_F u)j_1(zu)j_1(z_I u).$$

By using (V.3), we obtain

$$|I_{100}|^2\leqq\frac{|z_F{}^2|\,|z_I{}^2|z^2}{(27)^2}\times\frac{1}{x^{10}}[e^x(24-24x+12x^2-4x^3+x^4)-24]^2, \quad \text{(V.5)}$$

where

$$x=y_I+y_F.$$

In the absence of V_I, we have $x=0$ and

$$|I_{100}|^2\leqq(z_F)^2(z_I)^2(z)^2/[(27)^2 25]. \quad \text{(V.6)}$$

We shall also need the inequality

$$|j_s(x_F u+iy_F u)|\geqq|j_s(x_F u)|. \quad \text{(V.7)}$$

The latter follows from

$$j_s(z_F u)=\frac{1}{2z_F u}[\exp(iz_F u)M_s(z_F u)+\exp(-iz_F u)N_s(z_F u)],$$

where

$$M_s(z_F u)=\sum_{r=0}^{s}\frac{i^{r-s-1}(s+r)!}{r!(s-r)!(2z_F u)^r},$$

$$N_s(z_F u)=\sum_{r=0}^{s}\frac{(-i)^{r-s-1}(s+r)!}{r!(s-r)!(2z_F u)^r}.$$

It follows from (V.5), (V.6), and (V.7) that the yield of radiation for a given primary energy E_0 is higher with a complex potential than with a real one. It is also evident from (V.5) and from R-dependence of y_I and y_F that for a fixed E_0 the yield is increasing with mass number A. This result is in qualitative agreement with experiment.

The primary energy dependence of the experimental spectral curves implies the inequality

$$d\mathcal{a}/dE_0<0, \quad \text{(V.8)}$$

which can also be regarded as a restriction on V_R. The inequality (V.8) in itself is not enough to define the possible form of V_R, but it means that V_R is also a function of V_I. However, the form

$$V_I=aE_0{}^b$$

obtained for the imaginary part seems to reproduce the qualitative features of the ones discussed by others.[12]

In a numerical analysis of the series Q or Q_t, the number of waves required to be included can, approximately, be estimated from

$$l=R/\lambda\!\!\!^{-}_P,$$

where

$$\lambda\!\!\!^{-}_P=\frac{2.1\times10^{-14}AMc^2}{(A+1)E_0}\left[1+\frac{2Mc^2A}{(A+1)E_0}\right]^{-\frac{1}{2}}.$$

For 100-Mev primary energy, in the center-of-mass systems, in a collision with Be[9], we need to include about 6 waves to get an approximate numerical value for the series. The required number of waves does not decrease very fast with decreasing energy; for example, at 50-Mev incident energy one needs at least 4 waves

[12] A. M. Lane and C. F. Wandel, Phys. Rev. 98, 1524 (1955); W. B. Riesenfeld and K. M. Watson, Phys. Rev. 102, 1157 (1956).

to include in the analysis. There is, however, a possible simplification of the series for small final energies of the protons. In this case the wavelength in the final state is large enough to allow a further reduction in the number of the final-state angular momentum states.

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Scattering of K^+ Particles from Protons and Deuterons

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The direct coupling between K particles and π mesons, proposed by Schwinger, has been used to calculate the scattering of K^+ particles from nucleons and deuterium. It is shown that this interaction gives a reasonable explanation of the observed forward peaking in the angular distribution of K^+ particles scattered from protons. The theory predicts an inhibition of elastic scattering from the deuteron. The energy spectrum for K^+ particles inelastically scattered from deuterium has been calculated assuming a simple model for the deuteron. There are no experiments presently available for deuterium.

I. INTRODUCTION

RECENTLY[1] Schwinger has proposed a dynamical theory of mesons, nucleons, and hyperons in interaction. He has shown that one can obtain a qualitative understanding of phenomena involving production and decay of the new particles in terms of three basic strong couplings in addition to the weak interactions which are responsible for their instability. The three strong interactions are the well known pion-nucleon coupling characterized by a coupling constant $g_{N\pi}$, a direct coupling between K and π mesons, $g_{K\pi}$, and a coupling between nucleons, hyperons, and K mesons, g_K. It is argued that in the absence of all couplings the hyperons and nucleons form a degenerate mass multiplet and hence the direct interaction of a K particle with any pair of members of this multiplet is characterized by the single coupling parameter g_K.

Some of the most interesting aspects of Schwinger's proposals arise because of the direct $K-\pi$ coupling. Since the K particle has, in all probability, an isotopic spin of $\frac{1}{2}$ and an ordinary spin of zero, there is no way of coupling a K field of definite parity bilinearly to the pseudoscalar π field and still preserving the invariant scalar nature of the interaction. That the coupling must be bilinear in the K field is a consequence of the multiple-valuedness of a single field of half-integral spin (isotopic or ordinary) under rotations through 2π. The fact that the K field has spin zero makes it impossible to construct pseudoscalar matrices like the familiar γ_5 of the spin $\frac{1}{2}$ theory which can be balanced against the pseudoscalar meson field. On the other hand, the occurrence of both 2π and 3π decay modes of the K particle would seem to imply that its intrinsic parity is not a constant of the motion in the usual sense. In fact, if one introduces parity degrees of freedom for the K fields, making the intrinsic parity a dynamical variable, then, as has been shown by Schwinger,[1] one can construct a $K-\pi$ interaction which is properly invariant; see Eq. (1).

This interaction will manifest itself in the production of K particles in a reaction like $\pi^-+p\rightarrow K^0+\Lambda^0$. One mechanism contributing to this process will be the following: first $p\rightarrow K^++\Lambda^0$ by means of the g_K coupling, whence the K^+ absorbs the π^- directly leaving a residual Λ^0. The K^0 produced in this way will tend to move in the direction of the incident π^- with relatively high energy. A competing mode of production is the so-called "shaking off" transition in which the proton first absorbs the π^-, becoming a neutron, whereupon the neutron dissociates into a Λ^0 and a K^0. This mode will yield an angular distribution which tends to be symmetric in the center-of-mass system. The fact that the K^0's produced in association with Λ^0's have been observed to peak in the forward direction in the barycentric system[2] may indicate that the first production process dominates the second.

It is supposed that the relevant interactions effective in the associated production of K particles conserve parity symmetry. If $\phi_K{}^+$ and $\phi_K{}^-$ are the K fields of definite parity, then the parity-symmetric fields can be written as $\phi_{K1}=(\phi_K{}^++\phi_K{}^-)/\sqrt{2}$ and $\phi_{K2}=(\phi_K{}^+-\phi_K{}^-)/\sqrt{2}$. To conserve over-all parity symmetry, the Λ and Σ must also be created in states of definite parity symmetry. A K_1 is created in association with a Λ_1 or a Σ_1 while a K_2 is created with a Λ_2 or a Σ_2. As Schwinger has observed, the fact that Λ's and Σ's are created in

[1] J. Schwinger, lectures at Harvard University, Spring, 1956 and Stanford University, Summer, 1956 (unpublished); also Phys. Rev. **104**, 1164 (1956).

[2] J. Steinberger *et al.*, Phys. Rev. **103**, 1827 (1956).