PHYSICAL REVIEW VOLUME 105, NUMBER 4 FEBRUARY 15, 1957

Pair Production of *s*-Wave Pi Mesons*†

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(Received October 22, 1956)

A theory for pair production of s-wave π mesons is constructed along the lines of the Chew-Low-Wick formalism. A bilinear s-wave interaction of the form $(\lambda_0/\mu)\boldsymbol{\phi}\cdot\boldsymbol{\phi}+(\lambda/\mu^2)\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\boldsymbol{\pi}$ as used by Drell, Friedman, and Zachariasen is added to the p-wave interaction $(4\pi)^{\frac{1}{2}}(f/\mu)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\boldsymbol{\tau}\cdot\boldsymbol{\phi}$ as used by Chew and Low. It is shown that if the s-wave interaction is limited to the λ_0 term (scalar pair theory) the cross section for pair production of s waves vanishes.

By using both the λ_0 and λ terms, the cross section for inelastic scattering of a p-wave meson into two s-wave mesons, near threshold (total energy of the produced mesons ≤ 350 Mev), per unit energy of one of the produced mesons is determined to be of the order of millimicrobarns/Mev; for photoproduction the corresponding number is ~ 1000 times smaller.

A comparison is made with the work of Cutkosky and Zachariasen and it is concluded that if there is no meson-meson interaction s-wave pair production may be neglected except possibly as the very threshold.

I. INTRODUCTION

DRELL, Friedman, and Zachariasen[1] have shown recently how the fixed source theory of the p-wave pion-nucleon interaction of Chew and Low[2] can be extended to include s-wave pion-nucleon interactions. We report here an analysis of pair production of s-wave π mesons based on the work of above authors. In particular we study the inelastic scattering of a p-wave meson into two s-wave mesons and the photoproduction of s-wave meson pairs near threshold for these processes.

The Hamiltonian is taken to be

$$H=H_0+H';\quad H'=H_p'+H_s'. \tag{1}$$

Here H_0 is the Hamiltonian of the free meson field (the energy of a physical nucleon is taken to be zero), H_p' is the interaction Hamiltonian as used by CL, and H_s' is the s-wave interaction Hamiltonian added by DFZ.

The interaction Hamiltonian (1) serves as the basis for discussion of inelastic meson scattering in Secs. II and III. In Sec. IV we discuss photoproduction of s-wave meson pairs on the basis of the Hamiltonian (1) modified to include electromagnetic effects. A method for this modification is presented which preserves the gauge invariance of the theory without introducing any gauge currents if one works in the Coulomb ($\boldsymbol{\nabla}\cdot\mathbf{A}=0$) gauge. In Sec. V the equations derived in the preceding sections are solved approximately and various cross sections are calculated.

In view of the fact that H_p' is responsible for the interaction with the nucleon of p-wave mesons only and H_s' of s-wave mesons only, it is convenient to expand the meson field in terms of spherical waves. The various terms in the Hamiltonian (1) then become

$$H_0=\sum_p \omega_p \sum_q (-)^q\bigg[a_q^\dagger(p)a_{-q}(p)+\sum_{m=-1}^{+1}(-)^m a_q^\dagger(pm)a_{-q}(p-m)+\sum_{l=2}^{\infty}\sum_{m=-l}^{+l}(-)^m a_q^\dagger(plm)a_{-q}(pl-m)\bigg] \tag{2},$$

$$H_p'=\frac{f^0}{\mu}\sum_p \frac{Npv(p)}{(6\omega_p)^{\frac{1}{2}}}\sum_{q,m=-1}^{+1}(-)^q(-)^m\tau_q\sigma_m \times[a_{-q}(p-m)+a_{-q}^\dagger(p-m)], \tag{3}$$

$$\begin{aligned}H_s'=&\frac{\lambda_0{}^0}{\mu}\sum_{pp'}\frac{N^2}{4\pi}\frac{v(p)v(p')}{(4\omega_p\omega_{p'})^{\frac{1}{2}}}\sum_q(-)^q[a_q(p)a_{-q}(p')\\&+a_q(p)a_{-q}^\dagger(p')+a_q^\dagger(p)a_{-q}(p')+a_q^\dagger(p)a_{-q}^\dagger(p')]\\&+\frac{\lambda^0}{\mu^2}\sum_{pp'}\frac{N^2}{4\pi}\frac{v(p)v(p')}{(4\omega_p\omega_{p'})^{\frac{1}{2}}}\\&\times\{(\omega_p-\omega_{p'})\sum_q\tau_q[a_r(p)a_s(p')-a_s^\dagger(p')a_r^\dagger(p)]\\&+(\omega_p+\omega_{p'})\sum_q\tau_q[a_r(p)a_s^\dagger(p')-a_s(p')a_r^\dagger(p)]\},\\&(q,r,s=\text{cyclic permutations of }-1,0,+1).\end{aligned} \tag{4}$$

The phases of the creation and annihilation operators above are chosen so that the only nonvanishing commutator is given by

$$[a_q(plm),a_{q'}^\dagger(p'l'm')]=(-)^{q+m}\delta_{q,-q'}\delta_{m,-m'}\delta_{l,l'}\delta_{p,p'}. \tag{5}$$

We use the system of units in which $\hbar=c=1$. f^0 is the nonrenormalized p-wave coupling constant of CL,

* This work was assisted in part by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

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[1] Drell, Friedman, and Zachariasen, Phys. Rev. **104**, 236 (1956), hereafter referred to as DFZ.

[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570, 1579 (1956), hereafter referred to as CL. G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).

$\lambda_0{}^0$ and λ^0 are the nonrenormalized s-wave coupling constants of DFZ. N is a normalization constant which also enters into the expression for the density of states in phase space and which cancels out in the final expressions for cross sections. μ stands for the rest mass of the π meson and $\omega_p=(p^2+\mu^2)^{\frac{1}{2}}$ is the energy of a meson with momentum p. $v(p)$ is the Fourier transform of the nucleon source density $u(r)$: $v(p)=\int u(r)e^{i\mathbf{p}\cdot\mathbf{r}}d\mathbf{r}$. σ_m and τ_q, m, $q=-1$, 0, $+1$, are Pauli spin matrices acting in spin and isotopic spin spaces respectively. $a_q{}^\dagger(plm)$ is the creation operator for a meson of charge qe, energy ω_p, square of angular momentum $l(l+1)$ and z component of angular momentum m; whereas $a_q(plm)$ is the annihilation operator for a meson of charge $-qe$, energy ω_p, square of angular momentum $l(l+1)$ and z component of angular momentum $-m$. (Note the minus signs in these definitions.) Finally $a_q(p)$ is a shorthand notation for $a_q(p00)$ and $a_q(pm)$ is a shorthand notation for $a_q(p1m)$.

We note that $a_q(plm)$ and $(-)^{q+m}a_{-q}{}^\dagger(pl-m)$ are each others Hermitian conjugates, *not* $a_q(plm)$ and $a_q{}^\dagger(plm)$. The reason for this choice of phases is that now $a_q(plm)$ [as well as $a_q{}^\dagger(plm)$] behaves as the m component of an irreducible tensor of rank l as defined by Racah[3] under rotations in space; and as the q component of an irreducible tensor of rank 1 under rotations in isotopic spin space. The relation between $a_q(plm)$ and its Hermitian conjugate is also in agreement with the notation of Racah.

To complete the discussion of the notation, we turn now to the symbols used for the description of the states of the system. We use the symbols A, B, C to describe nucleons; other symbols used in a state vector refer to mesons. In particular we use the symbols R, S to describe s-wave mesons, and the symbol L to describe a p-wave meson. Each symbol stands for an aggregate of quantum numbers, thus R represents the aggregate:

$$\begin{aligned} R&=\text{the isotopic spin,}\\ \rho&=z\text{ component of the isotopic spin,}\\ R'&=\text{angular momentum (orbital or spin),}\\ \rho'&=z\text{ component of angular momentum,}\\ r&=\text{magnitude of linear momentum.}\end{aligned}\tag{6}$$

(The last quantum number is always assumed to be zero and therefore omitted, when referring to nucleons.) Thus, for example, when dealing with inelastic scattering we need a matrix element of the scattering matrix denoted by

$$\langle BRS^-|AL^+\rangle,\tag{7}$$

because the initial state of the system consists of a nucleon (A) and a p-wave meson (L), whereas the final state of the system consists of a nucleon (B) and two s-wave mesons (R and S). The superscripts $+$ and $-$ are used to denote the fact that the corresponding states are scattering eigenstates of the total Hamiltonian defined by the boundary condition at infinity of only outgoing or incoming waves respectively.

In a manner entirely analogous to CL and DFZ, we find for the states entering Eq. (7):

$$|AL^+\rangle=\{a_\lambda{}^\dagger(l\lambda') -(H-\omega_l-i\epsilon)^{-1}[H',a_\lambda{}^\dagger(l\lambda')]\}|A\rangle,\tag{8}$$

$$|BRS^-\rangle=\{a_\rho{}^\dagger(r)a_\sigma{}^\dagger(s)-(H-\omega_r-\omega_s+i\epsilon)^{-1} \times[H',a_\rho{}^\dagger(r)a_\sigma{}^\dagger(s)]\}|B\rangle.\tag{9}$$

II. SCALAR PAIR THEORY

It has been shown in the work of DFZ that both the $\lambda_0{}^0$ and λ^0 terms are necessary to account for the experimental s-wave phase shifts and therefore we will use both terms. However, we would like to show first that the $\lambda_0{}^0$ term alone leads to zero transition amplitude for the process that we are considering.

Substituting Eq. (9) into Eq. (7) gives

$$\langle BRS^-|AL^+\rangle=\langle B|(-)^{\rho+\sigma}\{a_{-\rho}(r)a_{-\sigma}(s) +[H',a_{-\rho}(r)a_{-\sigma}(s)](H-\omega_r-\omega_s-i\epsilon)^{-1}\}|AL^+\rangle.\tag{10}$$

In a manner analogous to CL and DFZ, one obtains

$$a_{-\rho}(r)a_{-\sigma}(s)|AL^+\rangle =(H+\omega_r+\omega_s-i\epsilon)^{-1}[H',a_{-\rho}(r)a_{-\sigma}(s)]|AL^+\rangle.\tag{11}$$

Hence Eq. (10) becomes

$$\langle BRS^-|AL^+\rangle=2\pi i\delta(\omega_l-\omega_r-\omega_s)(-)^{\rho+\sigma} \times\langle B|[H',a_{-\rho}(r)a_{-\sigma}(s)]|AL^+\rangle.\tag{12}$$

From Eq. (12) we see that the appropriate element of the transition matrix for our process is

$$-(-)^{\rho+\sigma}\langle B|[H',a_{-\rho}(r)a_{-\sigma}(s)]|AL^+\rangle.\tag{13}$$

This element of the transition matrix can be written as follows:

$$\begin{aligned}\langle B|&[H',a_{-\rho}(r)a_{-\sigma}(s)]|AL^+\rangle\\ &=\langle B|[a_{-\rho}(r),[H',a_{-\sigma}(s)]]+[H',a_{-\rho}(r)]a_{-\sigma}(s)\\ &\quad+[H',a_{-\sigma}(s)]a_{-\rho}(r)|AL^+\rangle\\ &=\langle B|[a_{-\rho}(r),[H',a_{-\sigma}(s)]]\\ &\quad+[H',a_{-\rho}(r)](H+\omega_s-\omega_l-i\epsilon)^{-1}[H',a_{-\sigma}(s)]\\ &\quad+[H',a_{-\sigma}(s)](H+\omega_r-\omega_l-i\epsilon)^{-1}[H',a_{-\rho}(r)]|AL^+\rangle\\ &=\langle B|[a_{-\rho}(r),[H',a_{-\sigma}(s)]]AL^+\rangle\\ &\quad+\sum_{N^+}\left\{\frac{\langle B|[H',a_{-\rho}(r)]N^+\rangle\langle N^+|[H',a_{-\sigma}(s)]AL^+\rangle}{E_N+\omega_s-\omega_l-i\epsilon}\right.\\ &\quad\left.+\frac{\langle B|[H',a_{-\sigma}(s)]N^+\rangle\langle N^+|[H',a_{-\rho}(r)]AL^+\rangle}{E_N+\omega_r-\omega_l-i\epsilon}\right\},\end{aligned}\tag{14}$$

where we have inserted a complete set of eigenstates of the Hamiltonian with the $+$ convention denoted by

[3] G. Racah, Phys. Rev. **62**, 438 (1942); Biedenharn, Blatt, and Rose, Revs. Modern Phys. **24**, 249 (1952).

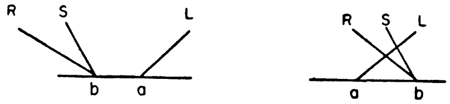

FIG. 1. Two-vertex diagrams.

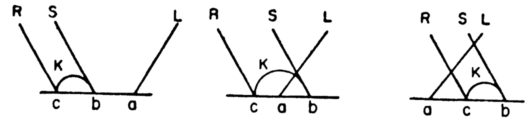

FIG. 2. Three-vertex diagrams.

N^+, and used two relations of the following kind:

$$a_{-\sigma}(s)|AL^+\rangle = (H+\omega_s-\omega_l-i\epsilon)^{-1}[H',a_{-\sigma}(s)]|AL^+\rangle. \quad (15)$$

In the special case of the scalar pair theory, one finds readily:

$$\langle M^+|[H',a_{-\sigma}(s)]|N^+\rangle = -U_s \sum_p U_p \langle M^+|a_{-\sigma}(p)+a_{-\sigma}^\dagger(p)|N^+\rangle, \quad (16)$$

with

$$U_x = (\lambda_0{}^0)^{\frac{1}{2}}(4\pi\mu\omega_x)^{-\frac{1}{2}}Nv(x). \quad (17)$$

Here $|M^+\rangle$ and $|N^+\rangle$ are any two eigenstates of the system with the $+$ convention. We construct next

$$a_{-\sigma}(p)|N^+\rangle = |(N-1_\sigma)^+\rangle\delta_{\sigma p}{}^N + (H+\omega_p-E_N-i\epsilon)^{-1} \times[H',a_{-\sigma}(p)]|N^+\rangle, \quad (18)$$

$$a_{-\sigma}^\dagger(p)|N^+\rangle = |(N+1_{-\sigma})^+\rangle + (H-\omega_p-E_N-i\epsilon)^{-1} \times[H',a_{-\sigma}^\dagger(p)]|N^+\rangle, \quad (19)$$

where $|(N-1_\sigma)^+\rangle\delta_{\sigma p}{}^N$ represents an eigenstate obtained from $|N^+\rangle$ by removing an s-wave meson of momentum p and charge σe [if the state $|N^+\rangle$ did not contain such a meson, then $|(N-1_\sigma)^+\rangle\delta_{\sigma p}{}^N=0$] and where $|(N+1_{-\sigma})^+\rangle$ represents an eigenstate containing in addition to what was in $|N^+\rangle$ an s-wave meson of momentum p and charge $-\sigma e$.

Using Eqs. (18) and (19), we obtain from Eq. (16):

$$\langle M^+|[H',a_{-\sigma}(s)]N^+\rangle = -U_s \sum_p U_p \Bigg\{ \langle M^+|(N-1_\sigma)^+\rangle\delta_{\sigma p}{}^N + \langle M^+|(N+1_{-\sigma})^+\rangle + \frac{\langle M^+|[H',a_{-\sigma}(p)]N^+\rangle}{E_M-E_N-i\epsilon+\omega_p} + \frac{\langle M^+|[H',a_{-\sigma}^\dagger(p)]N^+\rangle}{E_M-E_N-i\epsilon-\omega_p}\Bigg\}. \quad (20)$$

It follows from Eq. (20) that $U_s^{-1}\langle M^+|[H',a_{-\sigma}(s)]N^+\rangle$ is independent of the momentum s and therefore

$$\langle M^+|[H',a_{-\sigma}(s)]N^+\rangle = -\left\{\frac{U_s\sum_p U_p[\langle M^+|(N-1_\sigma)^+\rangle\delta_{\sigma p}{}^N + \langle M^+|(N+1_{-\sigma})^+\rangle]}{1-2\sum_p U_p{}^2\omega_p/[(E_M-E_N-i\epsilon)^2-\omega_p{}^2]}\right\}. \quad (21)$$

Disregarding the singular case when the value of $\lambda_0{}^0$ is such as to make the denominator in Eq. (21) equal to zero, we find that $\langle M^+|[H',a_{-\sigma}(s)]N^+\rangle=0$ unless the states $|M^+\rangle$ and $|N^+\rangle$ differ by one s-wave meson of charge σe, but are otherwise identical. Hence the summation over N^+ in Eq. (14) vanishes. What remains in Eq. (14) is the double commutator:

$$[a_{-\rho}(r),[H',a_{-\sigma}(s)]] = -(-)^\rho\delta_{\rho,-\sigma}U_sU_r. \quad (22)$$

This is a c number which can be pulled out and we are left with $\langle B|AL^+\rangle=0$ owing to orthogonality of these two eigenstates. This completes the proof that the transition amplitude for our process vanishes for a scalar pair interaction.

We observe that above result can be anticipated if the process in question is pictured in terms of a series of diagrams grouped together according to the number of vertices involved. We have the two-vertex diagrams pictured in Fig. 1, the three-vertex diagrams pictured in Fig. 2, etc. (Only one class of diagrams is shown, others can be obtained by interchanging mesons R and S.) The diagrams are to be read from right to left. The horizontal line represents the nucleon, the other lines are meson lines. At the vertex a the p-wave meson L is absorbed. Hence the operator in question as far as the nucleon is concerned is $\sigma\tau$. At all the other vertices s-wave mesons (R,S,K) are scattered or created in pairs. Hence the operator in question as far as the nucleon is concerned is simply unity (scalar pair theory). Therefore all the vertices commute and it does not matter whether we write ab or ba. The only difference between the contributions from diagrams with the same number of vertices is in the energy denominators and one sees easily that the contributions from all such diagrams add up to zero.

The crucial point in this "proof" by diagrams is the commutability of vertices, valid in the scalar pair theory. We note that if not all diagrams with a given number of vertices are considered, the mutual cancellation will not take place. This is what happens in approximation methods such as the Tamm-Dancoff method,[4] leading to erroneous results.

III. INELASTIC SCATTERING

We start again with the expression (7) for the relevant matrix element of the scattering matrix. In Sec. II we proceeded by replacing $\langle BRS^-|$ by Eq. (9)—this approach was dictated by some special features of the scalar pair theory. In the present case we proceed in the orthodox way of replacing $|AL^+\rangle$ by Eq. (8) to

[4] I. Tamm, J. Phys. (U.S.S.R.) 9, 449 (1945); S. M. Dancoff, Phys. Rev. 78, 382 (1950).

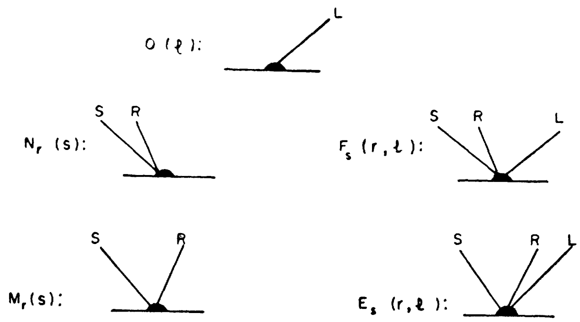

FIG. 3. The functions O, N, M, F, and E. The blob indicates the complete physical interaction.

obtain

$$\begin{aligned}\langle BRS^-|AL^+\rangle &= -2\pi i\delta(\omega_l-\omega_r-\omega_s)\langle BRS^-|[H',a_\lambda{}^\dagger(l\lambda')]A\rangle \\ &\equiv -2\pi i\delta(\omega_l-\omega_r-\omega_s)\langle BRS|F_s(r,l)|AL\rangle. \end{aligned} \quad (23)$$

We recognize that the coefficient of $-2\pi i\delta(\omega_l-\omega_r-\omega_s)$ is just the required matrix element of the transition amplitude in view of the relation between the scattering matrix and the transition amplitude matrix

$$S_{ab}=\delta_{ab}-2\pi i\delta(\omega_a-\omega_b)T_{ab}. \quad (24)$$

(We are dealing with an off-diagonal matrix element of S and therefore the δ_{ab} term does not appear.)

Instead of Eq. (9), we can use for $|BRS^-\rangle$ the equivalent expression:

$$|BRS^-\rangle=\{a_\rho{}^\dagger(r)-(H-\omega_r-\omega_s+i\epsilon)^{-1} \times[H',a_\rho{}^\dagger(r)]\}|BS^-\rangle. \quad (25)$$

Using Eq. (25) the transition amplitude becomes

$$\begin{aligned}\langle BRS|F_s(r,l)|AL\rangle=\langle BS^-|&[H',(-)^\rho a_{-\rho}(r)](H-\omega_r-\omega_s-i\epsilon)^{-1}[H',a_\lambda{}^\dagger(l\lambda')]+[H',a_\lambda{}^\dagger(l\lambda')] \\ &\times(H+\omega_r)^{-1}[H',(-)^\rho a_{-\rho}(r)]|A\rangle. \end{aligned} \quad (26)$$

Equation (26) is the basic equation of this problem; however it is not very useful unless certain approximations are made. As in CL and DFZ, we proceed by using closure to introduce into Eq. (26) a complete set of eigenstates of the Hamiltonian with the $-$ convention and then make two approximations:

the "one-meson approximation," (a)

the "$\frac{3}{2}$ $\frac{3}{2}$ approximation." (b)

(a) By the "one-meson approximation" we mean the same approximation as that used by CL and DFZ. In the present work, the actual details are different since we must deal with states such as $|BRS^-\rangle$ which is a two-meson state. However, the spirit of the one meson approximation is maintained by allowing only such two-meson intermediate states in which only one meson is rescattered, the other being identical with one of the two mesons *in* the final two-meson state.

(b) By the "$\frac{3}{2}$ $\frac{3}{2}$ approximation" we mean the assumption that all p-wave scattering phase shifts may be assumed to be equal to zero except for the phase shift corresponding to scattering in an eigenstate of total angular momentum J and total isotopic spin T equal to the eigenvalues $\frac{3}{2}$.

As a consequence of these approximations our basic Eq. (26) becomes

$$\begin{aligned}\langle BRS|F_s(r,l)|AL\rangle=&\sum_C\frac{\langle BSR|N_r(s)|C\rangle\langle C|O(l)|AL\rangle}{\omega_s+\omega_r} \\ &-\sum_C\frac{\langle B|O(l)|CL\rangle\langle CSR|N_r(s)|A\rangle}{\omega_s+\omega_r} \\ &-\sum_{CK}\frac{\langle CK|M_r(k)|BR\rangle^*\langle CKS|F_s(k,l)|AL\rangle}{\omega_k-\omega_s-i\epsilon} \\ &-\sum_{CK}\frac{\langle BS|E_s(k,l)|CKL\rangle\langle CKR|N_r(k)|A\rangle}{\omega_k+\omega_r}. \end{aligned} \quad (27)$$

Equation (27) is an integral equation for the function F. Of the quantities appearing in it the function O is related to the absorption of a p-wave meson and can be obtained from CL; it is defined by the relation

$$\langle C|O(l)|AL\rangle\equiv\langle C|[H',a_\lambda{}^\dagger(l\lambda')]A\rangle. \quad (28)$$

The functions M and N are related to the scattering and pair production of s-wave mesons and can be obtained from the work of DFZ; (see Figs. 3 and 4) they are defined by the relations

$$\langle CK|M_r(k)|BR\rangle\equiv\langle CK^-|[H',a_\rho{}^\dagger(r)]B\rangle, \quad (29)$$

$$\langle BSR|N_r(s)|C\rangle\equiv-\langle BS^-|[H',(-)^\rho a_{-\rho}(r)]C\rangle. \quad (30)$$

The function E, however, is unknown. It is defined by an equation similar to Eq. (26):

$$\begin{aligned}\langle BS|E_s(r,l)|ARL\rangle=-\langle BS^-|&[H',a_\rho{}^\dagger(r)](H+\omega_r-\omega_s-i\epsilon)^{-1}[H',a_\lambda{}^\dagger(l\lambda')]+[H',a_\lambda{}^\dagger(l\lambda')] \\ &\times(H-\omega_r+i\epsilon)^{-1}[H',a_\rho{}^\dagger(r)]|A\rangle. \end{aligned} \quad (31)$$

Subject to the same approximations as used in obtaining Eq. (27), Eq. (31) becomes

$$\begin{aligned}\langle BS|E_s(r,l)|ARL\rangle=&\sum_C\frac{\langle BS|M_r(s)|CR\rangle\langle C|O(l)|AL\rangle}{\omega_s-\omega_r+i\epsilon} \\ &-\sum_C\frac{\langle B|O(l)|CL\rangle\langle CS|M_r(s)|AR\rangle}{\omega_s-\omega_r+i\epsilon} \\ &-\sum_{CK}\frac{\langle BS|E_s(k,l)|CKL\rangle\langle CK|M_r(k)|AR\rangle}{\omega_k-\omega_r+i\epsilon} \\ &-\sum_{CK}\frac{\langle CKR|N_r(k)|B\rangle^*\langle CKS|F_s(k,l)|AL\rangle}{\omega_k+\omega_r}. \end{aligned} \quad (32)$$

Thus Eqs. (27) and (32) form a system of two coupled linear integral equations for the functions E and F which must be solved. (See Figs. 5 and 6.)

First, however, we dispose of the dependence of F and E on the magnetic quantum numbers which merely represent the geometry and not the physics of the problem. By making use of well-known orthogonality and symmetry properties of the Clebsch-Gordan and Racah coefficients,[3] one obtains in place of Eqs. (27) and (32) the following:

$$F_s(r,l)=\Lambda\frac{N_r^{\frac{1}{2}}(s)-N_r^{\frac{3}{2}}(s)}{\omega_r+\omega_s}O(l)-\sum_k\left\{\frac{M_r^*(k)F_s(k,l)}{\omega_k-\omega_r-i\epsilon}+\Delta\frac{N_r(k)E_s(k,l)}{\omega_k+\omega_r}\right\} \quad (33)$$

and

$$E_s(r,l)=\Delta\Lambda\frac{M_r^{\frac{1}{2}}(s)-M_r^{\frac{3}{2}}(s)}{\omega_s-\omega_r+i\epsilon}O(l)-\sum_k\left\{\frac{M_r(k)E_s(k,l)}{\omega_k-\omega_r+i\epsilon}+\Delta\frac{N_r^*(k)F_s(k,l)}{\omega_k+\omega_r}\right\}. \quad (34)$$

Equations (33) and (34) are matrix equations and the meaning of the various symbols is explained in the Appendix.

As stated, the functions O, M, and N are known. From the definitions (28), (A-1) and (A-4) as well as from Eqs. (3) and (5), we obtain

$$O(l)=\langle\tfrac{1}{2}\|f^0\tau\sigma\|\tfrac{1}{2}\rangle Nv(l)(24\omega_l)^{-\frac{1}{2}}l/\mu, \quad (35)$$

where $\langle\frac{1}{2}\|f^0\tau\sigma\|\frac{1}{2}\rangle$ is the reduced matrix element of $f^0\tau\sigma$, as defined by Racah,[3] taken between physical nucleon states. But this last quantity is simply a multiple, say f/f^0, of the same matrix element taken between bare nucleon states:

$$\langle\tfrac{1}{2}\|f^0\tau\sigma\|\tfrac{1}{2}\rangle\equiv\langle\tfrac{1}{2}\|f\tau\sigma\|\tfrac{1}{2}\rangle_{\text{bare}}=6f. \quad (36)$$

From the work of CL, $f^2=0.08$.

The functions M and N are not found as easily. DFZ derive a set of coupled nonlinear integral equations to determine these functions. Solving these equations approximately, they find

$$\begin{pmatrix}M_r^{\frac{1}{2}}(s)\\ M_r^{\frac{3}{2}}(s)\end{pmatrix}=\frac{N^2v(s)v(r)}{4\mu(\omega_s\omega_r)^{\frac{1}{2}}}\left\{2c_0\begin{pmatrix}1\\1\end{pmatrix}+\left[c_1\frac{\omega_r+\omega_s}{\mu}+c_2\frac{\omega_r-\omega_s}{\mu}\right]\begin{pmatrix}-1\\ \frac{1}{2}\end{pmatrix}\right\}, \quad (37)$$

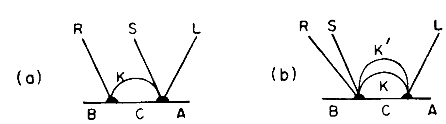

FIG. 4. The meaning of our one-meson approximation as applied to two-meson intermediate states: keep diagrams like (a) and disregard diagrams like (b).

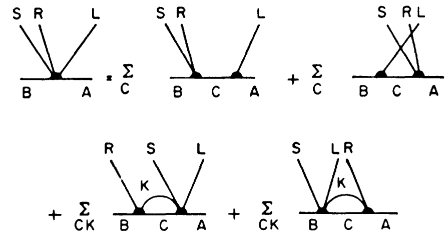

FIG. 5. Equation (27) in terms of diagrams.

$$\begin{pmatrix}N_r^{\frac{1}{2}}(s)\\ N_r^{\frac{3}{2}}(s)\end{pmatrix}=\frac{N^2v(s)v(r)}{4\mu(\omega_s\omega_r)^{\frac{1}{2}}}\left\{2c_0\begin{pmatrix}1\\1\end{pmatrix}-\left[c_1\frac{\omega_r-\omega_s}{\mu}+c_2\frac{\omega_r+\omega_s}{\mu}\right]\begin{pmatrix}-1\\ \frac{1}{2}\end{pmatrix}\right\}, \quad (38)$$

with

$$c_0=0.04;\quad c_1=0.14;\quad c_2\leq 0.01. \quad (39)$$

IV. PHOTOPRODUCTION

In the presence of electromagnetic fields the Hamiltonian H as given by Eqs. (2), (3), and (4) goes over into $H(\mathbf{A})$, where $\mathbf{A}$ is the vector potential. $H(\mathbf{A})$ must have a structure which is gauge-invariant. This means that the following equation must be satisfied[5]:

$$e^{iD}H(\mathbf{A})e^{-iD}=H(\mathbf{A}+\nabla G), \quad (40)$$

where

$$D=\int d\mathbf{r}G(\mathbf{r})\rho(\mathbf{r}). \quad (41)$$

Here $\rho(\mathbf{r})$ is the charge density of the system, given in this case by

$$\rho(\mathbf{r})=\rho_\pi(\mathbf{r})+\tfrac{1}{2}(1+\tau_0)e\delta(\mathbf{r}), \quad (42)$$

where $\rho_\pi(\mathbf{r})$ is the charge density of the mesons and $\frac{1}{2}(1+\tau_0)e\delta(\mathbf{r})$ is the charge density of the nucleon assumed to be located at the origin of the coordinate system.

The function $G(\mathbf{r})$ is a scalar gauge function—any electromagnetic field operator which commutes with $\mathbf{A}$.

If it were not for the source function $u(r)$ the transition $H\rightarrow H(\mathbf{A})$ could be accomplished by the standard prescription

$$\nabla\phi_q(\mathbf{r})\rightarrow(\nabla+ieq\mathbf{A})\phi_q(\mathbf{r}), \quad (43)$$

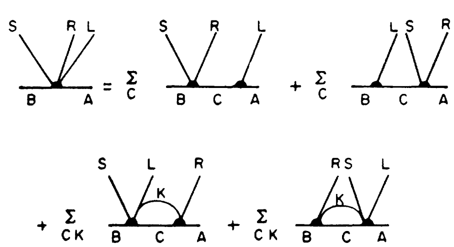

FIG. 6. Equation (32) in terms of diagrams.

[5] R. H. Capps, Phys. Rev. 99, 926 (1955).

where $\phi_q(\mathbf{r})$, $q=-1, 0, +1$, is the meson field amplitude. One easily verifies that the Hamiltonian modified according to Eq. (43) fails to satisfy Eq. (40).

It is well known that the manner in which an extended source theory is made gauge-invariant is not unique.[6] The standard manner is to introduce certain exponential factors as multipliers of the meson field amplitudes $\phi_q(\mathbf{r})$, the conjugate momenta $\pi_q(\mathbf{r})$, and the Pauli matrices τ_q, to imitate the operator properties of the nucleon field amplitudes $\bar{\psi}(\mathbf{r})\cdots\psi(\mathbf{r})$ which were eliminated in favor of the source function $u(r)$.[7]

We shall follow the same course except for taking a different exponential factor. Instead of the substitution (43), we take the following:

$$\begin{aligned} \phi_q(\mathbf{r}) &\rightarrow L(\mathbf{r})\phi_q(\mathbf{r}), \\ \pi_q(\mathbf{r}) &\rightarrow L(\mathbf{r})\pi_q(\mathbf{r}), \\ \tau_q &\rightarrow L(0)\tau_q, \\ \nabla\phi_q(\mathbf{r}) &\rightarrow L(\mathbf{r})[\nabla+ieq\mathbf{A}(\mathbf{r})]\phi_q(\mathbf{r}), \end{aligned} \tag{44}$$

where

$$L(\mathbf{r})\equiv\exp\left[-\frac{ieq}{4\pi}\int\frac{\nabla'\cdot\mathbf{A}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right]; \tag{45}$$

[for the nucleon located not at the origin but at $\mathbf{r}_0$, τ_q is modified not by $L(0)$ but by $L(\mathbf{r}_0)$].

One easily verifies that the Hamiltonian modified in accordance with (44) satisfies Eq. (40) and hence is gauge-invariant,[8] provided one restricts oneself to gauge transformations such that $\nabla G(\mathbf{r})\rightarrow 0$ as $\mathbf{r}\rightarrow\infty$. Since the Hamiltonian is gauge-invariant, we may choose a particular gauge to work in. The advantage of our formulation lies in the fact that if we now choose to work in the Coulomb gauge ($\nabla\cdot\mathbf{A}=0$) all the exponential factors reduce to unity.

Thus, in the Coulomb gauge,

$$H(\mathbf{A})=H+H'', \tag{46}$$

where H is the Hamiltonian given by Eqs. (2), (3), and (4) and

$$H''=\sum_q(-)^q ieq\left[\int\phi_q(\mathbf{r})\mathbf{A}(\mathbf{r})\cdot\nabla\phi_{-q}(\mathbf{r})d\mathbf{r} + (4\pi)^{\frac{1}{2}}(f^0/\mu)\int u(r)\tau_{-q}\boldsymbol{\sigma}\cdot\mathbf{A}(\mathbf{r})\phi_q(\mathbf{r})d\mathbf{r}\right]. \tag{47}$$

We are interested in the element of the transition amplitude corresponding to the absorption of a photon of type k and creation of two s-wave mesons R and S. This is given by

$$\langle BRS^-|H_k''|A\rangle, \tag{48}$$

where H_k'' is the matrix element of H'' taken between states of the radiation field of one photon of type k and no photons. (We are treating electromagnetic effects in perturbation theory.)

Just as ϕ_q was expanded in spherical waves to obtain Eqs. (2), (3), and (4), we now expand $\mathbf{A}(\mathbf{r})$ in spherical transverse vector waves regular at the origin. These are denoted by $\mathbf{M}_{lm}(\mathbf{r})$ and $\mathbf{N}_{lm}(\mathbf{r})$,[9] and we have

$$\mathbf{A}(\mathbf{r})=\sum_k N(2k)^{-\frac{1}{2}}\sum_{lm}(-)^m\{\mathbf{M}_{l-m}(\mathbf{r})[c(klm)+c^\dagger(klm)]+\mathbf{N}_{l-m}(\mathbf{r})[d(klm)+d^\dagger(klm)]\}, \tag{49}$$

where N is a normalization constant and the sum over l starts at 1. The c's and d's are annihilation operators for photons with properties analogous to the a's—meson annihilation operators. The parity of $c(klm)$, $c^\dagger(klm)$ is $-(-)^l$ whereas that of $d(klm)$, $d^\dagger(klm)$ is $(-)^l$.

Since the states $\langle BRS^-|$ and $\langle A|$ in Eq. (48) are both states of even parity and angular momentum $\frac{1}{2}$, H_k'' is obtained from H'' by simply replacing $\mathbf{A}(\mathbf{r})$ by the coefficient of $c(k1m)$, i.e.

$$N(2k)^{-\frac{1}{2}}\mathbf{M}_{1m_k}(\mathbf{r})=N(2k)^{-\frac{1}{2}}\nabla\times[\mathbf{r}Y_{1m_k}(\Omega_r)j_1(kr)]. \tag{50}$$

It then follows that if we expand ϕ_q in spherical waves the only contribution to $\langle BRS^-|H_k''|A\rangle$ comes from the p waves. Hence we may take for H_k'':

$$\begin{aligned} H_k''=\sum_q(-)^q ieqN(2k)^{-\frac{1}{2}}\Big\{ &(4\pi)^{\frac{1}{2}}(f^0/\mu)N \\ &\times\sum_{pm}(2\omega_p)^{-\frac{1}{2}}(-)^m\tau_{-q}[a_q(pm)+a_q^\dagger(pm)] \\ &\times\int u(r)j_1(pr)Y_{1-m}(\Omega_r)\boldsymbol{\sigma}\cdot\mathbf{M}_{1m_k}(\mathbf{r})d\mathbf{r} \\ &+N^2\sum_{pm,p'm'}(4\omega_p\omega_{p'})^{-\frac{1}{2}}(-)^{m+m'}[a_q(pm)+a_q^\dagger(pm)] \\ &\times[a_{-q}(p'm')+a_{-q}^\dagger(p'm')]\int j_1(pr)Y_{1-m}(\Omega_r) \\ &\times\mathbf{M}_{1m_k}(\mathbf{r})\cdot\nabla[j_1(p'r)Y_{1-m'}(\Omega_r)]d\mathbf{r}\Big\}. \end{aligned} \tag{51}$$

The explicit structure of H_k'' as given by Eq. (51) will not be used except for the following features:

(a) H_k'' contains operators for p-wave mesons only;

(b) H_k'' as far as rotations in space are concerned behaves like the m_k-component of an irreducible tensor of rank 1;

(c) H_k'' as far as rotations in isotopic spin space are concerned behaves like the 0-component of an irreducible tensor of rank 1.

Keeping the above features in mind, we now compare $\langle BRS^-|H_k''|A\rangle$ and $\langle BRS^-|[H',a_0^\dagger(km_k)]A\rangle$ and conclude that the following relation (similar to a relation

[6] L. L. Foldy and R. K. Osborne, Phys. Rev. **79**, 795 (1950).

[7] R. H. Capps and R. G. Sachs, Phys. Rev. **96**, 540 (1954); R. H. Capps and W. G. Holladay, Phys. Rev. **99**, 931 (1955).

[8] We note that if the theory is made gauge-invariant in a manner different from ours, additional currents appear; however, at low energies their contribution is negligible.

[9] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1865.

of CL) must exist:

$$\langle BRS^-|H_k''|A\rangle = -\frac{\frac{1}{2}(g_p-g_n)e/2M}{(4\pi)^{\frac{1}{2}}(f/\mu)(k/\omega_k)^{\frac{1}{2}}} \times\langle BRS^-|[H',a_0^\dagger(km_k)]A\rangle. \quad (52)$$

Here M stands for the mass of the nucleon and g_p (g_n) is the anomalous gyromagnetic ratio of the proton (neutron). (CL use here the full static magnetic moments, and not just the anomalous ones—the theory is not accurate enough to decide this question.)

Equation (52) yields immediately the photoproduction transition amplitude once the Eqs. (33) and (34) are solved.

V. RESULTS

We were unable to solve Eqs. (33) and (34) exactly but obtained approximate solutions. Near threshold, the use of the approximate solution is estimated to result in an error of less than 25%. If we define the function $f_s(r)$ by

$$F_s(r,l)\equiv\frac{N^2}{12\mu^2}\left(\frac{O(l)}{(\omega_r\omega_s)^{\frac{1}{2}}}\right)f_s(r), \quad (53)$$

then our solution can be written in terms of the $f_s(r)$ as follows:

$$\begin{array}{ccc} \omega_r+\omega_s= & 2.25\,\mu & 2.5\,\mu \\ f_s(r)= & \begin{pmatrix}-0.223\\0.087\\-0.114\\-0.013\end{pmatrix} & \begin{pmatrix}-0.241\\0.085\\-0.123\\-0.016\end{pmatrix}. \end{array} \quad (54)$$

We see from Eq. (54) that $f_s(r)$ is essentially constant near threshold, being independent of ω_r and ω_s. In obtaining the numbers given by Eq. (54) it was assumed that the cut-off functions $v(s)$, $v(r)$, etc., are equal to unity up to the cut-off energy $\omega_{\max}=4.5\mu$ and equal to zero for higher energies.

In order to calculate cross sections, it is necessary to form actual physical states from the eigenstates of the total isotopic spin so far considered. Thus, suppose that we are interested in the reaction

$$p+\pi^-\to n+\pi^-+\pi^+. \quad (55)$$

The initial state of the system can be written in terms of isotopic spin states as

$$|p\pi^-\rangle = -(\tfrac{2}{3})^{\frac{1}{2}}|\tfrac{1}{2}\rangle+(\tfrac{1}{3})^{\frac{1}{2}}|\tfrac{3}{2}\rangle, \quad (56)$$

and the final state can be written as

$$|n\pi^-\pi^+\rangle=(\tfrac{1}{2})^{\frac{1}{2}}|\tfrac{1}{2}(\tfrac{3}{2})\rangle+(\tfrac{2}{5})^{\frac{1}{2}}|\tfrac{3}{2}(\tfrac{3}{2})\rangle+(\tfrac{1}{10})^{\frac{1}{2}}|\tfrac{5}{2}(\tfrac{3}{2})\rangle, \quad (57)$$

where we omit the symbols for the z component of the isotopic spin and the quantity in brackets refers to the isotopic spin of the subsystem. Hence

$$F_{p\pi^-\to n\pi^-\pi^+}=-(\tfrac{1}{3})^{\frac{1}{2}}F_s^{\frac{1}{2}\frac{3}{2}}(r,l)+(2/15)^{\frac{1}{2}}F_s^{\frac{3}{2}\frac{3}{2}}(r,l). \quad (58)$$

TABLE I. $d\sigma/d\omega_r$, the cross section per unit energy of one of the produced mesons, in millimicrobarns/Mev.

Process \ E	2.25 μ	2.5 μ
$p+\pi^-\to p+\pi^-+\pi^0$	0.08	0.27
$\to n+\pi^-+\pi^+$	0.68	1.5
$\to n+\pi^0+\pi^0$	0.86	1.9
$p+\gamma\to p+\pi^0+\pi^0$	6.00×10^{-3}	16.3×10^{-3}
$\to p+\pi^++\pi^-$	0.39×10^{-3}	0.80×10^{-3}
$\to n+\pi^0+\pi^+$	0.02×10^{-3}	0.07×10^{-3}

Similarly one may change from eigenstates of the total angular momentum to states with plane-wave mesons. After this transformation, the density of final states per unit energy interval is given by

$$(d^2n/dEd\omega_r)d\omega_r=4V^2(2\pi)^{-4}rs\omega_r\omega_s d\omega_r, \quad (59)$$

where V is the volume in which the plane wave mesons are quantized and $E=\omega_s+\omega_r=$ energy of incident meson or photon (i.e., we neglect any recoil of the nucleon).

Expression (59) vanishes at the two limits $\omega_r=\mu$ and $\omega_s=\mu$, and reaches its peak value at $\omega_r=\omega_s=E/2$. Using this peak value we obtain the maximum value for the cross sections. In Table I we list $d\sigma/d\omega_r$ (cross section per unit energy of one of the produced mesons) for a number of possible reactions and for two values of E near threshold.

There are as yet no data available with which to compare Table I. Some experimental data exist for higher energies, in particular for the reaction $p+\gamma\to p+\pi^++\pi^-$.[10] Cutkosky and Zachariasen[11] obtain a good fit to these data by using the theory of Chew and Low (no interactions for s waves); their numbers are roughly 10 000 times larger than ours. One reason why our numbers are so much smaller is that in order to produce two s-wave mesons the photon must be absorbed by the nucleon, whereas an s- and a p-wave meson (as assumed by Cutkosky and Zachariasen) can be produced by having one of the mesons absorb the photon. For this reason our cross sections should be smaller than theirs by a factor of the order $(M/\mu)^2\sim50$. The fact that our cross sections are smaller yet must be blamed on the weakness of s-wave interactions as compared with p-wave interactions.

We conclude that the s-wave pair production may be neglected except possibly at the very threshold when phase space inhibits very strongly the production of an s-wave and a p-wave. We note that a meson-meson interaction (not considered in this work) could change the above results.

ACKNOWLEDGMENTS

The author takes pleasure in acknowledging his great indebtedness to Professor Sidney D. Drell for his

[10] R. M. Friedman and K. M. Crowe, Phys. Rev. **100**, 1799 (1955).

[11] R. K. Cutkosky and F. Zachariasen, Phys. Rev. **103**, 1108 (1956).

initiating and continuing stimulus and contributions to this work.

APPENDIX

The functions O, M, N, F, and E have been so defined that they behave as scalars under rotations in both space and isotopic spin space. Therefore their matrix elements taken between eigenstates of the total angular momentum J and total isotopic spin T are independent of the magnetic quantum numbers. Since the value of J is always $\frac{1}{2}$ (the final state being a state of a nucleon and two s-wave mesons), the elimination of the angular momentum quantum numbers is very simple. We rewrite all state vectors as eigenstates of the total angular momentum $\langle\frac{1}{2}\alpha'|$ and then ignore this trivial dependence by denoting

$$\langle\tfrac{1}{2}\alpha'|X|\tfrac{1}{2}\alpha'\rangle \text{ by } X. \tag{A-1}$$

Next we concentrate on the values of the isotopic spin T. To specify completely states consisting of a nucleon and two mesons, it is not enough to specify the total isotopic spin—we must in addition specify the isotopic spin of the subsystem consisting of the nucleon and one of the two mesons.

Let $\langle T\tau(T'\tau')|$ denote an eigenstate with total isotopic spin T (z component$=\tau$) formed from a subsystem with isotopic spin T' (z component$=\tau'$). Then we define

$$F_s^{TT'}(r,l)\equiv\langle T\tau(T'\tau')|F_s(r,l)|T\tau\rangle, \tag{A-2}$$

$$E_s^{TT'}(r,l)\equiv\langle T\tau|E_s(r,l)|T\tau(T'\tau')\rangle, \tag{A-3}$$

where the subsystem is formed with the meson whose symbol appears first inside the bracket (i.e., meson R in above definitions). Since matrix elements of O and N are always taken between states of total $T=\frac{1}{2}$, we define further

$$O(l)\equiv\langle\tfrac{1}{2}\tau|O(l)|\tfrac{1}{2}\tau\rangle, \tag{A-4}$$

$$N_r^{T'}(s)\equiv(-)^{T'-\frac{1}{2}}\left(\frac{2}{2T'+1}\right)^{\frac{1}{2}}\langle\tfrac{1}{2}\tau(T'\tau')|N_r(s)|\tfrac{1}{2}\tau\rangle. \tag{A-5}$$

The phase and normalization in Eq. (A-5) are chosen for convenience, the subsystem being formed with the meson whose symbol appears inside the angular brackets (i.e., meson S in above). Finally we define

$$M_r^T(s)\equiv\langle T\tau|M_r(s)|T\tau\rangle. \tag{A-6}$$

The possible values of T and T' are $\frac{1}{2}$ and $\frac{3}{2}$—hence there are four different functions $F^{TT'}$ and $E^{TT'}$. We define therefore the following four-row, one-column matrices:

$$F=\begin{bmatrix}F^{\frac{1}{2}\frac{1}{2}}\\F^{\frac{1}{2}\frac{3}{2}}\\F^{\frac{3}{2}\frac{1}{2}}\\F^{\frac{3}{2}\frac{3}{2}}\end{bmatrix},\quad E=\begin{bmatrix}E^{\frac{1}{2}\frac{1}{2}}\\E^{\frac{1}{2}\frac{3}{2}}\\E^{\frac{3}{2}\frac{1}{2}}\\E^{\frac{3}{2}\frac{3}{2}}\end{bmatrix}. \tag{A-7}$$

We also define the following diagonal matrices:

$$M=\begin{bmatrix}M^{\frac{1}{2}}&0&0&0\\0&M^{\frac{3}{2}}&0&0\\0&0&M^{\frac{1}{2}}&0\\0&0&0&M^{\frac{3}{2}}\end{bmatrix},\quad N=\begin{bmatrix}N^{\frac{1}{2}}&0&0&0\\0&N^{\frac{3}{2}}&0&0\\0&0&N^{\frac{1}{2}}&0\\0&0&0&N^{\frac{3}{2}}\end{bmatrix}. \tag{A-8}$$

The remaining matrices in Eqs. (33) and (34) are numerical matrices whose elements are combinations of Racah coefficients resulting from elimination of the magnetic quantum numbers:

$$\Lambda=-\tfrac{2}{9}\begin{bmatrix}4\\2\sqrt{2}\\1\\-\sqrt{5}\end{bmatrix};$$

$$\Delta=\tfrac{1}{9}\begin{bmatrix}1&2\sqrt{2}&8&4\sqrt{5}\\2\sqrt{2}&8&-2\sqrt{2}&-\sqrt{10}\\4&-\sqrt{2}&5&-2\sqrt{5}\\2\sqrt{5}&-\sqrt{(5/2)}&-2\sqrt{5}&4\end{bmatrix}. \tag{A-9}$$

The crossing matrix Δ has the usual property

$$\Delta\Delta=1. \tag{A-10}$$