Radiative Corrections to Pair Annihilation

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The e^6 corrections to the probability of direct two-photon pair annihilation are obtained by applying the substitution law to the Compton scattering radiative corrections. General results are given, as well as low- and high-energy limits. Divergence of the corrected matrix element for small relative velocity, due to failure of the Born approximation, is removed by a more accurate treatment of Coulomb effects based on the Sommerfeld factor. The results are applied to the singlet ground state of positronium, yielding an increase in the lifetime of 0.59%.

I. INTRODUCTION

THE e^6 corrections to the two-quantum annihilation of the electron-positron system can be obtained by appropriate application of the "substitution law"[1] to the work on Compton scattering of Brown and Feynman.[2] Features requiring a certain care are the infrared divergence, the inclusion of Coulomb interaction for direct annihilation at low energy and, for positronium, the extraction of the singlet cross section from a cross section averaged over spins.

A rigorous treatment of the electron-positron system requires a recognition of its two-body relativistic character. Such an approach has been successfully applied to a discussion of the energy levels of positronium.[3] We shall find that corrections of accuracy $e^2/\hbar c$ to the lifetime of singlet positronium can be consistently obtained by use of the one-particle kernel. This is because the Born approximation is satisfactory at high energies and a nonrelativistic center-of-mass treatment suffices, to our order of approximation, at low energies.

II. DIRECT ANNIHILATION

To obtain the absolute square of the matrix element for two-photon annihilation, averaged over spins and summed over polarizations, we apply the substitution law to Eqs. (27) through (30) of BF by changing the signs of the four-momenta of the incoming photon and the final electron and the sign of the trace. This is conveniently accomplished by defining anew the invariants κ and τ used by BF. Let (with $\hbar=c=1$)

$$m^2\kappa=2p_1\cdot q_1=2p_2\cdot q_2,\qquad m^2\tau=2p_1\cdot q_2=2p_2\cdot q_1, \tag{1}$$

where now p_1 and p_2 are the four-momenta of the electron and positron, respectively, and q_1 and q_2 are the four-momenta of the annihilation quanta.

The result of BF is expressed in terms of transcendental functions of these invariants. In particular, they introduce

$$h(y)=y^{-1}\int_0^y udu\coth u, \tag{2}$$

where

$$4\sinh^2 y=-(\kappa+\tau). \tag{2a}$$

In the center-of-momentum system we have

$$\sinh^2 y=-E^2/m^2,$$

where E is the energy of the electron or positron. Thus the parameter y becomes complex for the annihilation problem and $h(y)$ requires further interpretation. We define the real variable x by

$$4\cosh^2 x=\kappa+\tau, \tag{3}$$

and, accordingly,

$$y=x-i\pi/2. \tag{3a}$$

The signs in (3a) are fixed in accordance with the Feynman prescription for defining the hole theory propagators (namely, infinitesimal negative imaginary parts are added to the electron and photon masses) but the absolute square of the matrix element is independent of this sign choice. The path of integration in (2) is now a straight line from the origin to the point $y=x-i\pi/2$, or any equivalent path. All the relevant momentum integrals have been recalculated directly for the annihilation case, yielding agreement with substitution (3a) and the specified integration path. An example is given in Appendix A.

Applying this rule to $h(y)$ with the path shown in Fig. 1(a), we have

$$(x-\tfrac{1}{2}i\pi)h(x-\tfrac{1}{2}i\pi)=i\int_0^{-\frac{1}{2}i\pi}vdv\coth v-\tfrac{1}{2}i\pi\int_0^x du\tanh u+\int_0^x udu\tanh u. \tag{4}$$

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[1] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Cambridge, 1955).

[2] L. M. Brown and R. P. Feynman, Phys. Rev. **85**, 231 (1952) (referred to hereafter as BF). See also M. R. Schafroth, Helv. Phys. Acta **22**, 50 (1949) and **23**, 542 (1950).

[3] R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952).

After integrating the first term by parts, we get

$$(x-\tfrac{1}{2}i\pi)h(x-\tfrac{1}{2}i\pi) = -\tfrac{1}{2}i\pi\ln(2\cosh x)+\int_0^x udu\tanh u. \quad (5)$$

To evaluate $2yh(2y)$ we choose a similar path, shown in Fig. 1(b), but there is now a pole at $-i\pi$ which yields a term equal to one fourth the residue. The result is

$$(2x-i\pi)h(2x-i\pi)=-\tfrac{1}{2}\pi^2+\int_0^{2x} udu\coth u - i\pi\ln(2\sinh 2x). \quad (6)$$

Making the necessary substitutions, we obtain the result given below for the differential cross section for direct annihilation, to order e^6, averaged over spins and summed over polarizations:

$$d\sigma=d\sigma^0\{1-(e^2/\pi)[2(1-2x\coth 2x)\ln(\lambda_{\min}/m) -4x\coth 2x(2g(x)-h(2x)+\pi^2/4x) +\mathcal{G}(\kappa,\tau)+\mathcal{G}(\tau,\kappa)]\}, \quad (7)$$

where $d\sigma^0$ is the cross section in lowest Born approximation, $\lambda_{\min}$ is the usual fictitious photon mass, and

$$\begin{aligned}\mathcal{G}(\kappa,\tau)U=&[4(\kappa\tau)^{-1}\sinh 2x(1+2\cosh^2 x) +2x\tanh x]g(x)+\ln|\kappa|\Bigg\{4x\coth 2x\\ &\times\left[\frac{4}{\kappa^2}-\frac{1}{\kappa}-\frac{\tau}{2\kappa}-\frac{\kappa}{\tau}-1-4(\kappa\tau)^{-1}\sinh^2 x\right]\\ &-\frac{2x}{\sinh 2x}\left(\frac{\kappa-6}{\tau}\right)+\frac{3\tau}{2\kappa^2}(1+\kappa)+\frac{3}{\tau}+1-\frac{7}{\kappa\tau}+\frac{8}{\kappa}-\frac{8}{\kappa^2}\\ &-\frac{\tau^2-2\kappa+\kappa^2\tau}{2\kappa^2\tau(\kappa-1)}-\frac{2\kappa^2+\tau}{2\tau(\kappa-1)^2}\Bigg\}\\ &+\frac{\frac{1}{4}\pi^2-x^2}{\cosh^2 x}\left\{\frac{2}{\kappa}-\frac{7\kappa}{4}-\frac{3\kappa^2}{4\tau}\right\}-4x\coth x\left\{\frac{1}{2}-\frac{1}{\kappa}\right\}\\ &+4\left(\frac{1}{\kappa}+\frac{1}{\tau}\right)^2-\frac{12}{\kappa}-\frac{3\kappa}{2\tau}-\frac{2\kappa}{\tau^2}+\frac{1}{\kappa-1}\left(\frac{\kappa}{\tau}+\frac{1}{2}\right)\\ &+G_0(\kappa)\left\{\frac{\kappa^2}{\tau}+\frac{\tau}{\kappa^2}+\frac{\kappa}{\tau}+\kappa+\frac{\tau}{2}+\frac{2}{\kappa}-\frac{3}{\tau}-1\right\},\end{aligned} \quad (8)$$

with

$$U=\left(\frac{\kappa}{\tau}+\frac{\tau}{\kappa}\right)+4\left(\frac{1}{\kappa}+\frac{1}{\tau}\right)-4\left(\frac{1}{\kappa}+\frac{1}{\tau}\right)^2, \quad (8a)$$

$$g(x)=\frac{1}{x}\int_0^x udu\tanh u, \quad (8b)$$

$$h(x)=\frac{1}{x}\int_0^x udu\coth u, \quad (8c)$$

$$G_0(\kappa)=\frac{2}{\kappa}\int_1^{1-\kappa}\ln|1-u|du/u. \quad (8d)$$

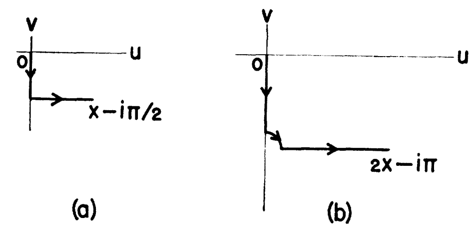

FIG. 1. Paths of integration in the complex plane (a) for $h(y)$ and (b) for $h(2y)$.

To the expression (7) must be added the cross section for direct three-quantum annihilation, integrated over the direction and energy of one photon, whose energy is assumed to be small. This results in the replacement of $\lambda_{\min}$ by the experimental energy resolution as explained, for example, in BF. Jauch and Rohrlich[4] have shown, following Bloch and Nordsieck,[5] that the combination of soft-photon emission and soft-photon radiative corrections eliminates completely the infrared divergence in a scattering process. However, an explicit calculation is necessary in order to obtain not merely the expansion term logarithmically dependent on k_m, the maximum excluded photon energy, but also the term independent of k_m. This "constant" term is an essential part of the e^6 corrections.

The calculation, which is similar to that in BF for the double Compton scattering, gives for three-quantum annihilation (note that $\tanh 2x=v$, the positron velocity in the laboratory system)

$$d\sigma_3=(e^2/\pi)d\sigma^0\{2(2x\coth 2x-1)\ln[(2k_m/\lambda_{\min})-\tfrac{1}{2}] +4x\coth 2x[1-h(2x)]\}. \quad (9)$$

This expression, which holds for $k_m\ll m$ in the reference frame in which the electron is at rest, is exactly that in BF, Eq. (39), with y replaced by x and $d\sigma_{\text{K.N.}}$ replaced by $d\sigma^0$. However, the substitution law cannot be applied here since $k_m\ll m$ is not a covariant restriction.

Adding Eq. (9) to Eq. (7) replaces

$$2(1-2x\coth 2x)[\ln(\lambda_{\min}/m)] -4x\coth 2x[\pi^2/4x+2g(x)-h(2x)] \quad (10)$$

by

$$2(1-2x\coth 2x)[\ln(2k_m/m)-\tfrac{1}{2}] -4x\coth 2x[\pi^2/4x+2g(x)-2h(2x)+1]. \quad (11)$$

The resulting cross section has the validity of the Born approximation, i.e., it is applicable providing $\xi=e^2/v\ll 1$. This restriction will be removed in the following section.

III. LOW ENERGY LIMIT

In the nonrelativistic limit, with the electron at rest and the positron velocity $v=\tanh 2x\ll 1$, we get

$$x\approx v/2,\quad U\approx 2,\qquad \mathcal{G}(\kappa,\tau)+\mathcal{G}(\tau,\kappa)\approx 3-\tfrac{1}{4}\pi^2, \quad (12)$$

[4] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta **27**, 613 (1954).
[5] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

and the differential cross section becomes

$$d\sigma_{\mathrm{NR}}=d\sigma_{\mathrm{NR}}{}^0[(1+\pi e^2/v)-(e^2/\pi)(5-\pi^2/4)]. \quad (13)$$

The infrared-divergent part and the compensating three-quantum cross section vanish in this limit and do not concern us here, but the result (13) diverges as v tends toward zero. This clearly exhibits the failure of the Born approximation in that part of the radiative corrections which arises from Coulomb interaction.

It is well known that a satisfactory approximate method of modifying a plane-wave cross section involving charged particles, to include the effects of Coulomb interaction at low energies, is to multiply by the "Sommerfeld factor." This factor is the absolute square of the Coulomb continuum wave function, evaluated for contact, and is for this case

$$|\psi(0)|^2=2\pi\xi(1-e^{-2\pi\xi})^{-1}, \quad \xi=e^2/v. \quad (14)$$

The differential cross section, corrected for Coulomb effects alone, is then

$$d\sigma_C{}^0=d\sigma^0\cdot 2\pi\xi(1-e^{-2\pi\xi})^{-1}, \quad (15)$$

which for $\xi\ll 1$ can be expanded as

$$d\sigma_C{}^0\approx d\sigma^0(1+\pi\xi), \quad (16)$$

identical with (13) within terms of order of the fine structure constant.[6]

We can obtain, therefore, a smooth result, valid at all energies to the desired accuracy (that is, of accuracy e^4), by subtracting $(\pi e^2/v)d\sigma^0$ from $d\sigma$ [Eq. (7) with the substitution of (11) for (10)] and multiplying the remainder by the Sommerfeld factor. This yields for $d\sigma_C$, the cross section corrected for Coulomb and radiative effects[7]:

$$\begin{aligned} d\sigma_C=d\sigma_C{}^0\{1-(e^2/\pi)[&2(1-2x\coth 2x)(\ln(2k_m/m)-\tfrac{1}{2}) \\ &-4x\coth 2x(1+2g(x)-2h(2x)) \\ &+\mathcal{G}(\kappa,\tau)+\mathcal{G}(\tau,\kappa)]\}. \end{aligned} \quad (17)$$

In the nonrelativistic limit we get

$$(d\sigma_C)_{\mathrm{NR}}=(d\sigma_C{}^0)_{\mathrm{NR}}[1-(e^2/\pi)(5-\tfrac{1}{4}\pi^2)]. \quad (18)$$

The NR cross section is thus reduced by 0.59%.

IV. LIFETIME OF POSITRONIUM GROUND STATE

We show now that the 0.59% decrease of the nonrelativistic direct annihilation cross section implies an increase of the same magnitude in the lifetime of the singlet ground state of positronium. Justification of the procedure here outlined will be found in Appendix B.

For a bound state, the matrix element for annihilation is calculated by integrating the appropriate (in this case the singlet) plane-wave annihilation matrix element, including the e^6 radiative corrections, over the momentum distribution of the state, i.e.,

$$\mathfrak{M}=\int \varphi_s(\mathbf{p})M_s(\mathbf{p})d\mathbf{p}. \quad (19)$$

The momentum distribution $\varphi_s(\mathbf{p})$ is strongly peaked at the Bohr momentum of positronium, $e^2m/2$. To obtain corrections of relative order e^2, therefore, it is sufficient to expand the plane-wave singlet matrix element $M_s(\mathbf{p})$ in powers of the momentum, retaining only the constant part, since terms linear in the momentum are not present. The integral over the momentum distribution then yields the value $\psi_s(0)$ of the singlet space wave function at contact. Taking the absolute square, we obtain a probability of annihilation proportional to

$$|\mathfrak{M}|^2=|\psi_s(0)|^2\cdot|M_s(0)|^2. \quad (20)$$

It is shown in Appendix B that, to the desired accuracy, $\psi_s(\mathbf{r})$ is identical with the Schrödinger wave function. $|M_s(0)|^2$ is four times the spin-averaged plane-wave result obtained in the previous section. This follows from the fact that the spin average can be expressed as an average over singlet and triplet states, but the triplet contribution vanishes[8] (including its radiative corrections) in the nonrelativistic limit. The arguments we have presented are well-known as applied to the lowest order result, but it is interesting that they apply also to the more general case being here considered. The usual calculation for the lifetime then leads to the result given in the first paragraph of this section.

V. NUMERICAL RESULTS AND EXTREME RELATIVISTIC LIMITS

For further examination of the radiative corrections, we define

$$\begin{aligned} X=\mathcal{G}(\kappa,\tau)+\mathcal{G}(\tau,\kappa)-1 \\ -4x\coth 2x[\tfrac{1}{2}+2g(x)-2h(2x)], \end{aligned} \quad (21)$$

so that the expression (17) for the cross section with radiative corrections, including the low-frequency part of the three-quantum annihilation, can be written as

$$\begin{aligned} d\sigma_C=d\sigma_C{}^0\{1-(e^2/\pi)[&2(1-2x\coth 2x) \\ &\times\ln(2k_m/m)+X]\}. \end{aligned} \quad (22)$$

Figure 2 illustrates the behavior of X for two cases of interest. The solid curves are calculated from the exact formula and the dotted curves from simpler formulas for the extreme relativistic (ER) limit.

[6] We wish to thank Professor Y. Nambu for a valuable conversation on this point.

[7] Coulomb effects are included by this method to all orders in e^2, though only, of course, approximately.

[8] The lowest order triplet term, being squared, vanishes as v^2; the interference term vanishes at least as e^2v. Though it is only in the limit of zero velocity that linear combinations of plane wave spinors become spin wave functions for the triplet state, Yang has shown that two-quantum annihilation from the triplet ground state of positronium is absolutely forbidden. This follows also in a simple way from the charge conjugation properties of the system. See C. N. Yang, Phys. Rev. 77, 242 (1950) and L. Wolfenstein and D. G. Ravenhall, Phys. Rev. 88, 297 (1953).

Case 1 corresponds to the emission of the two annihilation quanta 180° apart in the laboratory system. For extremely high energies this is the most probable case. The ER limit is defined by $\kappa \gg \tau \approx 1$. The ER limit of X is, if one drops terms of order $\kappa^{-1}(\ln\kappa)^2$,

$$X_1 = (\ln\kappa)^2 - \ln\kappa + 3 + \pi^2/3. \tag{23}$$

Case 2 is defined by $\kappa = \tau$, corresponding to the emission of the two photons at 90° to the positron momentum in the c.m. system. For moderate energies this is the most probable case. In the ER limit, given by $\kappa = \tau \gg 1$, we have for X, to order $1/\kappa$,

$$X_2 = \tfrac{3}{2}(\ln\kappa)^2 - (5/2)\ln 2\kappa + 5.97 + (2\kappa)^{-1}[18.5(\ln 2\kappa)^2 - 67.25\ln 2\kappa + 41.2]. \tag{24}$$

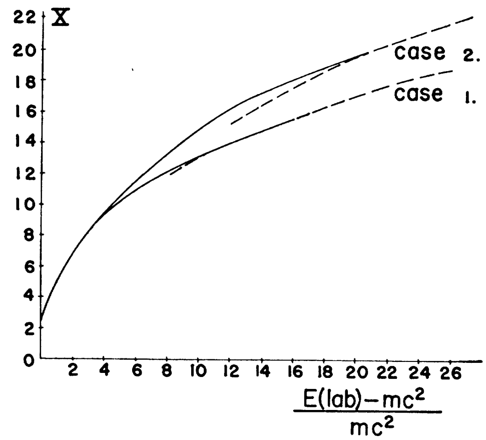

FIG. 2. X [Eq. (21)] as a function of the kinetic energy of the positron in the laboratory frame (electron at rest). Case 1: annihilation quanta 180° apart in the laboratory frame. Case 2: annihilation quanta at 90° to the positron momentum in the center-of-mass frame.

APPENDIX A

As an example of the methods used in the evaluation of the integrals for direct annihilation, we recalculate the integral H_0 defined by BF as (writing λ for $\lambda_{\min}$)

$$H_0 = 8i\int d^4k/(1)(2)(0)$$

$$\equiv 8i\int d^4k(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}(k^2-\lambda^2)^{-1}. \tag{A1}$$

This integral and all others which, in the annihilation case, contribute a term proportional to $1/v$ for small velocity v arise from the Feynman diagram, designated J by BF, which contains a virtual photon connecting the initial electron and positron lines. Diagrams of this type may be expected to contain contributions from the Coulomb interaction.

The integral H_0 is evaluated by Feynman[9]; however, his method[10] leads to the complex limit in the integrals $h(y)$ and $h(2y)$ in the annihilation problem, and requires clarification. We start from reference 7, Eq. (23a), which is

$$H_0 = \int_0^1 (dz/p_z^2)\ln(p_z^2/\lambda^2), \tag{A2}$$

with

$$p_z = zp_1 + (1-z)p_2,$$

from which (with momenta in units of m)

$$p_z^2 = Q^2(u^2-a^2) - i\delta, \tag{A3}$$

with

$$Q^2 = \tfrac{1}{4}(\kappa+\tau) = \cosh^2 x,$$
$$a = \tanh x, \quad u = 2z-1.$$

The term $-i\delta$ in (A3) has been added in accordance with the Feynman prescription that all masses are to be given an additional infinitesimal negative imaginary part. When one uses u as a new integration variable, and uses the symmetry of the integrand to integrate only over positive u, the pole at $u=a$ is displaced above the real axis and the logarithm in (A2) becomes

$$\ln[Q^2(u^2-a^2)/\lambda^2] \quad \text{when} \quad u>a,$$
$$\ln[Q^2(a^2-u^2)/\lambda^2] - i\pi \quad \text{when} \quad u<a.$$

Thus

$$H_0 = \int_0^1 du[Q^2(u^2-a^2)-i\delta]^{-1}\ln|Q^2(u^2-a^2)/\lambda^2| - i\pi\int_0^a du[Q^2(u^2-a^2)-i\delta]^{-1}. \tag{A4}$$

To evaluate the first term of (A4), we break the integration region into three parts, denoting the corresponding integrals by $H_0^{(1)}$, $H_0^{(2)}$, and $H_0^{(3)}$, respectively:

$$\begin{aligned}&(1)\ 0 \le u < a-\epsilon,\\ &(2)\ a-\epsilon \le u < a+\epsilon,\\ &(3)\ a+\epsilon \le u \le 1.\end{aligned} \tag{A5}$$

In $H_0^{(1)}$ we put $u/a = \tanh v$ and in $H_0^{(3)}$ we put $u/a = \coth v$, obtaining (as $\epsilon \to 0$)

$$H_0^{(1)} + H_0^{(3)} = 4x\,\mathrm{csch}2x\cdot\ln(\lambda\,\mathrm{csch}x) - 2\,\mathrm{csch}2x\left\{\int_0^\infty dv\ln(\mathrm{sech}^2 v) - \int_x^\infty dv\ln(\mathrm{csch}^2 v)\right\}. \tag{A6}$$

Using

$$\int_0^\infty dv\ln(\tanh^2 v) = -\tfrac{1}{4}\pi^2,$$

[9] R. P. Feynman, Phys. Rev. **76**, 769 (1949). See Eqs. (22a), (23a), (26a). Note that the sign of the left side of (23a) should be plus.

[10] Specifically, the difficulty arises from the substitution $2y-1 = \tan\alpha/\tan\theta$, in the notation of reference 9 (where y and α are integration variables). We evaluate (A1) avoiding this substitution.

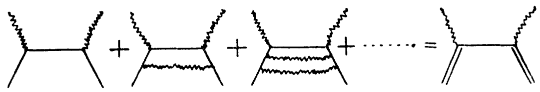

FIG. 3. Diagrams included in the ladder approximation for two-photon annihilation of a bound state.

and performing in (A6) an integration by parts, we have

$$H_0^{(1)}+H_0^{(3)}=\operatorname{csch}2x\left[4x\ln\lambda+\tfrac{1}{2}\pi^2-4\int_0^x vdv\coth v\right]. \quad (A7)$$

In $H_0^{(2)}$ we set $u-a=y$ and neglect y^2, yielding

$$H_0^{(2)}=\int_{-\epsilon}^{\epsilon}dy(2aQ^2y-i\delta)^{-1}[\ln(2aQ^2/\lambda^2)+\ln|y|]=i\pi\operatorname{csch}2x\cdot\ln(2aQ^2/\lambda^2)+(i\delta/2aQ^2)\int_0^{\epsilon}dy(y^2+\delta^2)^{-1}\ln|y|. \quad (A8)$$

The second term is easily evaluated by letting $y=\delta\tan\theta$ and dropping terms of order δ, giving finally (as $\epsilon\to 0$)

$$H_0^{(2)}=i\pi\operatorname{csch}2x\cdot\ln(\delta/\lambda^2). \quad (A9)$$

Returning to (A4), the second integral is

$$-i\pi\int_0^a du[Q^2(u^2-a^2)-i\delta]^{-1}=\operatorname{csch}2x[-i\pi\ln(\delta/2a)+\tfrac{1}{2}\pi^2] \quad (A10)$$

to order δ.

Thus we have for H_0:

$$H_0=\operatorname{csch}2x\cdot\left[4x\ln\lambda+\pi^2-4\int_0^x vdv\coth v+2\pi i\ln(2\sinh x/\lambda)\right]. \quad (A11)$$

This agrees with the application to Feynman's result of the rule given in Sec. II.

APPENDIX B

To obtain the matrix element for annihilation from a bound state, we shall sum selectively Feynman diagrams to all orders in the "ladder" approximation of Salpeter and Bethe,[11] as indicated in Fig. 3. The matrix element corresponding to the sum of the first l ladder diagrams can be written as

$$\mathfrak{M}=\sum_{n=0}^{l-1}\int\bar{v}_n(x')O(x',x)u_n(x)dxdx', \quad (A12)$$

where x and x' are points in space-time, $\bar{v}_n$ and u_n are iterated positron and electron wave functions, respectively, and $O(x',x)$ is a matrix representing the annihilation. For the lowest order approximation it can be taken as the matrix for annihilation of plane waves, with the intermediate state taken as free. Alternatively it can be taken to include radiative corrections.

It is convenient to use the charge conjugation operator C to define the charge conjugate spinor

$$u^C=C\bar{v}^T, \quad (A13)$$

where T represents the transpose. We have the following recurrence relationship:

$$u_n^C(x')\times u_n(x)=\int\int K_0(x',y')\Gamma(y')u_{n-1}^C(y')\times K_0(x,y)\Gamma(y)u_{n-1}(y)S(y',y)dy'dy, \quad (A14)$$

which corresponds to allowing an additional virtual photon to be emitted and absorbed. $\Gamma(x)$ is the vertex operator for emission or absorption of the photon, K_0 is the electron propagator, and S is the photon propagator. The cross in (A14) denotes the direct product.

Define

$$\psi_{ab}(x',x)=\sum_{n=0}^{\infty}u_n^C(x')\times u_n(x). \quad (A15)$$

The subscripts a and b designate an ordering of the spinor factors in the calculation of a matrix element, and can be considered to refer to the electron and positron, respectively. The transpose of the spinor associated with b, placed on the left, and the spinor associated with a, placed on the right, are to be understood in the calculation of any matrix element. With this understanding, (A12) can be written

$$-\mathfrak{M}=\int\psi_{ab}(x',x)C^{-1}O(x',x)dxdx'. \quad (A16)$$

Substituting (A14) in (A15), we obtain

$$\psi_{ab}(x',x)=\int\int K_{0b}(x',y')\Gamma_b(y')K_{0a}(x,y)\Gamma_a(y)\times S(y',y)\psi_{ab}(y',y)dy'dy+u_0^C(x')\times u_0(x). \quad (A17)$$

Following Salpeter and Bethe, we drop $u_0^C(x')\times u_0(x)$ as we are interested only in bound states. It is evident that Eq. (A17) is then the SB equation [Eq. (11a) of reference 11] in the ladder approximation.

In the center-of-mass system, with $\chi_{ab}(p)$ the four-dimensional momentum transform of the SB wave function, Eq. (A16) becomes

$$-\mathfrak{M}=\delta(q+q'-K)\int dp\chi_{ab}(p)C^{-1}\times[O_{ab}(p-\tfrac{1}{2}(q'-q))+O_{ab}(p+\tfrac{1}{2}(q'-q))], \quad (A18)$$

[11] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951) (referred to as SB).

where K is the four-momentum of the center-of-mass and q, q' are the four-momenta of the annihilation quanta. The second term in the square brackets arises from the interchange of q and q'.

As the positronium ground state wave function has characteristic momentum $e^2m/2$, it is sufficiently accurate to use for χ_{ab} the nonrelativistic singlet wave function, i.e., a spin wave function multiplied by the Schrödinger momentum distribution, and to expand the spin matrix element in powers of $\mathbf{p}$. There are no corrections of relative order e^2 arising from virtual annihilation because charge conjugation invariance forbids an intermediate state containing a single photon. The spin matrix element, as we have noted in Sec. IV is of the form const$+O(\mathbf{p}^2)$.[12]

Further contributions could, *a priori*, arise from the small components of the atomic wave functions. That is, terms in (A18) arising from the product of small and large components in χ_{ab}, taken with the lowest order of O_{ab}, must be considered. For an improved wave function, valid to $O(e^2)$ times the Pauli wave function, we can use the result given by Karplus and Klein[13]:

$$\varphi_C(\mathbf{r})=2e^2(2\pi)^{-2}\int d\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{r}m}\times(\mathbf{p}^2+\tfrac{1}{4}e^4m^2)^{-2}F_{\mathbf{p}}(t)\varphi_0(0), \quad \text{(A19)}$$

where $\varphi_0(0)$ is the Pauli wave function for singlet positronium evaluated at the origin and $F_{\mathbf{p}}(t)$ to the order of accuracy with which we are concerned is

$$F_{\mathbf{p}}(t)\cong\tfrac{1}{2}(1+\boldsymbol{\alpha}_a\cdot\mathbf{p}/2m)(1-\boldsymbol{\alpha}_b\cdot\mathbf{p}/2m)F'(t),$$

where $F'(t)$ contains no matrices and is of order unity. The term $(\boldsymbol{\alpha}_a-\boldsymbol{\alpha}_b)\cdot\mathbf{p}$ is equivalent to $(\boldsymbol{\sigma}_a-\boldsymbol{\sigma}_b)\cdot\mathbf{p}$ acting on the Pauli spin wave function of the singlet state, which yields $|\mathbf{p}|$ times the spin wave function for the triplet state. The corrections to the singlet matrix element from the small components of the Dirac wave function will then be of order $|\mathbf{p}|$ times the low-energy plane wave matrix element of the *triplet* state which is itself of order $|\mathbf{p}|$. When integrated over the momentum distribution, it will yield corrections of order e^2 times the ones we are considering.

[12] It should not be overlooked that what we have designated $O(\mathbf{p}^2)$ actually contains the term $v^2\ln(\lambda_{\min}/m)$, since selection rules rigorously forbid the emission of three quanta from the singlet ground state. The occurrence of this term constitutes a defect of the present treatment which would be corrected if we used a procedure which took into account binding effects in intermediate states [see T. Fulton and R. Karplus, Phys. Rev. **93**, 1109 (1954)]. Rohrlich's arguments [F. Rohrlich, Phys. Rev. **98**, 181 (1955)] do not seem to be relevant as they show merely that $\ln(\lambda_{\min}/m)$ has a coefficient which is small for positronium.

[13] Reference 3, Eqs. (A.9) and (4.4). See also E. E. Salpeter, Phys. Rev. **89**, 92 (1953).