Variational Calculations in Geon Theory

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Idealized spherical "geons," or gravitational-electromagnetic entities, of the type studied by Wheeler utilizing an electronic calculator, are here studied by using a simple adaptation of the Ritz variational principle. By using the simplest of trial functions, most of the relevant magnitudes are calculated with considerable accuracy.

INTRODUCTION

WHEELER has demonstrated the existence of certain nonsingular solutions of the coupled equations of classical relativity theory and electromagnetism.[1] Physically his solution corresponds to a spherical shell of light held together by its own gravitational field. Subsequently, the author has shown the existence of another set of solutions corresponding to beams of light highly concentrated by their own gravitational field.[2] Collectively these solutions are now called "geons" and they form the first nontrivial nonsingular solutions of the equations of classical general relativity.

In this paper we shall show how a great deal of information concerning geons can be ascertained without recourse to electronic digital computation. For this purpose we now apply the Ritz variational principle to the problem considered by Wheeler.[1]

IDEALIZED SPHERICAL GEON

In our discussion of spherical geons we shall follow Wheeler by using a line element of the form

$$ds^2 = +e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - e^{\nu(r)}dT^2. \tag{1}$$

The action of the combined gravitational and electromagnetic fields may be written as a single integral,

$$I = \int_0^\infty \left\{ \frac{-1}{16\pi c}\langle F_{\alpha\beta}F^{\alpha\beta}\rangle + \frac{c^3}{16\pi G}R \right\} e^{\frac{1}{2}(\lambda+\nu)}r^2 dr, \tag{2}$$

where the brackets $\langle\ \ \rangle$ indicate a suitable superposition of various possible modes of electromagnetic radiation of circular frequency Ω/c and an averaging over θ, ϕ, and T so that the Lagrangian is just a function of r.

In the case of the idealized spherical geon, $\langle F_{\alpha\beta}F^{\alpha\beta}\rangle$ may be easily expressed in terms of the expressions (31b) which Wheeler introduced in his geon paper:

$$\langle F_{\alpha\beta}F^{\alpha\beta}\rangle = 4\pi\{[r\phi]+[\theta\phi]+[T\phi]\}.$$

Furthermore, the curvature scalar,

$$R = \nu'' - \tfrac{1}{2}\lambda'\nu' + \tfrac{1}{2}\nu'^2 + \frac{2(\nu'-\lambda')}{r} + \frac{2(1-e^\lambda)}{r^2},$$

where primes indicate d/dr, may be expressed in the following form by letting $e^{\lambda+\nu} = Q^2$ and $e^{-\lambda} = 1 - 2M(r)/r$:

$$RQr^2 = 4M'Q + 2M''Qr + 2MQ' + 6Q'M'r - 2Q''r^2 - 4Q'r + 4MQ''r.$$

However, if $M(r)$ and $Q(r)$ tend toward constants as r goes to infinity, the divergence terms in RQr^2 integrate to zero. If one neglects a factor $+(c^3/2G\Omega)$, Eq. (2) may be written as follows:

$$I = \int_0^\infty \left\{ L'Q - LQ' - Q(1-2L/\rho)f'^2 + \frac{f^2}{Q(1-2L/\rho)} - Q\left(\frac{l^*}{\rho}\right)^2 f^2 \right\} d\rho, \tag{3}$$

where $\rho = \Omega r$, $L = \Omega M$, $f = (GNl^*/4c^4)^{\frac{1}{2}}\Omega F$, and primes indicate $d/d\rho$. F corresponds exactly to the function $R(r)$ defined in Wheeler's article on geons.[1] The expression (3) is valid when l, the order of spherical harmonic involved, is large compared to unity. When this is true, however, it is possible to expand the functions $L(\rho)$, $Q(\rho)$, $f(\rho)$, and ρ in a power series in $(l^*)^{-\frac{1}{3}}$. To see how this is done, let us first introduce a new function $J(\rho)$ defined by the equation

$$JK = 1 - (Ql^*/\rho)^2(1-2L/\rho),$$

where $K = 1/Q$. Then

$$I = \int_0^\infty \left\{ \frac{d}{d\rho}\left[\tfrac{1}{2}K^{-1}\rho - \tfrac{1}{2}K\rho(1-JK)\left(\frac{\rho}{l^*}\right)^2 \right] + \left[K^{-2} - (1-JK)\left(\frac{\rho}{l^*}\right)^2 \right]\rho\frac{dK}{d\rho} - K(1-JK)\left(\frac{\rho}{l^*}\right)^2 f'^2 + J(1-JK)^{-1}\left(\frac{l^*}{\rho}\right)^2 f^2 \right\} d\rho.$$

Now expand J, ρ, and f in a power series and consider only the first order terms in the action:

$$\begin{aligned} J &= (l^*)^{-\frac{2}{3}}j + \cdots, \\ \rho &= l^* + (l^*)^{\frac{1}{3}}r + \cdots \quad \text{(a new variable "}r\text{")}, \\ f &= (l^*)^{\frac{1}{3}}\varphi + \cdots, \\ I &= (-\tfrac{1}{2}jK^2 + \tfrac{3}{2}Kr - \tfrac{1}{2}K^{-1}r)\big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\{(jK - 3r + K^{-2}r)K' - K\varphi'^2 + j\varphi^2\}dr. \end{aligned} \tag{4}$$

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[1] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[2] F. J. Ernst, Jr., following paper [Phys. Rev. **105**, 1665 (1957)].

In order that the action be finite, it is necessary that jK have a certain asymptotic behavior, namely:

$$\lim_{r\to\pm\infty}\left[-\left(\frac{jK}{r}\right)+3-K^{-2}\right]=0.$$

Letting $\lim_{r\to\infty}K(r)=K_0$ and $\lim_{r\to-\infty}K(r)=1$, the asymptotic formulas become

$$\lim_{r\to\infty}\left(\frac{jK}{r}\right)=3-K_0^{-2},\quad \lim_{r\to-\infty}\left(\frac{jK}{r}\right)=2.$$

Assuming these asymptotic forms for the function jK, we may write the action integral in the simple form

$$I=\int_{-\infty}^{\infty}[(jK-3r+K^{-2}r)K'-K\varphi'^2+j\varphi^2]dr.$$

From this equation it is possible to show the significance of Wheeler's result that $K_0\approx 0.33$. In fact we can show that $K_0=\frac{1}{3}$ exactly. This means that $-g_{44}$ has throughout the interior of the geon the constant value $\frac{1}{9}$, and g_{11} has the value 1.

Assume that the electromagnetic field reaches a maximum at $r=r_0$ and falls sufficiently rapidly as $|r-r_0|$ increases so that the integrand of the action integral virtually vanishes except in a small region about $r=r_0$. A careful investigation will reveal that we have already assumed just about the same thing in writing minus infinity for the lower limit of the action integral anyway.

However, the field equations, which may be easily deduced from the action integral, in no way involve the independent variable r. It follows from this that if $K(\eta)$, $j(\eta)$, $\varphi(\eta)$ is a solution of the problem, so is $K(r_0+\eta)$, $j(r_0+\eta)$, $\varphi(r_0+\eta)$, where r_0 is any constant and may be arbitrarily designated "the position of the maximum electromagnetic field strength." In view of this property of the solution of the geon problem, we may require that the action be independent of the constant r_0. If one sets $r=r_0+\eta$, the action integral becomes

$$I=(4-3K_0-K_0^{-1})r_0+\int_{-\infty}^{\infty}\{(jK-3\eta+K^{-2}\eta)K'-K\varphi'^2+j\varphi^2\}d\eta,$$

where the primes now indicate $d/d\eta$. However, for the action to be independent of r_0, the position of the maximum electromagnetic field strength, it follows that $4-3K_0-K_0^{-1}=0$. This equation has the two roots, $K_0=1$ and $K_0=\frac{1}{3}$. The former corresponds to empty space; so, it is the latter value which is relevant to the present problem.

Since $\sqrt{g_{44}}\sim Q\sim K^{-1}$, we see that a clock at the center of a spherical geon will tick at exactly $\frac{1}{3}$ the rate of a clock far from the geon. Similarly Wheeler's Eqs. (57) and (58) may be written exactly:

$$\text{Radius of geon:}\quad r=\tfrac{1}{3}l^*/\Omega, \tag{W57}$$

$$\text{Mass of geon:}\quad M=(4/27)c^2l^*/G\Omega. \tag{W58}$$

Corresponding to the action,

$$I=\int_{-\infty}^{\infty}\{(jK-3\eta+K^{-2}\eta)K'-K\varphi'^2+j\varphi^2\}d\eta, \tag{5}$$

the Euler-Lagrange equations,

$$d^2\varphi/dx^2+j(x)K(x)\varphi(x)=0, \tag{W48}$$

$$dK/dx+\varphi^2=0, \tag{W49}$$

and

$$dj/dx=3-[1+(d\varphi/dx)^2]/K^2, \tag{W50}$$

are precisely the field equations which are analyzed by means of an I.B.M. calculator by Professor Wheeler. In these equations dx stands for $K^{-1}d\eta$. Notice that Eq. (W48) is very similar to the Schrödinger equation of quantum mechanics for a potential well $v=-jK$. Where v is positive the solution is exponential in character, while where v is negative the solution is sinusoidal in character. In Fig. 1 appear the results of Wheeler's calculations: the φ function, $-jK$, and K. Solutions $\varphi(x)$ with more maxima are also conceivable.

We shall now show in detail how it is possible to find the approximate maximum value of φ in the active region, the width of the active region, and the depth of the potential well without solving the field equations, by a simple application of the Ritz variational method.

As a trial solution of the problem, we shall assume that the action integrand vanishes except within a

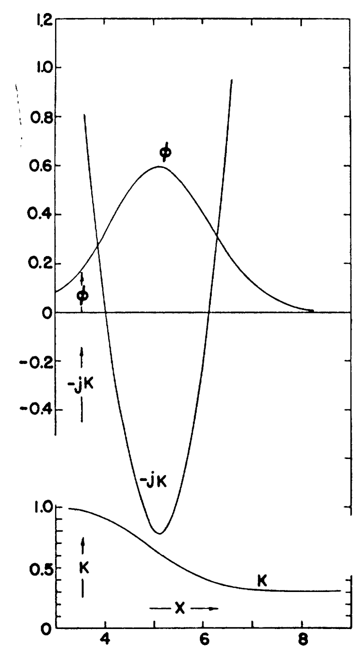

FIG. 1. The functions $\varphi(x)$, $-jK(x)=-D(x)$ and $K(x)$ according to Wheeler's machine calculation.

TABLE I. Comparison of results of very simple variational calculation with those of Wheeler's electronic machine calculation.

Results	Quantity	Significance	Ritz principle	Comparable values from Fig. 1
	α	$\frac{1}{2}$(well width)	1.14	1.33[a]
	D	well depth	0.76	0.90
	φ_0	electromagnetic field amplitude	0.59	0.59

[a] The figure $\alpha=1.33$ is only approximate because of the haziness of the edges of the active region and because of the difference of Professor Wheeler's abscissa x from that which is used in the present paper.

region $-\alpha\leqslant\eta\leqslant\alpha$ and that inside this region K varies linearly while $v=-jK$ is a constant, $-D$, and φ is sinusoidal. The adjustable parameters will then be α, D, and the maximum value of φ, which we shall call φ_0.

Within the range $-\alpha\leqslant\eta\leqslant\alpha$, we shall let

$$K(\eta)=\tfrac{1}{3}[2-(\eta/\alpha)],$$
$$jK(\eta)=D,\ \text{a constant}, \tag{6}$$
$$\varphi(\eta)=\varphi_0\cos(\pi\eta/2\alpha).$$

The calculation of the action (5) is straightforward and leads to the result

$$I=-\tfrac{2}{3}D+[3(\ln3)-4]\alpha+\alpha\varphi_0^2[\tfrac{3}{2}D(\ln3)-(\pi^2/6\alpha^2)]. \tag{7}$$

Extremizing the action I with respect to the constants D, φ_0, and α results in the following simple equations:

$$\alpha\varphi_0^2=(4/9)(\ln3)^{-1},$$
$$D\alpha^2=(\pi^2/9)(\ln3)^{-1},$$
$$(\varphi_0/\alpha)^2=(3/\pi^2)[4-3(\ln3)].$$

These three equations may be solved easily, so that it is possible to compare the results of our very simple Ritz variational calculation with the complete solution given in Fig. 1. (See Table I.) Furthermore, by the present method we have obtained a little better insight into why the constant K_0 has the value $\frac{1}{3}$.

Besides yielding φ_0 very well, the variational calculation using a square well diverges in the expected direction with regard to the value of D.

We feel that this example clearly shows the usefulness of variational calculations in finding solutions of the coupled equations of gravitation and electromagnetism. Extremely simple trial functions have been used intentionally to illustrate the effectiveness of this type of calculation.

This problem was suggested by Professor Wheeler in connection with my A.B. thesis at Princeton University, May, 1955 (unpublished).