Energy Equation of Magneto-Gas Dynamics*

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The general energy equation of a viscous, heat-conducting, and electrical-conducting fluid in magneto-gas dynamics has been derived. Various simplified forms of energy equations have been discussed, particularly for the cases with magneto-gas dynamic approximations. The fundamental equations of magneto-gas dynamics are also given.

I. INTRODUCTION

RECENTLY the study of magnetohydrodynamics of a compressible fluid, more properly magneto-gas dynamics,[1-4] has attracted great attention. However, nowhere do we find a complete account of the fundamental equations without simplifying assumptions injected from the outset. Practically all of the previous investigations use some approximate relation to replace the energy equation in the analysis. Baños[5] tried to derive an energy equation of magneto-gas dynamics for nonviscous and non-heat-conducting compressible fluid, but his result is incorrect because he derived the energy equation from the equation of motion. It is well known that in gas dynamics, the equations of energy and of motion are two independent relations. One cannot be derived from the other. As a result, in the energy equation of Baños, Eq. (15) of reference 5, the important terms involving the internal energy of the gas are missing. Hence his result does not give the complete energy equation. Since all the other fundamental equations of magneto-gas dynamics,[1-4] except the energy equation, have been discussed before, only the detailed derivation of the energy equation of magneto-gas dynamics for a viscous, heat-conducting, and electrically-conducting fluid is given in this paper. We shall discuss other simplified forms of the energy equation of magneto-gas dynamics. Finally, the whole set of the fundamental equations of magneto-gas dynamics is given.

II. DERIVATION OF THE ENERGY EQUATION

Consider the surface S of a volume V fixed in space. The law of conservation of energy requires that the difference in the rate of supply of energy to V and the rate at which energy goes out through S must be the net rate of increase of energy in V. We have then

$$\int_S u_i(\tau_{ij}n_j)dS-\int_S U_M\rho u_jn_jdS-\int_S\left(-\kappa\frac{\partial T}{\partial x_j}\right)n_jdS-\int_S S_jn_jdS-\int_S q_{Rj}n_jdS=\frac{\partial}{\partial t}\int_V U\rho dV, \quad (1)$$

where $i,j=1$, 2, or 3, and the summation convention is used, e.g., $u_jn_j=u_1n_1+u_2n_2+u_3n_3$.

The meanings of the various terms in Eq. (1) are as follows:

(*a*) $u_i=i$th component of the velocity vector $\mathbf{q}$ of the fluid; $n_j=j$th component of the outer normal of the surface S; $\tau_{ij}=ij$th component of the viscous stress tensor[6]:

$$\tau_{ij}=(\tau^M)_{ij}-p\delta_{ij},$$

where $p=$pressure of the fluid, $\delta_{ij}=0$ if $i\neq j$; $\delta_{ij}=1$, if $i=j$; and

$$(\tau^M)_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\tfrac{2}{3}\mu\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij},$$

μ being the coefficient of viscosity. Here $u_i(\tau_{ij}n_j)=$rate of energy produced by the system outside in contact with the system within, per unit area of the surface of S.

(*b*) $U=$total energy per unit mass of the fluid:

$$U=U_M+\frac{1}{\rho}(U_E+U_H)=\tfrac{1}{2}u_iu_i+P+I+\frac{1}{\rho}(U_E+U_H),$$

where U_M is the energy of the fluid other than the electromagnetic energy, which consists of: (i) $\frac{1}{2}u_iu_i=$kinetic energy per unit mass of the fluid; (ii) $P=$potential energy per unit mass of the fluid; (iii) $I=$internal per unit mass of the fluid$=C_vT$ for ideal gas, where C_v is the specific heat at constant volume and T is the absolute temperature.

The second term in Eq. (1) represents the energy flow by convection across the boundary S. In the analysis of magneto-gas dynamics, we assume that Maxwell's equations for the electromagnetic field hold true, and hence the electromagnetic energy is localized in the field and is not carried by the moving gas. Of course this assumption is only approximately correct, because the electric charges and electric current are carried by by the gas. However, in magneto-gas dynamics these contributions to the electric energy are negligibly small.

$U_E=\frac{1}{2}\epsilon E_iE_i=$electric energy per unit volume, where ϵ is the dielectric constant and E_i is the ith component of the electric field strength E. $U_H=\frac{1}{2}\mu_eH_iH_i=$magnetic energy per unit volume, where μ_e is the magnetic

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[1] E. Åström, Nature **165**, 1019 (1950).
[2] N. Herlofson, Nature **165**, 1020 (1950).
[3] H. C. van de Hulst, *Problems of Cosmical Aerodynamics* (Central Air Documents Office, Dayton, Ohio, 1951), pp. 46–56.
[4] S. Lundquist, Arkiv Fysik, **5**, 297 (1952).
[5] A. Baños, Jr., Phys. Rev. **97**, 1435 (1955).
[6] S. I. Pai, *Viscous Flow Theory, I. Laminar Flow* (D. Van Nostrand Company, Inc., New York, 1956).

permeability and H_i is the ith component of the magnetic field strength H. The mks unit system is used in this note.

(c) $x_j = j$th component of the spatial coordinate, and κ = coefficient of heat conductivity.

The third term in Eq. (1) represents the energy flow by heat conduction across the boundary S.

(d) $S_i = E_j H_k - E_k H_j = i$th component of the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ which represents the electromagnetic energy flowing across the boundary S, i.e., the fourth term of Eq. (1).

(e) $q_{Ri} = i$th component of the radiation energy flux, and

$$\partial q_{Ri}/\partial x_i = R_r - R_a,$$

where R_r is the rate of radiation energy emission per unit volume and R_a is the rate of radiation absorption per unit volume.

The fifth term in Eq. (1) is the rate of energy flow by radiation.

By Gauss' theorem, we may transform the surface integrals of Eq. (1) into volume integrals. We have then

$$\int_V \left\{ \frac{\partial}{\partial x_i}(u_i \tau_{ij}) - \frac{\partial (U_M \rho u_j)}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) - \frac{\partial (\rho U_M)}{\partial t} - \frac{\partial (U_E + U_M)}{\partial t} - \frac{\partial S_i}{\partial x_i} - \frac{\partial q_{Ri}}{\partial x_i} \right\} dV = 0. \quad (2)$$

Since the volume V is arbitrarily chosen, the integrand of Eq. (2) must be zero and we have

$$\frac{\partial}{\partial x_j}(u_i \tau_{ij}) - \frac{\partial (U_M \rho u_j)}{\partial x_j} - \frac{\partial (\rho U_M)}{\partial t} - \frac{\partial (U_E + U_H)}{\partial t} + \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) - \frac{\partial S_i}{\partial x_i} - \frac{\partial q_{Ri}}{\partial x_i} = 0. \quad (3)$$

This is the energy equation in magneto-gas dynamics of a viscous, heat-conducting, electrically-conducting, and compressible fluid.

Equation (3) may be simplified by the help of the following equations:

(i) The equation of continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} 0. \quad (4)$$

(ii) The equation of motion:

$$\rho \frac{D u_i}{D t} = -\rho \frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + F_i, \quad (5)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k},$$

and $F_i = i$th component of the electromagnetic[7] force $\mathbf{F}_e$, which is:

$$\mathbf{F}_e = \rho_e \mathbf{E} + \mu_e \mathbf{J} \times \mathbf{H}, \quad (6)$$

where ρ_e is the excess electric charge and $\mathbf{J}$ is the electric current density with components J_i.

(iii) The equation of electromagnetic energy:

$$\frac{\partial (U_E + U_H)}{\partial t} = -\frac{\partial S_i}{\partial x_i} - E_i J_i. \quad (7)$$

Combining Eqs. (3) to (7), we have the following energy equation of magneto-gas dynamics:

$$\rho \frac{DI}{Dt} = -p \frac{\partial u_i}{\partial x_i} + (\tau^M)_{ij} \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial T}{\partial x_i}\right) + \frac{i^2}{\sigma} - \frac{\partial q_{Ri}}{\partial x_i}, \quad (8)$$

where

$$\mathbf{i} = \mathbf{J} - \rho_e \mathbf{q} = \sigma(\mathbf{E} + \mu_e \mathbf{q} \times \mathbf{H}), \quad (9)$$

σ is the electric conductivity of the gas, and i^2/σ is the Joule heat; $\mathbf{i}$ is sometimes called the conduction current.

III. SIMPLIFIED FORMS OF ENERGY EQUATION

For the ordinary gas-dynamic problem, if the absolute temperature T is not enormously high, the energy flow due to radiation may be neglected. Furthermore, the compressible fluid which we investigate is usually assumed to be a perfect gas whose equation of state is

$$p = \rho R T, \quad (10)$$

where R is the gas constant. Under these two conditions, Eq. (8) may be written in the following form:

$$\frac{D C_p T}{Dt} = \frac{Dp}{Dt} + (\tau^M)_{ij} \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) + \frac{i^2}{\sigma}, \quad (11)$$

where $C_p = C_v + R$.

For ordinary magneto-gas dynamics, we consider the cases in which the velocity of the flow $\mathbf{q}$ is much smaller than the velocity of light c. Since the energy in the electric field is of the order of q^2/c^2 of the energy in the magnetic field,[8] the energy in the electric field may be neglected. Furthermore, since we shall not consider phenomena of very high frequency, the displacement current in Maxwell equations:

$$\nabla \times \mathbf{H} = \mathbf{J} + (\partial \epsilon \mathbf{E}/\partial t), \quad (12)$$

$$\nabla \times \mathbf{E} = -(\partial \mu_e \mathbf{H}/\partial t) \quad (13)$$

(where ∇ is the gradient operator), may be neglected, we have $\rho_e \cong 0$, and

$$\mathbf{i} = \nabla \times \mathbf{H}. \quad (14)$$

Under these conditions, the equation of motion (5),

[7] S. Chandrasekhar, Proc. Roy. Soc. (London) **A204**, 435 (1951).

[8] F. de Hoffmann and E. Teller, Phys. Rev. **80**, 692 (1950).

with $\partial P/\partial x_i=0$, becomes

$$\rho\frac{Du_i}{Dt}-\mu_e H_k\frac{\partial H_i}{\partial x_k}=-\frac{\partial}{\partial x_i}\left(p+\frac{\mu_e H_i H_i}{2}\right)+\frac{\partial(\tau^M)_{ij}}{\partial x_i}, \tag{15}$$

and the energy equation becomes

$$\rho\frac{Dh_0}{Dt}=\frac{\partial p}{\partial t}+\frac{\partial}{\partial x_j}[u_i(\tau^M)_{ij}]+\frac{\partial}{\partial x_i}\left(\kappa\frac{\partial T}{\partial x_i}\right)+(\nabla\times\mathbf{H})\cdot\left(\frac{\nabla\times H}{\sigma}-\mu_e\mathbf{q}\times\mathbf{H}\right), \tag{16}$$

where $h_0=C_pT+\frac{1}{2}u_iu_i$.

The electromagnetic equations may be reduced into a simple equation for the magnetic field $\mathbf{H}$ which is

$$\frac{\partial\mathbf{H}}{\partial t}=\nabla\times(\mathbf{q}\times\mathbf{H})-\nabla\times\left[\frac{1}{\mu_e\sigma}(\nabla\times\mathbf{H})\right]. \tag{17}$$

Hence we consider the interaction of the magnetic field $\mathbf{H}$ with the fields of gas dynamics under these conditions. This is why we use the name magneto-gas dynamics.

For a nonviscous and non-heat-conducting fluid, Eq. (16) becomes

$$\rho\frac{Dh_0}{Dt}=\frac{\partial p}{\partial t}+(\nabla\times\mathbf{H})\cdot\left[\frac{\nabla\times\mathbf{H}}{\sigma}-\mu_e\mathbf{q}\times\mathbf{H}\right]. \tag{18}$$

If there is no electromagnetic energy, Eq. (18) becomes

$$\rho\frac{Dh_0}{Dt}=\frac{\partial p}{\partial t}, \tag{19}$$

which is the well-known energy equation for the unsteady flow of an nonviscous compressible fluid. For steady flow, Eq. (19) becomes

$$u_j(\partial h_0/\partial x_j)=0. \tag{20}$$

Equation (20) shows that the stagnation enthalpy h_0 is constant along a streamline. However, with magnetic flow, from Eq. (18), the steady flow equation is

$$u_j\frac{\partial h_0}{\partial x_j}=\nabla\times\mathbf{H}\cdot\left[\frac{\nabla\times\mathbf{H}}{\sigma}-\mu_e\mathbf{q}\times\mathbf{H}\right]. \tag{21}$$

Hence, the stagnation enthalpy is not a constant along a streamline in magneto-gas dynamics.

For one-dimensional steady flow, i.e., flow in which all the variables depend on only one spatial coordinate, say x, Eq. (18) may be integrated. If only the x-component of the magnetic field $\mathbf{H}$ is different from zero, Eq. (18) after integration becomes

$$\rho u_1 h_0=\text{constant}. \tag{22}$$

In this case the magnetic field has no influence on the gas-dynamic phenomena.

If only H_2, the x_2 component of the magnetic field, is different from zero, then Eq. (18) with the help of Eq. (17), after integration, gives

$$\rho u_1 h_0-\mu_e H_2\left[\frac{1}{\mu_e\sigma}\frac{dH_2}{dx}-u_1H_2\right]=\text{constant}. \tag{23}$$

If we put $\sigma=\infty$, the result is the energy equation used by de Hoffmann and Teller in their study of magneto-hydrodynamic shock.[8]

IV. FUNDAMENTAL EQUATIONS OF MAGNETO-GAS DYNAMICS

For the problem of interaction of gas dynamics with electromagnetic fields, the unknowns are $\mathbf{H}$, $\mathbf{E}$, $\mathbf{J}$, ρ_e, $\mathbf{q}$, p, ρ, and T. The relations that govern these unknowns are: (*a*) Maxwell's equations (12) and (13), (*b*) the current density equation (9), and (*c*) the conservation of electric charge,

$$\nabla\cdot\mathbf{J}+\partial\rho_e/\partial t=0, \tag{24}$$

(*d*) the equation of motion (5), (*e*) the equation of continuity (4), (*f*) the equation of state (10), and (*g*) the equation of energy (8).

Under the magneto-gas dynamics approximations, the unknowns reduce to $\mathbf{H}$, $\mathbf{q}$, p, ρ, and T. The relations that govern these unknowns are: (*a*) the equation for magnetic field (17), (*b*) the equation of motion (15), (*c*) the equation of energy (16), (*d*) the equation of continuity (4), and (*e*) the equation of state (10).