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Spectrum of Turbulent Fluctuations Produced by Convective Mixing of Gradients

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Isotropic fluctuations of a passive scalar ψ produced by turbulent convection are investigated. The source of irregularities is considered to be the turbulent mixing of an established gradient of ψ. The mixing velocity field is described by Heisenberg's spectrum for homogeneous, isotropic turbulence. Replacing the (self-mixing) transfer of energy down the spectrum by an equivalent diffusion term, the local fluctuation spectrum becomes

$$S(k)=(\nabla\psi)^2\frac{1}{k^3}\frac{1}{[1+(k/k_s)^4]^{4/3}}\frac{1}{[1+(k/k_s)^{4/3}]^2},$$

where k_s is the viscosity cutoff wave number of the velocity field. In the inertial range ($k\ll k_s$) this result agrees with the spectrum deduced from purely dimensional arguments. Support for the above spectrum comes from the scattering of radio waves by dielectric fluctuations in the troposphere and ionosphere.

I. INTRODUCTION

IMPORTANT new modes of radio propagation via the ionosphere and troposphere have been identified with scattering by irregularities of the dielectric constant. The stochastic nature of signals so received suggest that the atmospheric fluctuations themselves form a random process in space and time. The physicist is asked to construct a model for the production of such irregularities and from it to predict their observed statistical properties. In so far as the dielectric constant is determined principally by water vapor content in the troposphere and electron density in the ionosphere, the problem is to describe fluctuations of a passive scalar quantity produced by the turbulent vector velocity field of the atmosphere's neutral fluid. Its solution evidently transcends radio scattering and is of interest to a variety of physical processes which are sensitive to fluctuations in the working agency.

Villars and Weisskopf[1] first imagined these fluctuations to be produced by pressure fluctuations in the neutral fluid which accompany its velocity pulsations. Variations in the density of the scalar were related to variations of the fluid and these (adiabatically) to variations in the pressure. A more recent view is that this mechanism is not strong enough to account for the observed irregularities, and it is now proposed that mixing of the scalar's established gradient by the neutral fluid's turbulent convection is the dominant mode.[2,3] Gradients of electron density associated with the ionospheric layers, for instance, give reasonable agreement with measured power levels at very high frequency (VHF).

This paper accepts the qualitative description of gradient mixing proposed by Villars and Weisskopf and attempts to describe the process analytically. The scalar's spatial variations are characterized by a diffusion equation in which the convection term's velocity vector is the turbulent solution(s) of the Navier-Stokes equation.

$$\frac{\partial V_\alpha(r,t)}{\partial t}+V_\beta(r,t)\frac{\partial}{\partial r_\beta}V_\alpha(r,t)=\nu\nabla^2 V_\alpha(r,t)-\frac{\partial}{\partial r_\alpha}\left(\frac{p(r,t)}{\rho_0}\right). \quad (1.1)$$

To this, we adjoin the incompressibility condition

$$\frac{\partial}{\partial r_\alpha}V_\alpha(r,t)=0, \quad (1.2)$$

which is valid so long as the velocity fluctuations are small compared with the local speed of sound—as they surely are. $\nu=\mu/\rho$ is the kinematic viscosity and one is to sum over repeated indices. This study will exploit previous (aerodynamical) studies of the neutral fluid's velocity field.[4]

We shall develop and illustrate the theory by identifying the passive scalar ψ with electronic density. If $N_0(r)$ is the stable plasma configuration, and $N(r,t)$ describes the space-time fluctuations induced in it by the turbulent convection, the irregularity is

$$\delta N(r,t)=N(r,t)-N_0(r). \quad (1.3)$$

A quantity of central interest for physical applications is the space Fourier transform

$$\eta(k,t)=\frac{1}{8\pi^3}\int_V d^3r e^{-i\mathbf{k}\cdot\mathbf{r}}\delta N(r,t). \quad (1.4)$$

The spectrum is defined in terms of η and the normalizing volume V by

$$\langle\eta(k,t)\eta(k',t)\rangle=\frac{S(k)}{4\pi k^2}\delta(k+k') \quad (1.5)$$

[1] F. Villars and V. F. Weisskopf, Phys. Rev. **94**, 232 (1954).
[2] F. Villars and V. F. Weisskopf, Proc. Inst. Radio Engrs. **43**, 1232 (1955).
[3] R. M. Gallet, Proc. Inst. Radio Engrs. **43**, 1240 (1955).
[4] See for instance, G. K. Batchelor, *The Theory of Homogenous Turbulence* (Cambridge University Press, New York, 1953).

Angular brackets indicate time and/or ensemble averaging. By inverting (1.4), one finds that the mean square fluctuation at a point is equal to the integral of $S(k)$ over all wave numbers:

$$\langle \delta N^2(r,t)\rangle = \int_0^\infty dk S(k). \tag{1.6}$$

Insofar as the Fourier transform (1.4) describes fluctuations which are localized in space to a dimension $l=k^{-1}$, the wave number k may be interpreted as an inverse blob size, and $S(k)$ describes how each blob size range in the decaying hierarchy contributes to the mean square fluctuation at a point. Our task is to deduce this function.

II. EQUATIONS OF TURBULENT MIXING

Imagine the electrons to be frozen into the neutral turbulent fluid (gas). Now consider what happens to an initial gradient of ionization. Motion of the neutral carrier transfers electrons from low- to high-density points on this gradient's profile and vice versa. These intruding cells appear as fluctuations against the ambient profile—and scatter accordingly. We imagine that this transfer is accomplished by turbulent convection, and that the resulting irregularities are erased by diffusion and recombination.

Let $N(\mathbf{r},t)$ be the local electron density in the mixing medium. Its total time change is related to the ionization rate $I(\mathbf{r})$, recombination coefficient α, and diffusion constant D by the continuity equation.[5]

$$\frac{\partial N(\mathbf{r},t)}{\partial t} + V_\alpha(\mathbf{r},t)\frac{\partial}{\partial r_\alpha}N(\mathbf{r},t) = I(\mathbf{r}) - \alpha N^2(\mathbf{r},t) + D\nabla^2 N(\mathbf{r},t). \tag{2.1}$$

The convective velocity $V_\alpha(\mathbf{r},t)$ is the divergence-free solution of (1.1). Since $V_\alpha(\mathbf{r},t)$ is a stochastic function itself, it induces statistical fluctuations in the mixed electron configuration. If there were no turbulence, a static profile $N_0(\mathbf{r})$ would be established satisfying,

$$0 = I(\mathbf{r}) - \alpha N_0^2(\mathbf{r}) + D\nabla^2 N_0(\mathbf{r}); \tag{2.2}$$

and such solutions have been discussed frequently.

We are interested here in the density fluctuation, (1.3). Subtracting (2.2) and (2.1) and neglecting nonlinear terms, we find that $\delta N(\mathbf{r},t)$ satisfies

$$\frac{\partial \delta N}{\partial t} + [2\alpha N_0 - D\nabla^2]\delta N = -V_\alpha \frac{\partial}{\partial r_\alpha}[N_0 + \delta N]. \tag{2.3}$$

It is quite a good approximation to hold N_0 constant in the recombination term and to let the gradient of N_0 be constant on the right hand side. Introducing Fourier transforms for δN and V_α, we find that $\eta(k,t)$ satisfies the integro-differential equation,

$$\frac{\partial \eta(k,t)}{\partial t} + [2\alpha N_0 + Dk^2]\eta(k,t) + \frac{dN_0}{dh} j_\alpha V_\alpha(k,t) + i\int d^3l k_\alpha V_\alpha(k-l,\, t)\eta(l,t) = 0, \tag{2.4}$$

where j_α is a unit factor in the direction of the (initial) ionization gradient, probably vertical.

If the integral or self-mixing term in this equation were negligible, one could proceed with an analytic solution; $\eta(k,t)$ then behaves like the output of an $R-L$ filter driven by a *random* input. As a rule,[6] one may neglect recombination effects in comparison with diffusion "damping," so that

$$\eta(k,t) = \frac{dN_0}{dh}\int_0^\infty du \exp(-Dk^2 u) j_\alpha V_\alpha(k,\, t-u). \tag{2.5}$$

To average this expression over the statistical V_α, we use a result of homogenous, isotropic turbulence theory[4]:

$$\langle V_\alpha(k,t) V_\beta(k',\, t+\tau)\rangle = \delta(k+k')\frac{E(k)}{4\pi k^2}\left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right]C(k,\tau). \tag{2.6}$$

$C(k,\tau)$ is the time correlation of fluctuations within a fixed wave number range, and is *not* known adequately at present. The spectrum of kinetic energy $E(k)$ adopted is Heisenberg's generalization[7] of Kologomoroff's similarity result ($k^{-5/3}$).

$$E(k) = \frac{V_0^2}{l_0^{2/3}}\frac{1}{k^{5/3}}\frac{1}{[k^4/k_s^4+1]^{4/3}}, \quad k > k_0 = \frac{1}{l_0}. \tag{2.7}$$

The smallest blob wave number k_s is given in terms of the large blob (energy input) parameters and ν.

$$k_s^4 = \tfrac{3}{8}(V_0^3/l_0\nu^3). \tag{2.8}$$

Combining Eqs. (1.5), (2.7), and (2.8), we find for the spectrum:

$$S(k) = E(k)\left(\frac{dN_0}{dh}\right)^2 \sin^2\phi \int_0^\infty du \int_0^\infty dv \times \exp[-Dk^2(u+v)]C(k,\, |u-v|). \tag{2.9}$$

In the *high* wave number range, a reasonable assumption[8] for the time correlation is[4]

$$C(k,\tau) = \exp(-\nu k^2 \tau), \tag{2.10}$$

[5] This approach was suggested by Professor Villars and Professor Weisskopf.

[6] In the lower E region, (80 km) for instance, $2\alpha N_0 = 10^{-3}$ sec^{-1} and $D = 10^4$ cm^2 sec^{-1}, but one is interested in values of $k = \frac{1}{2}$ meters $> [2\alpha N_0/D]^{\frac{1}{2}}$.

[7] W. Heisenberg, Z. Physik **124**, 628 (1948).

[8] This $C(k,\tau)$ implies infinite values for the fluid's (rms) acceleration and is hence quite improper for small τ, where it should have zero slope.

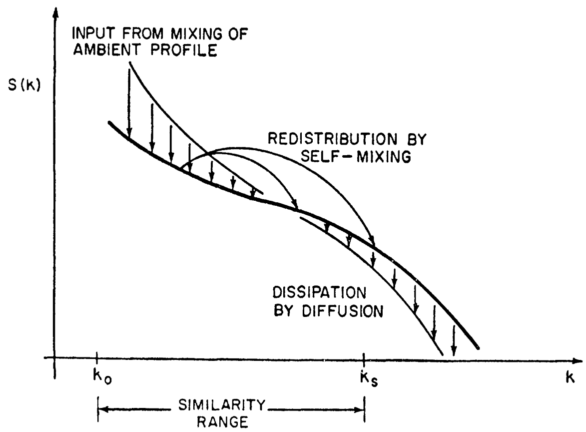

FIG. 1. Mechanisms which sponsor stable spectrum.

where ν is the kinematical viscosity appearing in (1.1). The integrations are now easily performed and the spectrum (2.7) introduced to give

$$S(k)=\left(\frac{dN_0}{dh}\right)^2\left(\frac{V_0^2}{l_0^{\frac{2}{3}}}\right)\left(\frac{\sin^2\phi}{D(D+\nu)}\right) \times\left(\frac{1}{k^{17/3}}\right)\frac{1}{[1+(k/k_s)^4]^{4/3}}. \tag{2.11}$$

The angle ϕ is that included between $\mathbf{k}$ and $\mathbf{j}$, and hence is very nearly zero for symmetrical scattering and a vertical gradient.[9]

The point of the matter here is that local gradients in the mixed plasma can be as steep as or steeper than the ambient profile and need not have the directional properties of the original gradient. The integral term in (2.4) cannot be omitted. The ambient profile provides a nonisotropic gradient source which is stirred into the spectrum $\eta(k,t)$ by blobs of all sizes simultaneously. This initial in-mixing retains most of the parental anisotropy, so that the resulting spectrum (2.11) does not scatter forward—as we found. The next step is one of self-mixing, which redistributes the fluctuation "energy" of the larger blobs to smaller ones. The integral term in (2.4) evidently describes this transfer mechanism, for it couples different wave number ranges together.[10] Since the velocity field itself is assumed isotropic, it acts progressively to erase the anistropy of the parent source and to create a stable isotropic scalar field. This redistribution or self-mixing continues until the fluctuations are damped at high wave numbers by diffusion. The spectrum's shape evidently depends critically on just how this energy is distributed and how long it takes a given input to reach isotropy by self-mixing.

These intuitive ideas are represented schematically in Fig. 1, which shows the stable spectrum and the mechanisms which maintain it. We should note an important difference between our turbulent mixing model and the more familiar decay of homogenous turbulence.[4] In the latter, one adds an external source of energy ϵ (cm^2/sec^3) at the largest eddy wave number k_0 to balance dissipation. The source of turbulent mixing appears explicitly in (2.4), however, and makes contributions to each wave number interval, according to the (decreasing) magnitude of $V(k,t)$. If one assumes that this energy enters the spectrum only at k_0, the input

$$\langle\delta N^2\rangle=l_0^2(dN_0/dh)^2, \tag{2.12}$$

would overestimate the mean square fluctuation, as Gallet has found.[3]

III. DIMENSIONAL SOLUTION FOR INERTIAL RANGE

In the inertial range $k_0<k<k_s$, one may use purely dimensional arguments to predict the spectrum. As the self-mixing proceeds, one may expect isotropy and thereby specialize Eq. (1.5).

$$\langle\eta(k,t)\eta(k',t)\rangle=\delta(k+k')S(|\mathbf{k}|)/4\pi k^2. \tag{3.1}$$

The mean square fluctuation at a given point in space is thus

$$\langle\delta N^2(r,t)\rangle=\int_0^\infty dkS(k). \tag{3.2}$$

The only quantity appearing in (2.4) which has the units of δN (i.e., electrons/cc) is dN_0/dh, so that $S(k)$ must be proportional to its square. Since D and ν are approximately equal,[11] the only other parameters are ϵ and ν, out of which one constructs (again by purely dimensional arguments) the characteristic speed and length of the velocity field.

$$v=(\epsilon\nu)^{\frac{1}{2}},\quad l=(\nu^3/\epsilon)^{\frac{1}{4}}. \tag{3.3}$$

We need only assume that

$$S(k)=(dN_0/dh)^2l^3\psi(kl)$$

for $\psi(x)$ a dimensionless function, to insure the dimensionality of (3.2). In the inertial range $k\ll k_s$ redistribution alone is important, so that the spectrum $S(k)$ ought to be independent of ν (or D). One satisfies this condition with the choice $\psi(x)=x^{-3}$.

$$S(k)=(dN_0/dh)^2(\text{const}/k^3); \tag{3.4}$$

This result is quite independent of the velocity field; except for the restriction $k>1/l_0$ on Heisenberg's spectrum (2.7). This result agrees with Villars and Weisskopf's treatment,[1] but is at variance with both Silver-

[9] See Sec. V.

[10] In analogy with the inertial (nonlinear) term in the Navier-Stokes equation [see Eq. (1.1)].

[11] $\nu\simeq D$ in the ionosphere since space charge effects bind the electrons to their ionized carrier's frictional experience. For other applications, one may need to discriminate between cutoff wave numbers for the scalar and velocity fields.

man[12] and Batchelor[13] who assumed a k-independent external source of turbulent input. [See (2.12).]

The loss of fluctuations by diffusion at high wave numbers is described by the second term in (2.4). Neglecting recombination, the power loss is computed as

$$P=2\int_0^\infty dkk^2S(k)=2D\left(\frac{dN_0}{dh}\right)^2\lambda_0, \tag{3.5}$$

where λ_0 is a constant determined by the spectrum's shape alone. On purely dimensional basis, one would have argued from (3.3) that

$$P=(dN_0/dh)^2\nu l,$$

which is again equivalent to (3.5) for $D=\nu$. The spectrum at high wave numbers evidently cannot be computed from dimensional arguments, since (3.5) is satisfied identically by (3.4). The iteration approach which led to (2.11) could be pursued to the next step by substituting (2.5) into the integral term of (2.4). This converges to k^{-11} [see (2.11)] in the high wave number range, but is not sound in the (k_s) transition range, where self-mixing is still quite the largest redistribution mechanism.

IV. TRANSITION RANGE

To bridge the gap between our similarity (k^{-3}) and dissipation (k^{-11}) results, we introduce the concept of a transfer diffusion $D'(k)$ to describe the removal of fluctuation energy from a wave number interval k to all higher k. Heisenberg[7] exploited this same concept in treating the nonlinear, inertial transfer of kinetic energy as an equivalent viscosity in the velocity equations (1.1), and so derived the transition spectrum (2.7). In this spirit, we rewrite (2.4) as

$$\frac{\partial\eta(k,t)}{\partial t}+[D+D'(k)]k^2\eta(k,t)=-\left(\frac{dN_0}{dh}\right)j_\alpha V_\alpha(k,t). \tag{4.1}$$

The equivalent diffusion constant $D'(k)$ should depend principally on the convective velocity field, so that by purely dimensional arguments one has

$$D'(k)=\gamma\int_k^\infty d\lambda[E(\lambda)/\lambda^3]^{\frac{1}{2}}, \tag{4.2}$$

where γ is an absolute constant of order unity. One can also argue that $\eta(l,t)$ in (2.4) can be brought outside the integral as $\eta(k,t)$, since $V_\alpha(k-l,\,t)$ is largest—and hence most effective as a convective mixer—for k near l. The further condition $k_\alpha V_\alpha(k,t)=0$ tells one that small values of l contribute little to the integration. When $\eta(k,t)$ is so removed, it is not difficult to infer (4.2).

The transfer diffusion evidently depends on the inertial range of the velocity spectrum $E(k)$, so that one may substitute from (2.7), and find

$$D'(k)=(3\gamma/4l_0^{\frac{1}{3}})(V_0/k^{4/3}). \tag{4.3}$$

When this result is inserted into (4.1), one sees that $\eta(k,t)$ may be expressed exactly as in (4.5), except now with an additional damping term. Using the results (2.5) through (2.8) and dropping the viscosity ν when it appears with D (equivalent to an infinite time correlation) for convenience, we have

$$S(k)=\left(\frac{dN_0}{dh}\right)^2\frac{V_0^2}{l_0^{\frac{2}{3}}}\frac{\sin^2\phi}{[D+\frac{3}{4}\gamma V_0/l_0^{\frac{1}{3}}k^{4/3}]^2}\times\frac{1}{[1+k^4/k_s^{\,4}]^{4/3}}. \tag{4.4}$$

If k is small, this gives the similarity k^{-3} expression (3.4), and at high numbers it approaches the k^{-11} result adduced in (2.11). It would appear that we have succeeded in describing the diffusion transition range (k_s), where both self-mixing and dissipation are competing.

The persistant angular (ϕ) dependence in (4.4) requires a word of explanation. By replacing the self-mixing integral term by a transfer diffusion constant, we have destroyed the directional properties of this term. At this stage however, we are quite sure that considerable self-mixing has taken place, since the direct input mechanism at these wave numbers is small. To enforce this physical picture on the over-simplified representation (4.1), we average over all angles ϕ relating to the original gradient orientation. This is quite important for vertical gradients associated with ionospheric layers, and probably not so critical for meteor trails with random orientation.

Our result (4.4) can be cast in more convenient form by recalling the definition (2.8) of the cutoff wave number. For our purposes, one may set $D=\gamma\nu$, so that the transition to high wave numbers depends only on k/k_s.

$$S(k)=\left(\frac{dN_0}{dh}\right)^2\frac{1}{k^3}\frac{1}{[1+(k/k_s)^{4/3}]^2}\frac{1}{[1+k^4/k_s^{\,4}]^{4/3}},\quad k>k_0=1/l_0. \tag{4.5}$$

This is the central result of our study. It is interesting that it depends only on the velocity field which sponsors the fluctuations through its viscosity cutoff wave number k_s as given by Eq. (2.8). The fluctuation intensity depends rather on the steepness of the mixed profile. In so far as the deduction of (4.5) relied solely on averages over spatial volumes of size $l_0\simeq1000$ meters (i.e., largest blob size), the above expression may be considered a local formula. One should therefore expect large fluctuations at points of rapid change on the profile (inversion layers, shears), which contradicts neither one's intuition nor experiment.

[12] R. A. Silverman, J. Appl. Phys. **27**, 699 (1956).

[13] G. K. Batchelor, Cornell University Electrical Engineering Research Report 262, September 15, 1955 (unpublished).

V. EXPERIMENTAL SUPPORT

For radio-wave scattering at very high frequency by electron fluctuations in the ionosphere's lower E region, it has been shown[1] that the received power is proportional to[14]

$$\sigma = r_0^2(2\pi)^6 \cdot \frac{1}{V}\langle|\eta(k,t)|^2\rangle = r_0^2 S\frac{(k)}{k^2}\cdot 2\pi^2, \qquad (5.1)$$

where $r_0 = 2.8\times 10^{-13}$ cm is the classical electron radius. The vector $\mathbf{k}$ is the difference between the upgoing and downcoming propagation vectors, and is almost always oriented along the vertical. Its magnitude depends on the radiation's wavelength (λ) and scattering angle θ.

$$|\mathbf{k}| = (4\pi/\lambda)\sin(\tfrac{1}{2}\theta). \qquad (5.2)$$

Comparison of simultaneous transmissions over the same path with different frequencies and scaled aerials permits one to map out $S(k)$ by (5.1). For the National Bureau of Standards' experiments,[15] it has been found[16] that a good fit to the data is obtained with (4.5) if k_s is chosen as (1.5 meters)$^{-1}$.

The troposphere's dielectric constant is set mostly by concentrations of water vapor. In this case, the ionization rate must be replaced by gravitational and buoyant forces, while the recombination term is rigorously zero. It has been found for uhf and shf waves scattered by tropospheric fluctuations that the measured power is proportional to

$$P \propto \lambda^{-4}S(k)\frac{1}{k^2}, \qquad (5.3)$$

where k has the meaning of (5.2). In the troposphere k_s^{-1} is of the order of millimeters,[17] so that only the similarity reduction of (4.5) is of interest.

$$P \propto (dn/dh)^2\lambda/[\sin(\tfrac{1}{2}\theta)]^5. \qquad (5.4)$$

A linear dependence on wavelength is definitely *favored* by Norton's careful analysis[18] of numerous broad-beam, tropospheric scatter links. A useful correlation between scattered power and the refractive gradient computed from radiosonde records has also been obtained.[19]

In the measurement of dielectric fluctuations directly with microwave refractometers (resonant cavities) mounted in aircraft, there is no sensible correlation between gust (velocity) forces acting on the aircraft and intensity of refractive fluctuations. This experience at least does not contradict the minor role played by the turbulent velocity field (i.e., V_0, l_0) in the spectrum (4.5).

ACKNOWLEDGMENT

The writer is grateful to Professor Felix Villars for helpful discussion of this problem.

[14] Note:

$$\delta(0) = \frac{1}{8\pi^3}\int_V d^3r e^{i\mathbf{r}\cdot 0} = \frac{V}{8\pi^3}.$$

[15] Bailey, Bateman, and Kirby, Proc. Inst. Radio Engrs. **43**, 1181 (1955).

[16] A. D. Wheelon, J. Geophys. Research (March, 1957).

[17] A. D. Wheelon, Proc. Inst. Radio Engrs. **43**, 1381 (1955).

[18] Norton, Rice, and Vogler, Proc. Inst. Radio Engrs. **43**, 1488 (1955).

[19] B. R. Bean and F. M. Meaney, Proc. Inst. Radio Engrs. **43**, 1419 (1955).