PHYSICAL REVIEW VOLUME 105, NUMBER 5 MARCH 1, 1957

Linear and Toroidal Geons

FREDERICK J. ERNST, JR.*
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey, and University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico

(Received August 16, 1956)

It has been argued by Wheeler that the coupled equations of the electromagnetic field and the gravitational field of general relativity,

$$R_{ik}-\tfrac{1}{2}g_{ik}R=(8\pi G/c^4)T_{ik},$$
$$T_i{}^k=(1/4\pi)(F_{ij}F^{kj}-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_i{}^k),$$
$$(-g)^{-\frac{1}{2}}(\partial/\partial x^j)(-g)^{\frac{1}{2}}F^{ij}=0,$$
$$F_{ij}=\partial A_j/\partial x^i-\partial A_i/\partial x^j,$$

should admit a set of completely singularity-free solutions, A_i, g_{jk} (i, j, $k=1, 2, 3, 4$), with the following properties: 1. The gravitational mass originates solely from the energy stored in the electromagnetic field. In particular there are no material masses present. 2. No charges or currents are present, and $A_4=0$ everywhere. 3. The other components of the electromagnetic vector potential A_i are vanishingly small except within a *toroidal* region of space. Physically the electromagnetic field consists of light waves circling the torus in either direction. Such a torus of electromagnetic field energy is called a toroidal geon. An exact and detailed mathematical treatment of the general toroidal geon problem would be extremely complicated, requiring the solution of a set of coupled nonlinear partial differential equations. However, in the present paper it is shown how toroidal geons of large major radius to minor radius ratio may be studied by a simple method of approximation, providing one has a complete knowledge of the so-called linear geons, the electromagnetic field energy of which is confined to an infinitely long circular cylinder rather than to a torus. A detailed mathematical treatment of linear geons proves to be possible, as is demonstrated in this paper. The electromagnetic field potentials A_i $i=(1, 2, 3)$ of a toroidal geon or of a linear geon possess the same general nature as the electromagnetic field potentials encountered in the solution of classical toroidal and cylindrical wave guide problems. In this paper the case is considered where the electromagnetic field of the linear geon is a monochromatic standing wave vibrating in the lowest transverse-electric mode of the system. The field equations are derived from a variational principle, and these equations are solved numerically. The results are not surprising, as the general form of the unknown functions can be ascertained by quite simple considerations. These results give the foundation material for a proposed later treatment of toroidal geons.

1. INTRODUCTION

MAXWELL'S theory of the classical electromagnetic field may be easily expressed in terms of the general theory of relativity. Let A_j be a covariant vector, to be interpreted as an electromagnetic 4-potential; then an electromagnetic field tensor may be defined by the relations,

$$F_{ij}=\partial A_j/\partial x^i-\partial A_i/\partial x^j. \tag{1}$$

The second Maxwell's systems of equations is then defined by the tensor equation, resulting from this,

$$\partial F_{ij}/\partial x^k+\partial F_{jk}/\partial x^i+\partial F_{ki}/\partial x^j=0, \tag{2}$$

and the first of Maxwell's systems of equations is defined by the tensor-density relation

$$\partial \mathfrak{F}^{ij}/\partial x^j=\mathfrak{J}^i, \tag{3}$$

in which

$$\mathfrak{F}^{ij}=(-g)^{\frac{1}{2}}g^{in}g^{jm}F_{nm},\quad \mathfrak{J}^i=(-g)^{\frac{1}{2}}\rho dx^i/ds.$$

We shall assume that the current-density is identically zero throughout space-time, so that $\mathfrak{J}^i=0$. Therefore, we shall be dealing always with a space-time free of all atomic electricity and matter. Mass will arise only as a concomitant of the energy of the singularity-free electromagnetic field.

Since Maxwell's equations depend upon the metric tensor g_{ik}, in the presence of a gravitational field the solution of those equations will be different than the solution in free space. However, strictly speaking there is no such thing as "free" space, since the very presence of an electromagnetic field implies the existence of a gravitational field. Because electromagnetic fields are ordinarily weak, they ordinarily produce a gravitational field and a curvature in the metric small enough to be neglected. However, in the present paper we propose to consider very strong electromagnetic fields, for which a considerable deviation from flat space-time is induced. The problem will be that of finding solutions of the coupled gravitational and electromagnetic field equations, in the absence of material bodies, charges, and currents. One such solution has already been found by numerical integration of the field equations.[1] The electromagnetic field energy is essentially confined to a spherical shell of radius R. The gravitational field outside the shell is an ordinary Schwarzschild field corresponding to a mass M, where $M=4c^2R/9G$; c is the velocity of light and G is the gravitational constant. Such a ball of light Wheeler calls a spherical geon or gravitational-electromagnetic entity.

Wheeler has argued that spherical geons must be unstable, tending to transform into another form of gravitational-electromagnetic entity with the electromagnetic field energy essentially confined to a toroidal region of space. He suggested this study of the nature of toroidal geons.[2]

2. TOROIDAL GEON

The electromagnetic field of a toroidal geon is very similar to that within a toroidal wave guide. Each of

* Now at University of Wisconsin, Madison, Wisconsin.

[1] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[2] F. J. Ernst, senior thesis, Princeton (May 2, 1955) (unpublished).

these fields is characterized by possessing two orthogonal sets of normal modes of vibration. In the limiting case of a toroidal wave guide of infinitely large major radius (i.e., a straight wave guide of circular cross section) and in the limiting case of a toroidal geon of infinitely large major radius (i.e., a linear geon) the two orthogonal sets of normal modes are designated TE (transverse-electric) and TM (transverse-magnetic): in the former there is no component of the electric field along the axis of the guide or geon while in the latter there is no component of the magnetic field along the axis. Even in the case of the toroidal configurations of finite major radius we shall use the terminology TE and TM to designate the two orthogonal sets of normal modes, although the words are not quite appropriate. Since the simple substitution $E\rightarrow H$, $H\rightarrow -E$ transforms the complete set of TE modes into the complete set of TM modes, it is only necessary that we consider TE modes in our discussion of the geon, thus simplifying the problem somewhat.

We shall specialize the problem of the toroidal geon by assuming that the electromagnetic field consists of a standing wave which is monochromatic and vibrating in the lowest TE mode of the system. We could equally well consider the case where three times as much energy, for example, runs in the $+\phi$ direction as in the $-\phi$ direction, but this would unduly complicate the analysis. Similarly one could consider higher modes of vibration, corresponding—in ray language—to photons executing spirals about the line of energy concentration.

Let us adopt a cylindrical coordinate system (x_1,x_2,ϕ,T) where the x_1-axis is the axis of symmetry of the toroidal geon, x_2 measures distance from the x_1-axis, and ϕ measures the angle about the x_1-axis. Weyl has shown that any axially symmetric static gravitational field can be completely described by a line element of the form

$$ds^2=+U(dx_1^2+dx_2^2)+Vd\phi^2-WdT^2, \qquad (4)$$

where U, V, and W are functions of x_1 and x_2 alone and where T is the cotime (i.e., time multiplied by the velocity of light, c).[3] Of course, the energy of the toroidal geon will show ripples with a spacing in the ϕ direction of one-half wavelength $\frac{1}{2}\lambda$, where $\lambda/2\pi=\lambdabar=1/k$. This is unavoidable as long as we are unwilling to superpose solutions with a continuous spectrum of frequencies. However, if $\frac{1}{2}\lambda$ is much smaller than the minor radius of the torus, the gravitational field will be almost exactly axially symmetric because of the long-range character of gravitational forces. This long-range character will also help to smooth out the temporal variations of the gravitational field because of the finite velocity of light. Hence, by restricting ourselves in such a way that $\frac{1}{2}\lambda$ is smaller than the minor radius of the torus, we may make use of Weyl's simple form of the line element (4).

Weyl's line element (4) is expressed in terms of cylindrical coordinates. It is of advantage to reexpress it in terms of coordinates particularly appropriate to the toroidal geometry of the present problem. Letting $x_1=\rho\sin\theta$ and $x_2=\alpha+\rho\cos\theta$, where α is approximately the major radius of the geon, we can rewrite the line element in the form

$$ds^2=\rho^{-2}R^2e^{-2\psi}\{d\rho^2+\rho^2[d\theta^2+\Omega^{-2}R^{-2}e^{4\psi}(1+D)\alpha^2d\phi^2]\}-e^{2\psi}dT^2, \qquad (5)$$

where R, ψ, and D are functions of ρ and θ alone, and the constant $\Omega=1/[c$ (the frequency of the radiation)$]$.

The electromagnetic field equations (3) together with the gravitational field equations may be obtained by means of the variational principle[1]

$$\delta\int\int\int\int[(c^3/16\pi G)R-(1/16\pi c)\langle F_{ij}F^{ij}\rangle](-g)^{\frac{1}{2}}dV=0. \qquad (6)$$

Here the F_{ij} are defined by Eq. (1) while R is the space-time curvature scalar

$$R=g^{ik}\left[\frac{\partial\Gamma^i{}_{ik}}{\partial x^j}-\frac{\partial\Gamma^i{}_{ij}}{\partial x^k}+\Gamma^i{}_{ik}\Gamma^m{}_{jm}-\Gamma^m{}_{ij}\Gamma^i{}_{km}\right] \qquad (7)$$

and

$$\Gamma^k{}_{ij}=\tfrac{1}{2}g^{km}\left[\frac{\partial g_{mj}}{\partial x^i}+\frac{\partial g_{im}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^m}\right]. \qquad (8)$$

Naturally the g_{ij}'s represent the coefficients of the quadratic form (5). The signs $\langle\ \rangle$ about $F_{ij}F^{ij}$ signify that $F_{ij}F^{ij}$ is to be averaged over the coordinates ϕ and T. The field equations which result from carrying out the indicated variation are rather involved partial differential equations. No attempt has been made to solve them exactly, for hydrodynamics offers instances where the same type of problem has been encountered and solved by a simple method of approximation.[4] Here, as there, consider the case of an annular source of gravitation whose minor radius is very small compared to its major radius. At distances large with respect to the minor radius of the toroidal geon the electromagnetic field can be considered vanishingly small and the gravitational field can be considered identical to the gravitational field due to an infinitely thin ring of energy. The latter field, however, can be described rather simply in terms of elliptic functions. The ratio of the minor radius to the major radius is assumed to be so small that the exterior field just described is valid for distances quite small with respect to the major radius of the geon. As one approaches closer to the torus, however, the gravitational field will diverge from that of an infinitely thin ring. This region of space we shall call the transition region. We may consider the electromagnetic field relatively small in this region also. Right at the torus itself, however, the electromagnetic field will suddenly become very large.

[3] H. Weyl, Ann. Physik **54**, 117 (1917).

[4] H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, 1932), sixth edition, p. 707.

In this interior region the torus looks very much like an infinitely long cylinder, and hence we expect it is reasonable to assume the electromagnetic and gravitational fields in the interior region may be considered identical to the corresponding fields in a linear geon, that is, in a straight beam of light. In the present paper we are primarily concerned with the study of the fields which occur within the interior region of the toroidal geon.

3. LINEAR GEON

In the interior region of the toroidal geon the line element (5) may be simplified, because there R, ψ, and D are merely functions of ρ. We define a new independent variable z such that $dz=\alpha d\phi$ and a new independent variable r such that

$$d(\ln\rho)/dr=\Omega^{-1}(1+D)^{\frac{1}{2}}R^{-1}e^{2\psi}. \quad (9)$$

As a result of this transformation, the line element (5) may be written

$$ds^2=\Omega^{-2}(1+D)e^{2\psi}(dz^2+dr^2)+R^2e^{-2\psi}d\theta^2-e^{2\psi}dT^2, \quad (10)$$

where R, ψ, and D are now considered to be functions of r alone. This line element (10) is in the general form (4) given by Weyl for axially symmetric static gravitational fields.

In the case of the linear geon the lowest mode of TE radiation is characterized in the (z,r,θ,T) coordinate system by the vector potential components.

$$\begin{aligned} A_z&=0, & A_r&=0, \\ A_\theta&=B(r)\sin z\cos\Omega T, & A_T&=0, \end{aligned} \quad (11)$$

where we have chosen the unit of length is such a way that the wavelength along the z axis is exactly 2π. Using (10) and (11) to evaluate Eq. (6), we obtain the action principle of the linear geon,

$$\delta\int_0^\infty\left\{-\tfrac{1}{2}R^{-1}e^{2\psi}\left[\left(\frac{dB}{dr}\right)^2-DB^2\right]+4\frac{dR}{dr}\frac{d\psi}{dr}-2R\left(\frac{d\psi}{dr}\right)^2+\frac{dR}{dr}\frac{d}{dr}[\ln(1+D)]\right\}dr=0. \quad (12)$$

By carrying out the variations with respect to the four unknown functions, B, R, ψ, and D, we obtain the field equations of the linear geon:

$$d^2B/dr^{*2}+R^{-2}e^{4\psi}DB=0, \quad \text{where} \quad dr^*/dr=Re^{-2\psi}, \quad (13)$$

$$d^2R/dr^2=\tfrac{1}{2}R^{-1}e^{2\psi}(1+D)B^2, \quad (14)$$

$$\frac{d}{dr}\left(R\frac{d\psi}{dr}\right)-\frac{d^2R}{dr^2}=\tfrac{1}{4}R^{-1}e^{2\psi}\left[\left(\frac{dB}{dr}\right)^2-DB^2\right], \quad (15)$$

$$4\frac{d^2\psi}{dr^2}+2\left(\frac{d\psi}{dr}\right)^2+\frac{d^2}{dr^2}[\ln(1+D)]=\tfrac{1}{2}R^{-2}e^{2\psi}\left[\left(\frac{dB}{dr}\right)^2-DB^2\right]. \quad (16)$$

At large distances r from the axis of the beam of light, the function B will tend rapidly to zero. From Eq. (14), we see that $R(r)$ tends then to a linear function of r, which without loss of generality we may take to be simply r. Then

$$\begin{aligned} &R\to r, \\ &\psi\to 2M\ln(r/a), \\ &D\to\Omega^2(r/a)^{8M^2-8M}-1, \end{aligned} \quad (17)$$

where $M=G/c^2$ times the mass per "unit length" and where a is a constant analogous to the free additive constant in the Newtonian potential function $\psi_N=2M\ln(r/a)$. The values of a, Ω, and M cannot be determined in the case of a linear geon that is really infinitely long.

Each solution $B(r)$, $R(r)$, $\psi(r)$, and $D(r)$ of Eqs. (13) through (16) possesses an asymptotic form (17) uniquely determined by the three constants, M, Ω, and a. It might appear at first sight that we would have to numerically integrate Eqs. (13) through (16) for every possible value of the three constants. However, there exists a scaling law for linear geons, such that every solution of the field equations may be obtained from those solutions for which $a=1$. Observe that the transformation

$$\begin{aligned} &B\to a^{2M}B, \\ &R\to R, \\ &\psi\to\psi-2M\ln a, \\ &D\to D, \\ &r\to r, \end{aligned} \quad (18)$$

leaves the field equations (13) through (16) unchanged in form, but it changes the asymptotic form of the solution from

$$\begin{aligned} &R\to r, \\ &\psi\to 2M\ln r, \\ &D\to\Omega'^2r^{8M^2-8M}-1, \end{aligned}$$

to

$$\begin{aligned} &R\to r, \\ &\psi\to 2M\ln(r/a), \\ &D\to\Omega^2(r/a)^{8M^2-8M}-1, \end{aligned}$$

where

$$\Omega a^{4M-4M^2}=\Omega'. \quad (19)$$

TABLE I. Summary of solutions by electronic computer of the eigenvalue problem for the linear geon. Column 1: number of the curves in Figs. 1, 2, 3, and 4. Column 2: mass per unit length (unit $=c^2/G$). Column 4: characteristic vibration frequency of the standing electromagnetic wave (unit $=c/\lambda$).

No.	M	$4M-4M^2$	Ω'
1	0.15075	0.512	2.895
2	0.132	0.458	2.66
3	0.109	0.388	2.48
4	0.080	0.294	2.41
5	0.042	0.161	2.45

Corresponding to each solution of the field equations which we find, we can deduce values of the constants $4M-4M^2$ and Ω' in Eq. (19). At this time we have numerically solved the field equations five times. For each of the five solutions we have graphed $a^{-2M}B(r)$, $R(r)$, $\psi(r)+2M\ln a$, and $\delta(r)\equiv\ln(1+D)$. The circular frequency is given by Eq. (19), where the constants $4M-4M^2$ and Ω' are tabulated in Table I.

Finally it should be recalled that the unit of distance in all the preceding equations has been fixed by requiring that the wavelength in the z direction be exactly 2π. This means that all distances (e.g., r, a, and R) have been expressed in terms of the reduced wavelength of the radiation in the z direction. The reduced wavelength, $\lambdabar=\lambda/2\pi$, is equal to the wavelength divided by 2π. It is also true that the frequency Ω has been expressed in terms of $c/\lambdabar$, where c is the velocity of light. Finally, the mass per unit length M has been expressed in terms of c^2/G as a unit, where G is the Newtonian gravitational constant. In this way all the quantities which enter our equations have been expressed as dimensionless quantities.

Example.—To find the peak rms electric field strength $E_{\text{peak rms}}$ and the values of the various metric components at the point of maximum field strength. Suppose it is given that the reduced wavelength of the radiation $\lambdabar=10^8$ cm, the frequency $\Omega=7.72$ sec^{-1}, the mass per unit length $M=2.04\times10^{27}$ g/cm, and $a=10^{12}$ cm. However, the unit of frequency which we employ is $c/\lambdabar=300$ sec^{-1}, while the unit of mass per unit length is $c^2/G=1.35\times10^{28}$ g/cm and the unit of a is $\lambdabar=10^8$ cm. Hence, in dimensionless units, we have $\Omega=2.57\times10^{-2}$, $M=0.151$, and $a=10^4$. Furthermore, by Eq. (19), $\Omega'=\Omega a^{4M-4M^2}=2.891$. It is clear from Table I that solution No. 1 is the appropriate one in this case.

From Figs. 1–4 we see that the maximum of $B(r)$ occurs at $r=2.6\lambdabar=2.6\times10^8$ cm and that here $a^{-2M}B(r)=0.37$, $R(r)=2\lambdabar=2\times10^8$ cm, $\psi(r)+2M\ln a=0.4$, and $\delta(r)\equiv\ln(1+D)=0.4$. It follows that $E_{\text{peak}}=c^2G^{-\frac{1}{2}}\lambdabar^{-1}D^{\frac{1}{2}}\times R^{-\frac{1}{2}}e^{-\psi}B=0.935\times10^{16}$ cgs units. For the metric components, we have in cgs units (assuming the use of cotime rather than time as the fourth component) the following:

$$g_{11}=g_{22}=\Omega^{-2}(1+D)e^{2\psi}=\Omega^{-2}e^{\delta}e^{2\psi}=200,$$
$$g_{33}=R^2e^{-2\psi}=(2\lambdabar)^2e^{-2\psi}=20.2\lambdabar^2=20.2\times10^{16}\text{ cm}^2,$$
$$g_{44}=-e^{2\psi}=-0.198.$$

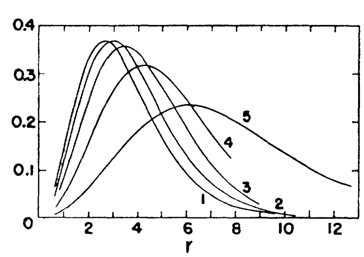

FIG. 1. The electromagnetic field strength function $B(r)$, where r is expressed in terms of the reduced wavelength $\lambdabar$.

The details of the numerical calculation are contained in the Appendix to this paper. The calculation was performed with Dr. John Gammel at the Los Alamos Scientific Laboratory.

4. EXTERIOR REGION

In Sec. 3 we discussed a method of numerically finding the values of the functions $B(r)$, $R(r)$, $\psi(r)$, and $D(r)$ in the interior region of a toroidal geon. By solving the differential equation (9), we can obtain r as a function of ρ, and hence we can obtain B, R, and D as functions of ρ in the interior region of the toroidal geon. We have, therefore, found a method of evaluating the line element (5) in the interior region. The evaluation of the line element (5) in the transition region will be left to a later publication. Let us now consider the simpler exterior region, in which we have to find the gravitational field of a thin ring of energy.

Weyl has indicated the way to solve the thin-ring problem.[3] He claims that a line element of the form

$$ds^2=e^{2(\gamma-\psi)}(dx_1^2+dx_2^2)+x_2^2e^{-2\psi}d\phi^2-e^{2\psi}dT^2, \quad (20)$$

where ψ and γ are functions of x_1 and x_2, is a solution of Einstein's field equations $R_{ik}=0$ *only if*[5] ψ satisfies Laplace's equation,

$$\frac{\partial^2\psi}{\partial x_1^2}+\frac{1}{x_2}\frac{\partial}{\partial x_2}\left(x_2\frac{\partial\psi}{\partial x_2}\right)=0, \quad (21)$$

and γ satisfies the two-dimensional Poisson equation,

$$\frac{\partial^2\gamma}{\partial x_1^2}+\frac{\partial^2\gamma}{\partial x_2^2}=-\left[\left(\frac{\partial\psi}{\partial x_1}\right)^2+\left(\frac{\partial\psi}{\partial x_2}\right)^2\right]=-Q(x_1,x_2), \quad (22)$$

with the boundary conditions $\gamma(x_1,0)=0$ and $\gamma(x_1,\infty)=0$. In the case of an infinitely thin ring, the solution of (21) may be expressed in terms of the elliptic function $K(k)$. We have

$$\psi(x_1,x_2)=-\frac{Gm}{\pi\alpha}\frac{kK(k)}{(x_2/\alpha)^{\frac{1}{2}}}, \quad (23)$$

where

$$k=k(x_1,x_2)=[4\alpha x_2/(x_1^2+x_2^2+\alpha^2+2\alpha x_2)]^{\frac{1}{2}}, \quad (24)$$

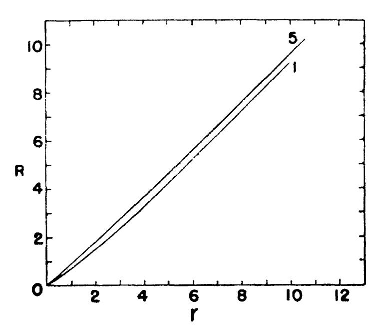

FIG. 2. The function $R(r)=(-g_{33}g_{44})^{\frac{1}{2}}$, where r is expressed in terms of the reduced wavelength $\lambdabar$.

[5] . . . but not necessarily if, for certain subsidiary equations must be satisfied.

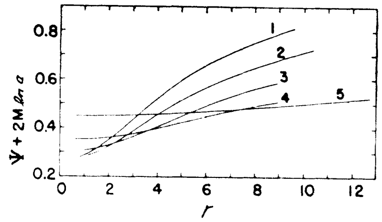

FIG. 3. The gravitational potential $\psi(r)=\frac{1}{2}\ln(-g_{44})$, where r is expressed in terms of the reduced wavelength λ.

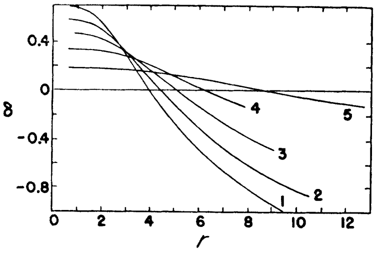

FIG. 4. The function $\delta(r)=\ln(1+D)$ $=\ln(-\Omega^2 g_{11}/g_{44})$, where r is expressed in terms of the reduced wavelength λ.

and

$$K(k)=\int_0^{\frac{1}{2}\pi}\frac{d\omega}{[1-k^2\sin^2\omega]^{\frac{1}{2}}}. \qquad (25)$$

The constant m is the total mass of the ring.

In turn, if one utilizes the theory of Green's functions, the solution of Eq. (22) may be written

$$\gamma=\int\int Q(y_1,y_2)\ln[(x_1-y_1)^2+(x_2-y_2)^2]^{\frac{1}{2}}dy_1dy_2. \qquad (26)$$

We conclude that finding the exterior gravitational field of a toroidal geon reduces to the evaluation of the integral (26). It is not clear now whether or not $\gamma(x_1,x_2)$ can be evaluated in terms of elliptic functions, but in any event the integral can be computed numerically.

5. SUMMARY AND CONCLUSIONS

The graphs contained in Sec. 3. constitute a solution of Einstein's and Maxwell's equations free from singularities. Physically the solution represents an intense electromagnetic field localized within a cylinder by the gravitational field which it creates. Drawing an analogy with Lamb's treatment of the problem of a rotating annulus in hydrodynamical theory, we suspect that it is permissible to extend this solution to include the case of the toroidal geon, a toroidal-shaped concentration of electromagnetic field energy. The field far from the torus and the field within the torus have been treated in this paper. The field in the intermediate region has yet to be calculated, but there is hope of an easy solution via the methods employed in hydrodynamics.

I would like to thank Professor John A. Wheeler for suggesting the problem of the toroidal geon, and for his willing advice and helpful criticism. Much of the work involved in setting up Eqs. (A1) through (A5) for numerical calculation was shared by Dr. John Gammel of the Los Alamos Scientific Laboratory of New Mexico. I am very grateful to Dr. Gammel for his assistance in coding the problem for machine calculation.

APPENDIX. NUMERICAL INTEGRATION OF THE FIELD EQUATIONS

The field equations (13) through (16) may be written in the form

$$B''+(e^{\delta}-1)e^{-2\lambda}B=0, \qquad (A1)$$

$$(l\lambda')'+\tfrac{1}{2}(B')^2+\tfrac{1}{2}e^{-2\lambda}B^2=0, \qquad (A2)$$

$$l''+\tfrac{1}{2}(B')^2-\tfrac{1}{2}(e^{\delta}-1)e^{-2\lambda}B^2=0, \qquad (A3)$$

$$\delta''+\lambda'\delta'+(2l)^{-1}[(B')^2+(1+3e^{\delta})e^{-2\lambda}B^2] +4l^{-1}l'\lambda'-(3/2l^2)(l')^2-2(\lambda')^2=0, \qquad (A4)$$

where primes signify differentiation with respect to r^*, $l=R^2e^{-2\psi}$, $\lambda=\ln R-2\psi$, and $\delta=\ln(1+D)$. By using Eq. (A1), Eq. (A3) may be readily integrated twice, yielding the simple relation

$$l=l_1r^*-\tfrac{1}{4}B^2 \qquad (A5)$$

between l and B. Here l_1 may be any positive constant, but since the simultaneous multiplication of l by a constant n and B by the constant $n^{\frac{1}{2}}$ leaves the field equations (A1) through (A4) unchanged in form, we can now work with any value of l_1 (say 1.75) in Eq. (A5); and then, after the entire numerical integration has been performed for this l_1 (1.75), by a simple transformation we may obtain the l and B functions corresponding to any other value of l_1. In particular, in order to end up with the asymptotic form (17) it will be necessary to make $l_1=2-4M$. Only with this value of l_1 will $R(r)$ tend to r for large values of r.

We shall first expand the functions B, l, λ, and δ in terms of power series about $r^*=0$:

$$\begin{aligned} B&=b_1r^*+b_2(r^*)^2+b_3(r^*)^3+\cdots,\\ l&=l_1r^*+l_2(r^*)^2+l_3(r^*)^3+\cdots,\\ \lambda&=\tfrac{1}{2}\ln(r^*L_0^2)+\lambda_1r^*+\lambda_2(r^*)^2+\cdots,\\ \delta&=\delta_0+\delta_2(r^*)^2+\cdots. \end{aligned} \qquad (A6)$$

Substituting (A6) into the field equations (A1), (A2), (A4), and (A5), one obtains the following relations among b_1, b_2, b_3, l_1, l_2, l_3, L_0, λ_1, λ_2, δ_0, and δ_2:

$$\begin{aligned} \delta_0&=\ln(1+D_0),\\ b_2&=-D_0b_1/2L_0^2,\\ l_2&=-\tfrac{1}{4}b_1^2,\\ l_3&=-D_0l_2/L_0^2,\\ \lambda_1&=3l_2/2l_1,\\ b_3&=-(2b_2/3)[D_0/4L_0^2+\lambda_1],\\ \lambda_2&=(\lambda_1/3)[(1-\tfrac{3}{2}D_0)/L_0^2-\lambda_1],\\ \delta_2&=(2\lambda_1/3L_0^2)(2+D_0). \end{aligned} \qquad (A7)$$

From the form of Eqs. (A7) it is clear that to specify completely any solution of the field equations which possesses the form (A6) in the neighborhood of $r^*=0$, only four constants l_1, L_0, D_0, and b_1 must be specified. Most assignments of values to these four constants, however, will correspond to solutions unacceptable on physical grounds—i.e., solutions in which the wave functions $B(r^*)$ tends to plus or minus infinity as $r^*\rightarrow\infty$. For a given set of values of l_1, L_0, and D_0 there will be some maximum value of b_1 beyond which the function $B(r^*)$ turns out to be everywhere positive (i.e., for $r^*>0$) and tends to plus infinity as $r^*\rightarrow\infty$. For slightly lower values of b_1 the function $B(r^*)$ will cross the r^*-axis just once, tending to minus infinity as $r^*\rightarrow\infty$. For the intermediate value of b_1 the curve $B(r^*)$ will approach the r^* axis exponentially as $r^*\rightarrow\infty$ but never actually will become negative. This value of b_1 is the "eigenvalue" which we seek, for it corresponds to the lowest mode of vibration of the system. In practice we found that it is not at all difficult to localize the eigenvalue b_1 by successive numerical integrations of the field equations.

It is not very difficult to derive difference equations which in the limit of infinitely small step size Δr^* reduce to Eqs. (A1), (A2), (A4), and (A5). For this purpose we define for any point r_0^* two other points, $r_{-1}^*=r_0^*-\Delta r^*$ and $r_1^*=r_0^*+\Delta r^*$, and at these points define

$$B_{-1}=B(r_{-1}^*),\quad B_0=B(r_0^*),\quad B_1=B(r_1^*),$$
$$\mathfrak{L}_{-1}=l(r_{-1}^*),\quad \mathfrak{L}_0=l(r_0^*),\quad \mathfrak{L}_1=l(r_1^*),$$
$$\Lambda_{-1}=\lambda(r_{-1}^*),\quad \Lambda_0=\lambda(r_0^*),\quad \Lambda_1=\lambda(r_1^*),$$
$$\Delta_{-1}=\delta(r_{-1}^*),\quad \Delta_0=\delta(r_0^*),\quad \Delta_1=\delta(r_1^*).$$

In terms of these quantities the difference equations may be written

$$B_1=2B_0-B_{-1}-(\Delta r^*)^2(e^{\Delta_0}-1)e^{-2\Lambda_0}B_0,$$
$$\mathfrak{L}_1=l_1r_1^*-\tfrac{1}{4}B_1^2,$$
$$\Lambda_1=\frac{8\mathfrak{L}_1\Lambda_0-\Lambda_{-1}(4\mathfrak{L}_0-\mathfrak{L}_1+\mathfrak{L}_{-1})-\frac{1}{2}(B_1-B_{-1})^2-2(\Delta r^*)^2e^{-2\Lambda_0}B_0^2}{4\mathfrak{L}_0+\mathfrak{L}_1-\mathfrak{L}_{-1}},$$
$$\Delta_1=\frac{8\Delta_0-\Delta_{-1}(4-\Lambda_1+\Lambda_{-1})-4H}{4+\Lambda_1-\Lambda_{-1}},$$
$$H=\frac{1}{2\mathfrak{L}_0}\left\{\frac{(B_1-B_{-1})^2}{4}+(1+3e^{\Delta_0})e^{-2\Lambda_0}B_0^2(\Delta r^*)^2\right\}-(E+F)(\Delta r^*/r_0^*+\tfrac{1}{2}E+\tfrac{3}{2}F),\qquad \text{(A8)}$$
$$E=(\tfrac{1}{2}\ln r_1^*-\Lambda_1)-(\tfrac{1}{2}\ln r_{-1}^*-\Lambda_{-1}),$$
$$F=\tfrac{1}{2}\left\{\frac{\mathfrak{L}_1r_0^*}{\mathfrak{L}_0r_1^*}-\frac{\mathfrak{L}_{-1}r_0^*}{\mathfrak{L}_0r_{-1}^*}\right\}.$$

TABLE II. The values of M, a, and Ω for the five sets of power series coefficients chosen.

D_0	0.2	0.4	0.6	0.8	1.0
L_0	1	1	1	1	1
l_1	1.75	1.75	1.75	1.75	1.75
b_1	0.0765	0.210	0.3674	0.5352	0.7092
M	0.042	0.080	0.109	0.132	0.15075
a	0.172	0.548	0.694	0.726	0.717
Ω	1.85	2.02	2.15	2.30	2.44

The rather complicated expression for Δ_1 was adopted after it was discovered that the simpler expressions that were first employed did not give good results.

Notice that Eqs. (A8) allow one to calculate B, l, λ, and δ at any point r^* if the functions are already known at the two previous points $r^*-\Delta r^*$ and $r^*-2\Delta r^*$. Once this process is begun, one can keep iterating in this manner until one reaches as large values of r^* as is desired. The only problem lies in starting the iteration, and for this purpose we use the power series expansion (A6) valid for small r^*. In practice we found that letting $r_{-1}^*=0.1$ and $\Delta r^*=0.01$ initially and increasing the step size as the iteration proceeded gave quite satisfactory results. The values of B, l, λ, and δ were calculated at the points $r^*=0.10$ and $r^*=0.11$ by means of the power series. Then the values of B, l, λ, and δ at $r^*=0.12$ were calculated by means of the difference equations (A8) as well as by the power series. If the two sets of values agreed, we continued to apply (A8) until large r^* values were reached and the character of the solution had been ascertained. Simultaneously we solved the equation $dr^*/dr=e^{\lambda}$ to discover the relationship between r and r^*.

The values of M, a, and Ω were computed from the functions $l(r^*)$, $\lambda(r^*)$, and $\delta(r^*)$ at large values of r^*, where $B(r^*)$ is essentially zero and where the line element is of the form

$$ds^2=-(r/a)^{8M^2-4M}(dz^2+dr^2)-r^2(r/a)^{-4M}d\phi^2+(r/a)^{4M}dT$$

(see Table II). Of course it was necessary to multiply $l(r^*)$ by $n=(2-4M)/1.75$ and $B(r^*)$ by $n^{\frac{1}{2}}$ in order to compensate for the fact that we arbitrarily set $l_1=1.75$ in all cases while we should have set $l_1=2-4M$. Finally, the functions $R(r)$, $\psi(r)$, and $\delta(r)$ were computed and plotted. We observed that the peak value of $B(r)$ did not vary considerably during the last stages of the eigenvalue search. Hence we believe that only the tail of the function $B(r)$ is inaccurately portrayed in the graphs.