Multiple Scattering by Quantum-Mechanical Systems*

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A simple description is given of the quantum-mechanical theory of multiple scattering. The separation of the scattered wave into "coherent" and "incoherent" parts is discussed in greater generality than has been done previously. Applications to transport theory are described. Specific calculations are made of the refractive index of a medium which is "polarized" by the scattered particle (Lorentz-Lorenz formula) and also of a medium which has correlated structure (critical opalescence). Other applications are given.

I. INTRODUCTION

IN several previous papers, a technique was developed for the description of the scattering of a given particle by a system of particles.[1–3] The derivations given were somewhat complex and lengthy. In the present discussion, a much simpler and more straightforward development will be given of the previous results. At the same time, a more comprehensive study of the separation of the scattered wave into "coherent" and "incoherent" parts[1] will be made.

In an important series of papers, Brueckner and his collaborators[4–6] have applied techniques of scattering theory to the formulation of the energy eigenvalue problem for many-particle quantum-mechanical systems in connection with a theory of the structure of atomic nuclei.[7] An extension of these methods has been made to statistical mechanics,[8] including a "nearest neighbor" expansion for the energy of a homogeneous system.[9] These techniques for calculating the energy of a many-particle system are in many respects similar to those used for the multiple scattering problem. There are important differences, however, which we wish to emphasize.

To aid in the understanding of the multiple scattering problem, we should like to discuss a number of applications. These include a description of the propagation of a wave packet through a sequence of scatterings, a derivation of a "Lorentz-Lorenz formula" for the index of refraction, and a discussion of the energy levels of the π-mesonic atom.[10]

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[2] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).
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[4] Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954).
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[6] K. Brueckner, Phys. Rev. **100**, 36 (1955).
[7] A discussion of the Brueckner method has recently been given by H. A. Bethe, Phys. Rev. **103**, 1353 (1956)[8]
[8] K. M. Watson, Phys. Rev. **103**, 489 (1956).
[9] W. B. Riesenfeld and K. M. Watson, Phys. Rev. **102**, 492 (1956).
[10] A knowledge of references 1, 2, and 3 is not necessary for the reading of the present paper.

II. FORMAL DEVELOPMENT OF THE MULTIPLE SCATTERING PROBLEM

The notation of reference 2 will be followed insofar as possible. The scattering medium consists of a *large* number N of equivalent particles.[11] This medium is supposed to have an angular momentum J with a z component $J_z=M$. Its states will be represented by the indices "(γ,M)" and its wave functions by

$$g_{\gamma,M}(\xi), \tag{1}$$

where ξ is some appropriate set of many-particle coordinates. If H_N is the Hamiltonian for the medium, then

$$H_N g_{\gamma,M}=W_\gamma g_{\gamma,M}, \tag{2}$$

where W_γ is the energy of the medium when it is in the state (γ,M). It will be convenient to suppose that the energy W_γ is not degenerate, except for the $(2J+1)$ values of M. This restriction is in no sense important for our following discussion, however.

The incident particle is described by a complete set of plane wave functions λ_q, where $q=(\mathbf{q},\nu)$ and $\mathbf{q}$ is its momentum and ν is its spin orientation. Then

$$h\lambda_q=\epsilon_q\lambda_q. \tag{3}$$

Here h is the kinetic energy operator for this particle.

The interaction of the incident particle with the target medium is written as

$$V=\sum_{\alpha=1}^{N} V_\alpha, \tag{4}$$

where V_α is its interaction with the αth particle in the medium. If we now define

$$H_0\equiv H_N+h, \tag{5}$$

the Schrödinger equation which describes the scattering is

$$(H_0+V)\Psi_a=E_a\Psi_a. \tag{6}$$

Equation (6) has the boundary condition that at large distances from the scatterer

$$\Psi_a\rightarrow g_{\gamma_0,M_0}\lambda_{q0}\equiv|a\rangle. \tag{7}$$

[11] This restriction may easily be relaxed. We shall usually suppose our units chosen so that $\hbar=1$ in the following.

Here (γ_0, M_0) represents the "ground state" (or, more generally, the *initial state*) of the scattering medium. The scattering will, in general, lead to changes in both the states (γ_0, M_0) and q_0.

In the usual manner, the Møller wave matrix Ω is introduced as follows:

$$\Psi_a = \Omega g_{\gamma_0, M_0} \lambda_{q0} = \Omega | a \rangle. \tag{8}$$

A relatively complete discussion of the formal features of the scattering was given in Sec. II of reference 2. This included a very general definition of the "optical model" problem, or that of determining the refractive index of the medium. We shall not attempt to reproduce that discussion here.

The Schrödinger equation (6) for Ψ_a may be converted, as is customary,[12] into an integral equation for Ω:

$$\Omega = 1 + \frac{1}{a} V \Omega, \tag{9}$$

where

$$a \equiv E_a + i\eta - H_0. \tag{10}$$

(Here η is the usual[12] positive, infinitesimal parameter introduced for performing integrations across the pole of a^{-1}.)

We wish, at the outset, to separate the solution Ω to Eq. (9) into parts "coherent" and "incoherent" with respect to the incident wave. For this purpose, we write

$$\Omega = F \Omega_C, \tag{11}$$

and define F and Ω_C as the solutions to the equations:

$$F = 1 + \frac{1}{a - \mathcal{O}} (V - \mathcal{O}) F, \quad \Omega_C = 1 + \frac{1}{a} \mathcal{O} \Omega_C. \tag{12}$$

The matrix $\mathcal{O}$ will be specified presently; for the moment, we define it to be *diagonal* with respect to the indices "γ" describing the medium:

$$\begin{aligned} (\gamma' M' | \mathcal{O} | \gamma M) &= \delta_{\gamma' \gamma} (\gamma M') | \mathcal{O} | \gamma M), \\ (\gamma' M' | \Omega_C | \gamma M) &= \delta_{\gamma' \gamma} (\gamma M' | \Omega_C | \gamma M). \end{aligned} \tag{13}$$

One readily verifies that Eqs. (11) and (12) provide a solution to the Schrödinger equation (9). We shall refer to Ω_C and $(F-1)\Omega_C$ as the "coherent" and the "incoherent" waves, although this departs somewhat from customary notation. For instance, if the spin orientation of the scattering medium or of the scattered particle is changed, as may be permitted by Eqs. (13), the scattered wave will not interfere with the incident wave.[13] For experimental reasons, it seems desirable, nevertheless, to keep the "spin flip" amplitudes in $\mathcal{O}$ and Ω_C.[14] Were the degeneracy of the energy W_γ greater than we have assumed, transitions among the extra states might also be kept in $\mathcal{O}$ if so desired.

To continue, we wish to rewrite the first of Eqs. (12) for F as

$$F = 1 + \frac{1}{a - \mathcal{O}} PVF, \tag{14}$$

where P is an operator acting on VF which is yet to be specified.[15] In order that Eq. (14) provide a solution to the first of Eqs. (12), we must have

$$(1-P) VF\Omega_C | a \rangle = \mathcal{O} F \Omega_C | a \rangle, \tag{15}$$

as is easily verified by substitution of (14) into (12). Here we have anticipated that F is to operate on $\Omega_C | a \rangle$, according to Eqs. (8) and (11).

Equation (15) represents the fundamental restriction on the operators P and $\mathcal{O}$ which must be satisfied if Eq. (14) is to provide a solution to the original Schrödinger equation.

Equation (14) is now of convenient form for introducing a "multiple scattering" solution.[1] Indeed, we easily may verify by substitution that Eq. (14) is exactly satisfied by

$$\begin{aligned} F &= 1 + \frac{1}{d} \sum_{\alpha=1}^{N} P t_\alpha F_\alpha, \\ F_\alpha &= 1 + \frac{1}{d} \sum_{\beta \neq \alpha}^{N} P t_\beta F_\beta, \\ t_\alpha &= V_\alpha + V_\alpha \frac{1}{d} P t_\alpha. \end{aligned} \tag{16}$$

Here

$$d = a - \mathcal{O}, \tag{17}$$

and the two-body interactions V_α were introduced in connection with Eq. (4). In the next section, we shall discuss the interpretation of these equations, including the "two-body" scattering operators t_α.

To satisfy Eq. (15), a number of choices for P are possible. For instance, to obtain the solution of reference 1, we set

$$P = P_{ND}, \quad P_{ND} t_\alpha = I_\alpha, \tag{18}$$

where P_{ND} acting on t_α *vanishes* for elements diagonal in the γ states and is otherwise unity. Explicitly,

$$\begin{aligned} P_{ND} (\gamma' M' | t_\alpha | \gamma M) &= (\gamma' M' | t_\alpha | \gamma M) \quad \text{for} \quad \gamma' \neq \gamma \\ &= 0 \quad \text{for} \quad \gamma' = \gamma. \end{aligned} \tag{19}$$

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[12] See, for instance, B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950) or M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[13] We might, of course, have defined $\mathcal{O}$ to be also diagonal with respect to M' and ν if this were desirable.

[14] As was observed in reference 2, this seems particularly useful for the description of the elastic scattering of particles by nuclei. More recently, this point has been developed in greater detail: for example, E. Fermi, Nuovo cimento **11**, 407 (1954).

[15] A similar operator and technique have been used in references 1 and 2, but in a quite different notation. The present notation was introduced in reference 9.

P_{ND} does not, however, act on the states q of the scattered particle. Now,

$$(1-P_{ND})t_\alpha \equiv t_{C\alpha}, \qquad (19')$$

where $t_{C\alpha}$ represents the matrix elements of t_α which are *diagonal* with respect to the states γ.

To see if we can satisfy Eq. (15), we note the general relation[16]

$$VF=\sum_\alpha t_\alpha F_\alpha. \qquad (20)$$

Thus

$$(1-P_{ND})VF\Omega_C|a\rangle=\sum_\alpha t_{C\alpha}F_\alpha\Omega_C|a\rangle. \qquad (21)$$

This may be re-expressed as (for brevity, we shall frequently omit matrix indices other than γ)

$$\langle\gamma|\sum_\alpha t_{C\alpha}F_\alpha|\gamma_0\rangle=\langle\gamma|\sum_\alpha t_\alpha|\gamma\rangle\left\langle\gamma\left|1+\sum_{\alpha_1\neq\alpha}\frac{1}{d}I_{\alpha_1}+\sum_{\alpha_1\neq\alpha}\frac{1}{d}I_{\alpha_1}\sum_{\alpha_2\neq\alpha_1}\frac{1}{d}I_{\alpha_2}+\cdots\right|\gamma_0\right\rangle$$

$$=\langle\gamma|\sum_\alpha{}' t_\alpha|\gamma\rangle\langle\gamma|F|\gamma_0\rangle. \qquad (22)$$

Here $\sum_\alpha{}'$ means that we omit from the sum over α that *one term* which occurred in the *previous* scattering in F (if there was a previous scattering). The last step in Eq. (22) involves only a reordering of the sums taken.

From Eqs. (21) and (22) it is clear that Eq. (15) is satisfied if we take (again suppressing indices other than γ)

$$\langle\gamma'|\mathcal{O}|\gamma\rangle=\delta_{\gamma'\gamma}\langle\gamma|\sum_\alpha{}' t_\alpha|\gamma\rangle. \qquad (23)$$

When N is large we can usually simplify Eq. (23) in that $\sum_\alpha{}'$ can be replaced by $\sum_\alpha$. In doing this we violate Eq. (22) by adding a redundant term of $O(1/N)$. That is, the added terms are of the form

$$\sum_\alpha\langle\gamma|t_{C\alpha}|\gamma\rangle\left\langle\gamma\left|\frac{1}{d}I_\alpha\cdots\Omega_C\right|a\right\rangle\simeq O(1/N), \qquad (24)$$

since the t_α's are of $O(1/N)$ (this will be demonstrated in Sec. III). Then we may take

$$\langle\gamma'M'q'|\mathcal{O}|\gamma Mq\rangle=\sum_\alpha t_{C\alpha}=\delta_{\gamma'\gamma}\langle\gamma M'q'|\sum_\alpha t_\alpha|\gamma Mq\rangle. \qquad (25)$$

Equation (25) appears to be an adequate approximation to Eq. (23) except for media having a definite crystalline structure. In this case a *second* scattering from the *omitted* particle in Eq. (23) may be heavily weighted in Eq. (25). When this happens the *correct* Eq. (23) must be used.

With Eq. (25) we have obtained the form of the multiple scattering equations introduced in reference 1 and used subsequently.

Except for the obvious possible choice $P=1$, $\mathcal{O}=0$, Eq. (18) provides the simplest form of P which has been found. We may, however, easily generalize Eq. (18) to a class of operators which control virtual states. For example, we may forbid the repetition of a γ state after one or two scatterings, one or two or three scatterings, etc. Carrying this to the limit of all previous scatterings, we obtain

$$P=P_0, \qquad (26)$$

where P_0 forbids any repetition whatsoever of a γ state which has occurred after *any* previous scattering (that is, following any previous t matrix). In other words, $\langle\gamma|P_0t_\alpha(1/d)t_\beta\cdots|\gamma_0\rangle$ vanishes for terms in which γ appears as an intermediate state following a t operator. When γ has not appeared previously, $P_0t_\alpha=t_\alpha$ in this expression.

The converse of this is expressed by the statement that

$$\langle\gamma|(1-P_0)\sum_\alpha t_\alpha F_\alpha|\gamma_0\rangle$$

has the state γ occurring someplace prior to the last scattering. It may occur at one previous scattering, before the second previous scattering, etc. Formally, this may be written explicitly as [see Eq. (20)]

$$\langle\gamma|(1-P_0)VF|\gamma_0\rangle=\langle\gamma|(1-P_0)\sum_\alpha t_\alpha F_\alpha|\gamma_0\rangle$$

$$=\sum_\alpha\langle\gamma|t_\alpha|\gamma\rangle\left\langle\gamma\left|1+\sum_{\alpha_1\neq\alpha}\frac{1}{d}P_0t_{\alpha_1}+\cdots\right|\gamma_0\right\rangle$$

$$+\sum_\alpha\sum_{\alpha_1\neq\alpha}\left\langle\gamma\left|t_\alpha\frac{1}{d}P_0t_{\alpha_1}\right|\gamma\right\rangle$$

$$\times\left\langle\gamma\left|1+\sum_{\alpha_2\neq\alpha_1}\frac{1}{d}P_0t_{\alpha_2}+\cdots\right|\gamma_0\right\rangle+\cdots. \qquad (27)$$

Now, each of the factors on the right is just $\langle\gamma|F|\gamma_0\rangle$ except for the omission of a single scattering. Just as was done in Eq. (22), we may rearrange the sums so as to make these factors equal to $\langle\gamma|F|\gamma_0\rangle$ by omitting the term from the first factor instead. (The origin of this restricted summation is of course due to the restriction that no *two successive* scatterings may be from the same particle.)

Then Eq. (27) may be rewritten as

$$\langle\gamma|(1-P_0)VF|\gamma_0\rangle=\left\{\left\langle\gamma\left|\sum_\alpha t_\alpha\left[1+\sum_{\alpha_1\neq\alpha}\frac{1}{d}P_0t_{\alpha_1}+\sum_{\alpha_1\neq\alpha}\frac{1}{d}P_0t_{\alpha_1}\sum_{\alpha_2\neq\alpha_1}\frac{1}{d}P_0t_{\alpha_2}+\cdots\right]'\right|\gamma\right\rangle\right\}\langle\gamma|F|\gamma_0\rangle$$

$$=\langle\gamma|\sum_\alpha t_\alpha F_\alpha{}'|\gamma\rangle\langle\gamma|F|\gamma_0\rangle. \qquad (28)$$

Here the notation $\cdots'|\gamma\rangle$ means that we must restrict the summation to the left in such a manner that the *first* scattering in $F_\alpha|\gamma\rangle$ does not occur from the *same particle* as the *last* scattering in $\langle\gamma|F|\gamma_0\rangle$. [This is of course just what we did in Eq. (23).]

[16] Equation (20) is easily verified, using Eqs. (16).

Again, the single scattering omitted in Eq. (28) is of the order of

$$\sum_\alpha \langle\gamma|t_\alpha|\gamma\rangle\left\langle\gamma\left|\frac{1}{d}P_0t_\alpha\cdots\right|\gamma_0\right\rangle \simeq O\left(\frac{1}{N}\right), \quad (29)$$

except for crystalline media. Thus we may ordinarily ignore the restricted summation in Eq. (28).

It is clear from Eq. (28) that Eq. (15) is satisfied by taking

$$\langle\gamma'M'q'|\mathcal{O}|\gamma Mq\rangle = \delta_{\gamma'\gamma}\langle\gamma M'q'|\textstyle\sum_\alpha t_\alpha F_\alpha'|\gamma Mq\rangle \equiv \mathcal{V}_C. \quad (30)$$

This is just the "optical model potential," as introduced in reference 2.

Other forms for the operator P are possible. For instance P_{ND}, P_0, etc. might have been defined with respect to states other than the states γ. To illustrate this we consider a medium with particle-particle correlations extending over distances comparable to, or larger than, the wavelength of the incident particle. Let us suppose the energy of the incident particle to be large compared to the spacing of states of the medium. Following a collision which "knocks a particle from the medium," the medium requires a certain relaxation time to settle into a new eigenstate. This process is described by a "wave packet" of γ states. The scattered particle than travels in an "optical potential" appropriate to the "wave-packet state" following the scattering—and because of the assumed correlations, this potential may be different near the "hole" left by the scattering than elsewhere in the medium.[17] This modified optical model potential is described most naturally by defining P_{ND} as nondiagonal with respect to this wave-packet state.

In several respects, our formalism is more general than is indicated by our application. For example, the index α may refer to some other property of a system than its "particles."

It is evident, in accordance with Eq. (11), that the operator P_0 effects a separation into "coherent" and "incoherent" waves (as discussed above). That is, when $P=P_0$, F satisfies

$$\langle\gamma|F|\gamma\rangle = 1. \quad (31)$$

(We again suppress the indices $(M,\mathbf{q},\nu)$, since P_0 does not affect these.) Consequently,

$$\langle\gamma|\Omega|\gamma\rangle = \langle\gamma|\Omega_C|\gamma\rangle \quad (32)$$

and

$$\Psi_{Ca} \equiv \Omega_C|a\rangle \quad (33)$$

describes the elastic scattering. The wave function of the medium, $g_{\gamma 0}$, is a factor of Eq. (33) and may be removed by defining

$$\Phi_{Ca} = (g_{\gamma_0}, \Psi_{Ca}). \quad (34)$$

From Eq. (12), we see that Φ_{Ca} satisfies[2] the single-particle "Schrödinger equation"

$$[h+\mathcal{V}_C]\Phi_{Ca} = \epsilon_{q0}\Phi_{Ca}, \quad (35)$$

depending only on the variables $(M,\mathbf{q},\nu)$. Here $\mathcal{V}_C = \langle M'\mathbf{q}'\nu'|\mathcal{V}_C|M\mathbf{q}\nu\rangle$ defines the "optical model."

For a large, uniform medium, $\mathcal{V}_C$ takes the approximate form

$$\mathcal{V}_C \simeq \delta(\mathbf{q}'-\mathbf{q})\langle M'\mathbf{q}\nu'|v_C|M\mathbf{q}\nu\rangle. \quad (36)$$

When $\mathcal{V}_C$ does not depend on the spin orientations (M,ν), we may also define an *index of refraction* n for the medium. In the nonrelativistic case, $h=q^2/2M$, $\epsilon_{q0}=q_0^2/2M$, and Eq. (35) reduces to

$$q^2+2Mv_C = q_0^2,$$

or

$$(q/q_0)^2 \equiv n^2 = 1-(2M/q_0^2)v_C. \quad (37)$$

(The relativistic case was treated in references 1 and 2.)

Similarly, $(F-1)$ describes the *inelastic* scatterings. Since $\mathcal{O}=\mathcal{V}_C$ appears in the propagator d^{-1}, we see that between inelastic scatterings, the particle "propagates in a dispersive medium."

Since $\mathcal{V}_C$ is in general complex, Eq. (37) implies that the "momentum vector" $\mathbf{q}$ is also complex. It is important to note that the complete set of plane wave functions λ_q (Eq. (3)) in terms of which our operators are described does not, of course, involve complex $\mathbf{q}$'s. Equation (37) appears only when we evaluate integrals over the λ_q states. Since the wave propagates as $e^{i\mathbf{q}\cdot\mathbf{x}}$ in the medium, it is convenient to consider the complex $\mathbf{q}$ as a momentum, however.

III. INTERPRETATION OF THE SCATTERING EQUATIONS

In this section, we should like to discuss the operators t_α and to make some descriptive comments concerning the inelastic scattering operator F.

We have remarked in connection with Eqs. (24) and (28) that the t_α are of $O(1/N)$. It seems apparent from Eq. (16) that a useful choice for P and $\mathcal{O}$ will give t_α and V_α the same order of magnitude—and we shall assume this to be the case (unless V_α has singular matrix elements). Let us also suppose the scattering medium to occupy a volume $\mathcal{V}$. The average particle density in the medium is then

$$\rho_0 \equiv N/\mathcal{V}, \quad (38)$$

which we consider to be not necessarily "large or small." The local density of particles in the scattering medium is then (we consider $\mathbf{z}_\alpha$ to be the space coordinate of the αth scatterer)

$$\rho(z_\alpha) = N\int g_{\gamma_0}^* g_{\gamma_0}[d\tau]_\alpha, \quad (39)$$

[17] It is a pleasure to acknowledge that this phenomenon was called to my attention by Professor M. A. Ruderman. A more complete discussion is given in the appendix.

where by $[d\tau]_\alpha$ we mean that d^3z_α has been omitted from the volume element of integration. We also consider that

$$\rho(z_\alpha)\simeq\rho_0, \tag{40}$$

when $\mathbf{z}_\alpha$ is in $\mathcal{V}$.

The order of magnitude of $t_{C\alpha}$ is then, since we are taking $t_\alpha=O(V_\alpha)$:

$$\begin{aligned}\langle p'|t_{C\alpha}|p\rangle &\simeq\langle\gamma_0 p'|V_\alpha|\gamma_0 p\rangle\\ &=\frac{1}{(2\pi)^3}\int g_{\gamma_0}{}^*g_{\gamma_0}d\tau V(\mathbf{x}-\mathbf{z}_\alpha)e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}d^3x\\ &=\frac{(p'|V|p)}{N}\int\rho(z_\alpha)d^3z_\alpha\exp[-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{z}_\alpha],\end{aligned} \tag{41}$$

where

$$(p'|V|p)=(2\pi)^{-3}\int d^3xV(x)e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}. \tag{42}$$

Actually, of course, we should replace $(p'|V|p)$ by the exact $\langle p'|t|p\rangle$ in Eq. (41).

Except for crystalline media, when $(\mathbf{p}'-\mathbf{p})$ satisfies a Bragg condition, the integral in Eq. (41) is of order unity compared to N; so

$$t_C=O(1/N),$$

as was stated in Sec. II. Even for crystalline materials, the integral in Eq. (41) is effectively of order unity when either $\mathbf{p}'$ or $\mathbf{p}$ is an integration variable (as was the case in our applications in Sec. II). Finally, the matrix elements of the t_α for inelastic scattering will not be of a larger order of magnitude than those of t_C.

The discussion just given shows that we can replace the defining Eq. (16) for t_α by

$$t_\alpha=V_\alpha+V_\alpha\frac{1}{d}t_\alpha. \tag{43}$$

That is, we can in general omit the operator P. The reason for this is that P deletes certain γ states from the sum over virtual states. Any finite number (actually, any number $\ll N$) of such states will not affect the value of Eqs. (16) and (43). To see this, we write Eq. (16) as

$$t_\alpha=V_\alpha+V_\alpha\frac{1}{d}t_\alpha+V_\alpha\frac{1}{d}(1-P)t_\alpha,$$

and consider the last term as a perturbation. We have just seen that $t_\alpha\simeq V_\alpha=O(1/N)$. Thus the last term above is of $O(1/N^2)$ unless the number of γ-states permitted by $(1-P)$ is of $O(N)$. This is manifestly not so for $P=P_{ND}$. It will also not be true for $P=P_0$ unless the scattered particle makes a number of *inelastic* scatterings which is of the order of the entire number of particles in the medium. For cases of practical interest to a multiple scattering theory, this would not seem likely to occur. Henceforth, we shall, therefore, consider Eq. (43) as replacing Eq. (16).

It will also be sometimes possible to at least partially ignore the P operator even when it stands between two t_α's.

The approximation of Eq. (43) is closely related to the "impulse approximation" of Chew, Wick, and Goldberger[18] and would seem to be valid whenever the impulse approximation is. Indeed, we should like to reformulate the impulse approximation as that by which Eq. (16) for t_α is replaced by the two-body equation

$$\begin{aligned}t_\alpha{}^0&=V_\alpha+V_\alpha\frac{1}{d^0}t_\alpha{}^0,\\ d^0&\equiv\epsilon_{q0}{}^0+i\eta-h-h_\alpha-\langle\gamma|\mathcal{O}|\gamma\rangle,\\ \epsilon_{q0}{}^0&\equiv\epsilon_{q0}-(W_\gamma-W_{\gamma0}).\end{aligned} \tag{44}$$

Here we suppose γ to be the state of the medium at the time the scattering begins to take place and h_α to be the kinetic energy of particle α. For the impulse approximation to be valid, this state γ is not necessarily left unchanged. That is, we suppose there to be many γ states which lie close together in energy and which describe the recoil of the particle α—and which leave the remainder of the medium essentially unchanged. Then the evaluation of $\langle\gamma|\mathcal{O}|\gamma\rangle$ for the single state γ is valid in Eq. (44).

To see the conditions under which Eq. (44) is valid, we define

$$\Delta W\equiv-[d-d^0]. \tag{45}$$

ΔW then represents the *excitation* of the medium (excepting the kinetic energy of particle α) during the single scattering. Expanding Eq. (16) for t_α to first order in ΔW and using Eq. (44), we easily obtain [neglecting P, as implied by Eq. (43)]

$$t_\alpha\simeq\left[1+t_\alpha{}^0\frac{1}{d^0}\Delta W\frac{1}{d^0}\right]t_\alpha{}^0. \tag{46}$$

The second factor in square brackets is just the Chew-Wick-Goldberger[18] correction term. Its order of magnitude is expected to be[18]

[fluctuation in potential energy of particle α due to the remainder of the medium]$\times$[energy of the incident particle $(=\epsilon_{q0})$]$^{-1}$. (47)

When this ratio is small, the impulse approximation is valid—which means that we neglect the effect of the medium on particle α during the scattering. For the applications in this paper, we shall assume the validity of the impulse approximation. (At the same time, we shall not bother usually to keep the superscript 0 on $t_\alpha{}^0$.)

Finally, the scattering operator $t_\alpha{}^0$ may lead to transitions of the medium from a state γ to a state γ'.

[18] G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952); G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

According to the impulse approximation, we must evaluate the two-body operator t_α^0 from Eq. (44) *before* evaluating the matrix element[19]

$$(\gamma'|t_\alpha^0|\gamma)\equiv(g_{\gamma'},t_\alpha^0 g_\gamma). \tag{48}$$

A rather obvious first correction to the impulse approximation is to add to h_α in d^0 [Eq. (44)] the *dispersive* energy of particle α moving in the medium of the other particles. In this case, one must find the depth of the "optical model potential well" for particle α. This correction will often not greatly increase the difficulty of the calculation. [The first-order ΔW correction in Eq. (46) will now vanish.]

Even when the impulse approximation is not valid, the operators t_α of Eq. (43) have physical meaning as the scattering operators from *bound* (rather than *free*) particles. They are also "physical observables" in the sense that for "thin media," or for "glancing collisions," the wave matrix is given by a single term such as $\Omega=1+(1/a)t_\alpha$, etc. In principle, of course, one needs to know the wave functions g_γ of the medium in order to evaluate the t_α when the impulse approximation is not valid. In the absence of simplifying models, this will in general considerably increase the difficulty of obtaining the t_α.

To provide a simple illustration of the scattering Eqs. (16), we consider a large, uniform medium consisting of scattering particles which are much heavier than the scattered particles and which are arranged at random positions in the medium. We also suppose the scattering particles to have states of excitation separated by energies small compared to that of the scattered particle. Using the form of the scattering equations given by Eq. (18), we have

$$\Psi=\left[1+\frac{1}{d}\sum_\alpha I_\alpha+\frac{1}{d}\sum_\beta I_\beta\frac{1}{d}\sum_{\alpha\neq\beta}I_\alpha+\cdots\right]\Omega_C|a\rangle. \tag{49}$$

For massive scatterers, we may take I_α to have the form

$$(\mathbf{k}|I_\alpha|\mathbf{k}_0)=(\mathbf{k}|t|\mathbf{k}_0)\exp[-i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{z}_\alpha], \tag{50}$$

where $\mathbf{k}_0$ and $\mathbf{k}$ are the respective momenta of the incident particle before and after the scattering and $(\mathbf{k}|t|\mathbf{k}_0)$ is the scattering matrix for a scatterer located at the point $\mathbf{z}_\alpha=0$. If we neglect the recoil energy of the (heavy) scattering particle, neglect possible spin interactions, and set $\mathcal{V}_C\simeq Nt_C$ [see Eqs. (25) and (36)]:

$$d\simeq d^0=\epsilon_{q0}-\frac{k^2}{2M}-v_C(k),\quad v_C(k)=(2\pi)^3N(\mathbf{k}|t|\mathbf{k}). \tag{51}$$

In accordance with Eq. (7), we set

$$|a\rangle=(2\pi)^{-\frac{3}{2}}e^{i\mathbf{q}_0\cdot\mathbf{x}}. \tag{52}$$

By omitting the wave function g_{γ_0}, our wave function Ψ is an operator involving the coordinates $(\mathbf{z}_1\cdots\mathbf{z}_N)$ of the scatterers. At a later stage, we may reinsert the g_{γ_0} by performing an appropriate average over the positions of the scatterers.

The coherent wave, Φ_{Ca}, is[1]

$$\Phi_{Ca}\equiv\Omega_C|a\rangle=(2\pi)^{-\frac{3}{2}}e^{i\mathbf{k}_0\cdot\mathbf{x}}, \tag{53}$$

where $\mathbf{k}_0$ is in the direction of $\mathbf{q}_0$ and k_0 is the root of the equation

$$\frac{k_0^2}{2M}+v_C(k_0)=\epsilon_{q0}. \tag{54}$$

k_0 has a positive imaginary part and may be written in terms of real and imaginary parts as

$$k_0\equiv k_{0R}+(i/2\lambda). \tag{55}$$

When $|v_C|\ll q_0^2/2M$, we have

$$1/\lambda=N\sigma, \tag{56}$$

where σ is the total scattering cross section for the incident particle on one of the target particles.[20]

Now, one easily verifies, as usual in scattering theory, that for $q_0|\mathbf{x}-\mathbf{z}_\alpha|\gg1$,

$$\frac{1}{d}I_\alpha\Phi_{Ca}=\exp(i\mathbf{k}_0\cdot\mathbf{z}_\alpha)\frac{\exp(ik_0R_\alpha)}{(2\pi)^{\frac{3}{2}}R_\alpha}f(\hat{n}_{R\alpha},\hat{n}_0). \tag{57}$$

Here

$$\mathbf{R}_\alpha\equiv\mathbf{x}-\mathbf{z}_\alpha,\quad(\alpha=1,2,\cdots N), \tag{58}$$

$$\hat{n}_0\equiv\mathbf{k}_0/k_0,\quad\hat{n}_{R\alpha}=\mathbf{R}_\alpha/R_\alpha, \tag{59}$$

and the scattering amplitude f is

$$f(\hat{n}_{R\alpha},\hat{n}_0)=-(2\pi)^2M^*(k_0\hat{n}_{R\alpha}|t|k_0\hat{n}_0). \tag{60}$$

Finally, M^* is the "effective mass,"

$$M^*\equiv M\bigg/\left(1+\frac{M}{k_0}\frac{d\mathcal{V}_C}{dk_0}\right). \tag{61}$$

We suppose a second scattering to take place at $\mathbf{x}=\mathbf{z}_\beta$ and define a "small distance" $\mathbf{y}$ by

$$\mathbf{x}=\mathbf{z}_\beta+\mathbf{y}. \tag{62}$$

Then Eq. (57) becomes

$$\frac{1}{d}I_\alpha\Phi_{Ca}=\exp(i\mathbf{k}_0\cdot\mathbf{z}_\alpha)\frac{\exp(i\mathbf{k}_{\beta\alpha}\cdot\mathbf{y})}{(2\pi)^{\frac{3}{2}}}\times\frac{\exp(ik_0|\mathbf{z}_\beta-\mathbf{z}_\alpha|)}{|\mathbf{z}_\beta-\mathbf{z}_\alpha|}f(\hat{n}_{\beta\alpha},\hat{n}_0). \tag{63}$$

We have introduced $\hat{n}_{\beta\alpha}$ as the unit vector in the direction of $(\mathbf{z}_\beta-\mathbf{z}_\alpha)$ and $\mathbf{k}_{\beta\alpha}\equiv\mathbf{k}_0\hat{n}_{\beta\alpha}$.

[19] It appears somewhat ambiguous as to whether one should take γ, γ' or some combination for evaluating d^0, according to Eq. (44). Actually, if the impulse approximation is valid, this should make no difference.

[20] See, for example, reference 2. Equation (56) follows from the "optical theorem," which states that

$$\mathrm{Im}(q_0|t|q_0)=-[2(2\pi)^3]^{-1}(q_0/M)\sigma.$$

For the second scattering [for example, see the second term in Eq. (49)] we let $(1/d)I_\beta$ operate on the state $(2\pi)^{-\frac{3}{2}}\exp(i\mathbf{k}_{\beta\alpha}\cdot\mathbf{y})$. By analogy with Eq. (57), we obtain

$$\frac{1}{d}I_\beta\frac{1}{d}I_\alpha\Phi_{C a}=\exp(i\mathbf{k}_0\cdot\mathbf{z}_\alpha)\frac{\exp(ik_0R_\beta)}{(2\pi)^{\frac{3}{2}}R_\beta}f(\hat{n}_{R\beta},\hat{n}_{\beta\alpha})\times\frac{\exp(ik_0|\mathbf{z}_\beta-\mathbf{z}_\alpha|)}{|\mathbf{z}_\beta-\mathbf{z}_\alpha|}f(\hat{n}_{\beta\alpha},\hat{n}_0). \quad (64)$$

In this manner, we may obtain as many terms as we need in Eq. (49). We have assumed, of course, that the average distance between scatterings is large compared to q_0^{-1}.

Instead of a plane wave, we may choose a wave packet for $|a\rangle$[21] (here t is the time)

$$|a\rangle\equiv g(|\mathbf{x}-\mathbf{z}_\alpha-\mathbf{V}t|)\exp\{i[\mathbf{q}_0\cdot(\mathbf{x}-\mathbf{z}_\alpha)-\epsilon_{q0}t]\}.$$

Here $g(u)$ is a "smooth" function which vanishes for $u\gg R$, the "radius of the wave packet." We suppose that $R\gg q_0^{-1}$ and $R^2\gg\hbar^2D/MV$, where $\mathbf{V}=\mathbf{q}_0/M\simeq\mathbf{k}_0/M$ and D is the distance which the wave packet travels during the time which we observe it. Then the spread in the wave packet is negligible. We may also suppose the density of the medium to be low enough that the mean free path $\lambda\gg R$.

Then it is easily shown that

$$\frac{1}{d}I_\alpha\Phi_{Ca}\simeq\frac{\exp(ik_0R_\alpha)}{(2\pi)^{\frac{3}{2}}R_\alpha}\exp(-i\epsilon_{q0}t)f(\hat{n}_{R\alpha},\hat{n}_0)\times g(|R_\alpha\hat{n}_0-\mathbf{V}t|). \quad (65)$$

This tells us that the first scattered wave from α appears at time $t=0$ and spreads out as a spherical wavelet moving with speed V and local amplitude

$$f(\hat{n}_{R\alpha},\hat{n}_0)\frac{\exp(ik_0R_\alpha)}{(2\pi)^{\frac{3}{2}}R_\alpha}.$$

A second scattering from β will appear when this ripple passes the point $\mathbf{z}_\beta$. This in turn will appear as a secondary spherical wavelet. These will give rise to tertiary waves, etc. The wavelets from such successive scatterings are obtained on repeating the calculation which led to Eq. (65), i.e., we let the expression (65) represent the "incident wave" for the next scattering, etc.

The connection with classical transport phenomena is easily obtained, as was shown in reference 1. When $\lambda\gg q_0^{-1}$, we may neglect the *interference* of wavelets scattered from different particles in the expression

$$\rho_s\equiv|\Psi|^2$$

for the density of scattered particles. Using the integral equations (16), the appropriate classical transport equation is obtained.[22] The quantum-mechanical theory of transport phenomena is contained in the general transport equation satisfied by ρ_s.

[21] Extensive use of wave packets in the development of scattering theory has been made by Francis Low (unpublished lectures at the University of Illinois, 1953).

IV. OPTICAL MODEL POTENTIAL

We consider the choice $P=P_0$ of Eq. (26) in this section. Then the elastic scattering is described by $\Omega_C|a\rangle$ and the inelastic scattering by $(F-1)\Omega_C|a\rangle$ as was mentioned in connection with Eq. (35).

The "optical model potential" $\mathcal{V}_C$ is given by Eq. (30). The appearance of $\mathcal{V}_C$ in d in F and the F_α tells us that the particle propagates between scatterings in the *dispersive medium* as determined by $\mathcal{V}_C$. Thus, for instance, Eq. (30) is a nonlinear integral equation for $\mathcal{V}_C$.

A zeroth approximation (the one used in Sec. III) is obtained on setting $F_\alpha=1$ in Eq. (30). Then

$$\mathcal{V}_C=\langle\gamma|\textstyle\sum_\alpha t_\alpha|\gamma\rangle\delta_{\gamma'\gamma}, \quad (66)$$

which is just Eq. (25). Corrections to this equation arise because of correlations between scatterers in the scattering medium. These correlations may be *induced* by the scattered particle or may be an intrinsic property of the medium. An example of the former type of correlation is the "dielectric polarization" in the Lorentz-Lorenz formulas for the refractive index of a gas. The manner in which the latter correlations affect $\mathcal{V}_C$ was discussed in reference 2. In general, these phenomena will be interrelated.

To illustrate the effect of polarizing the medium, we shall use the model of Sec. III, supposing the scatterers to be much more massive than the scattered particle, to have closely spaced excited states, and to be randomly distributed in a large, isotropic uniform medium. As before, we shall take

$$d=\epsilon_{q0}-\epsilon(k)+i\eta,\quad \epsilon(k)=\frac{k^2}{2M}+v_C(k), \quad (67)$$

and suppose $\mathbf{k}_0$ to satisfy

$$\epsilon(k_0)=\epsilon_{q0}. \quad (68)$$

We define

$$\Psi_\alpha=e^{i\mathbf{k}_0\cdot\mathbf{x}}+\frac{1}{d}P_0\sum_{\beta\neq\alpha}t_\beta\Psi_\beta, \quad (69)$$

so by Eq. (30)

$$\langle\mathbf{k}'|v_C|\mathbf{k}_0\rangle=\textstyle\sum_\alpha\langle\gamma_0,\mathbf{k}'|t_\alpha\Psi_\alpha|\gamma_0,\mathbf{k}_0\rangle. \quad (70)$$

This may be written explicitly as

$$\langle\mathbf{k}'|v_C|\mathbf{k}_0\rangle=\int\prod_{\nu=1}^{N}d^3z_\nu|g_{\gamma_0}(\mathbf{z}_1,\cdots\mathbf{z}_N)|^2\times\langle\mathbf{k}'|\textstyle\sum_\alpha t_\alpha\Psi_\alpha|\mathbf{k}_0\rangle, \quad (71)$$

using the wave function g_{γ_0} of Eq. (2).

[22] In reference 1, this was done for the special case of point scatterers. The same method works in general, however.

To calculate Ψ_α, we suppose the states of the scattering medium to be so dense that the chance of repeating a previous state is negligible for all but very small angle scatterings. The scatterings at sufficiently small angles will not lead to a change of state of the medium and must be explicitly excluded by the P_0 operator. Also, the mean free path for scattering is considered to be much greater than k_0^{-1}.

To evaluate Eq. (71), we must solve Eq. (69) for Ψ_α. It is apparent that we need Ψ_α only for $\mathbf{x}-\mathbf{z}_\alpha$, which is the position at which the next scattering occurs. Let us make the assumption that Ψ_α has the form

$$\Psi_\alpha = e^{i\mathbf{k}_0\cdot\mathbf{x}}Q, \tag{72}$$

where Q is independent of α for $\mathbf{x}\simeq\mathbf{z}_\alpha$. As will be seen, this implies a homogeneous medium. Also, as in Sec. III, we suppose the distance between scatterings to be large compared with k_0^{-1} and

$$(\mathbf{k}|t_\alpha|\mathbf{k}_0) = (\mathbf{k}|t|\mathbf{k}_0)\exp[-i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{z}_\alpha]. \tag{73}$$

Then we obtain

$$\begin{aligned} S_\beta &\equiv \frac{1}{d}P_0 t_\beta e^{i\mathbf{k}_0\cdot\mathbf{x}} \\ &= \int \frac{d^3k\,\exp(i\mathbf{k}\cdot\mathbf{R}_\beta)}{\epsilon_{q0}+i-\epsilon(k)}(\mathbf{k}|t|\mathbf{k}_0)\exp(i\mathbf{k}_0\cdot\mathbf{z}_\beta) \\ &\simeq f(\hat{n}_{R\beta},\hat{n}_0)\frac{\exp(ik_0R_\beta)}{R_\beta}\exp(i\mathbf{k}_0\cdot\mathbf{z}_\beta), \end{aligned} \tag{74}$$

just as in Eq. (57). Here $\mathbf{R}_\beta$, $\hat{n}_{R\beta}$, f, etc. have been defined by Eqs. (58), (59) and (60).

Equations (72) and (74) may now be substituted into Eq. (69) and the resulting equation solved for Q:

$$Q = [1-\Delta]^{-1}, \tag{75}$$

where

$$\Delta = \sum_{\beta\neq\alpha} P_0 f(\hat{n}_{R\beta},\hat{n}_0)\frac{\exp(ik_0R_\beta)}{R_\beta}\exp(-i\mathbf{k}_0\cdot\mathbf{R}_\beta). \tag{76}$$

In the expression for Δ, we suppose that $\mathbf{x}=\mathbf{z}_\alpha$, so $\mathbf{R}_\beta=(\mathbf{z}_\alpha-\mathbf{z}_\beta)$. By our assumption that the medium is homogeneous, we conclude that Δ, and thus Q, is independent of α—as was assumed in connection with Eq. (72).

Equations (72), (75), and (76) provide the solution for Ψ_α, so Eq. (71) may be evaluated as

$$v_C = \sum_\alpha \int \prod_{\nu=1}^{N} d^3z_\nu |g_{\gamma o}(\mathbf{z}_1\cdots\mathbf{z}_N)|^2 \times \frac{\exp[-i(\mathbf{k}'-\mathbf{k}_0)\cdot\mathbf{z}_\alpha](\mathbf{k}'|t|\mathbf{k}_0)}{1-\Delta}. \tag{77}$$

If we assume complete *randomness* in particle positions we may set $|g_{\gamma o}|^2=\mathcal{V}^{-N}$ when all the z's are within the volume $\mathcal{V}$ and $|g_{\gamma o}|^2=0$ otherwise. Also, we may take

$$\int_{\mathcal{V}} d^3z_\alpha \exp[-i(\mathbf{k}'-\mathbf{k}_0)\cdot\mathbf{z}_\alpha] \simeq (2\pi)^3\delta(\mathbf{k}'-\mathbf{k}_0),$$

and change to the $\mathbf{R}_\beta$ for the remaining variables to obtain

$$v_C \simeq \left[\frac{N}{\mathcal{V}}(2\pi)^3\delta(\mathbf{k}'-\mathbf{k}_0)(\mathbf{k}_0|t|\mathbf{k}_0)\right] \times \frac{1}{\mathcal{V}^{N-1}}\int \frac{\prod_\nu d^3R_\nu}{1-\Delta(R_1\cdots R_N)}. \tag{78}$$

(There is no R_α in the integral above, of course.) The refractive index n may be obtained from Eq. (37).

Exact evaluation of the integral in Eq. (78) does not appear feasible. The only approximate treatment which we shall discuss is the rather crude one of substituting

$$\sum_{\beta\neq\alpha} = \rho_0\int d^3z_\beta = \rho_0\int d^3R_\beta$$

in Eq. (76) for Δ. Here $\rho_0=N/\mathcal{V}$, as before.

Then

$$\begin{aligned} \Delta &\simeq \rho_0 \int d^3R_\beta \frac{\exp(ik_0R_\beta)}{R_\beta}[f(\hat{n}_{R\beta},\hat{n}_0)\exp(-i\mathbf{k}_0\cdot\mathbf{R}_\beta)] \\ &= 2\pi\rho_0\int\frac{dR_\beta}{ik_0}\exp(ik_0R_\beta)\{f(-\hat{n}_0,\hat{n}_0)\exp(ik_0R_\beta) \\ &\qquad - f(\hat{n}_0,\hat{n}_0)\exp(-ik_0R_\beta)\}+\cdots. \end{aligned} \tag{79}$$

We now recall the presence of the operator P_0 in Eq. (71). P_0 instructs us to discard all scatterings which do not lead to a *change of state* of the scattering medium. The second term in curly braces in (79) describes *forward* scattering, which will not excite the scattering medium, and must, therefore, be omitted.

Because k_0 has a positive imaginary part, we may set the upper limit in the integral equal to infinity to obtain

$$\Delta \simeq \frac{\pi\rho_0}{k_0^2}f(-\hat{n}_0,\hat{n}_0). \tag{80}$$

From Eq. (77), we evaluate v_C as

$$(\mathbf{k}'|v_C|\mathbf{k}_0) = \delta(\mathbf{k}'-\mathbf{k}_0)\left[(2\pi)^3\rho_0\frac{(\mathbf{k}_0|t|k_0)}{1-\Delta}\right]. \tag{81}$$

From Eq. (37), we obtain the refractive index as

$$n^2-1 = 4\pi\left[\frac{M}{M^*}\rho_0\frac{f(\hat{n}_0,\hat{n}_0)}{q_0^2}\right]\Big/\left[1-\frac{\pi}{k_0^2}\rho_0 f(-\hat{n}_0,\hat{n}_0)\right]. \tag{82}$$

This is similar in structure to the Lorentz-Lorenz formula for the refractive index of an isotropic medium.

We must recognize that a number of approximations have been made in obtaining Eq. (82). First of all, we have made an approximation in treating the P_0 operator on which $(\mathbf{k}_0|t|\mathbf{k}_0)$ operates in Eq. (78). In other words, we must suppose this represents only "nearly forward scattering." Next, we have kept only the first term in an expansion (obtained by partial integration) in $(k_0R_\beta)^{-1}$.[23] A more accurate treatment is beyond our present scope.

For a medium in which correlations between pairs of particles are important, but for which the induced polarization is negligible, we may write, approximately

$$\mathcal{V}_C\simeq\langle\gamma_0|\sum_\alpha t_\alpha|\gamma_0\rangle+\sum_{\alpha_1}\sum_{\alpha_2\neq\alpha_1}\left\langle\gamma_0\left|t_{\alpha_1}\frac{1}{d}P_0t_{\alpha_2}\right|\gamma_0\right\rangle, \tag{83}$$

$$d\simeq\epsilon_{q0}+i\eta-\epsilon(k).$$

We shall neglect the term v_C in ϵ as well as spin interactions. In carrying out the sum over virtual states in the second term above, we must omit the state γ_0. Thus,[2]

$$\left\langle\mathbf{k}'\gamma_0\left|t_{\alpha_1}\frac{1}{d}P_0t_{\alpha_2}\right|\mathbf{k}_0\gamma_0\right\rangle$$

$$\simeq\int d^3z_{\alpha1}d^3z_{\alpha2}d^3k[P(\mathbf{z}_{\alpha1},\mathbf{z}_{\alpha2})-P(\mathbf{z}_{\alpha1})P(\mathbf{z}_{\alpha2})]$$

$$\times\left[\frac{(k'|t|k)(k|t|k_0)}{\epsilon_{q0}+i\eta-\epsilon(k)}\right]\exp[-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{z}_{\alpha1}]$$

$$\times\exp[-i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{z}_{\alpha2}]. \tag{84}$$

If the $P(\mathbf{z}_{\alpha1})$ is the probability of finding particle α_1 at the point $\mathbf{z}_\alpha$, $P(\mathbf{z}_{\alpha1},\mathbf{z}_{\alpha2})$ is the joint probability of finding particles α_1 and α_2 at points $\mathbf{z}_{\alpha1}$ and $\mathbf{z}_{\alpha2}$, respectively.

We may conveniently write

$$P(\mathbf{z}_{\alpha1},\mathbf{z}_{\alpha2})-P(\mathbf{z}_{\alpha1})P(\mathbf{z}_{\alpha2})=P(z_{\alpha1})g(r),\quad \mathbf{r}\approx\mathbf{z}_{\alpha1}-\mathbf{z}_{\alpha2}, \tag{85}$$

for an extended isotropic medium. $g(r)$ represents the pair correlation function for the scattering medium.

Some discussion of the real part of Eq. (84) was given in reference 2. When the phase shifts are sufficiently small that the t's are real, we obtain for the imaginary part of Eq. (84):

$$\sum_{\alpha_1}\sum_{\alpha_2\neq\alpha_1}\mathrm{Im}\left\langle\gamma\left|t_{\alpha1}\frac{1}{d}P_0t_{\alpha2}\right|\gamma\right\rangle$$

$$=\delta(\mathbf{k}'-\mathbf{k}_0)[-\tfrac{1}{2}\rho_0v_R]\int d\Omega_k\frac{d\sigma}{d\Omega}\int d^3r[\rho_0g(r)]e^{-i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}},$$

where the differential scattering cross section is

$$\frac{d\sigma}{d\Omega}=\frac{(2\pi)^4}{v_R}\int d^3k\delta(\epsilon-\epsilon_0)(k_0|t|k)(k|t|k_0),$$

and v_R is the velocity of the scattered particle. Combining this with the first term of Eq. (83), we obtain for the imaginary part of the refractive index:

$$\mathrm{Im}(n)\simeq\frac{1}{2k_0}\rho_0\int d\Omega\frac{d\sigma}{d\Omega}\left[1+\rho_0\int d^3rg(r)e^{-i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}}\right]. \tag{86}$$

This expression for the absorption coefficient is familiar from the theory of *critical opalescence* in optics.[24]

If we extend the series (83), we obtain a dependence on three-particle correlation functions, etc. It is a characteristic of our description of multiple scattering, that the structure of the *scattering medium* is assumed to be *known* in the calculation of the scattering. At just this point, the formal similarity between the scattering theory and Brueckner's[4–6] theory of the nuclear many-body problem breaks down, since the aim of the latter method is to obtain a detailed theory of the medium. Brueckner's work has recently been extended[9] to include a method for the evaluation of the equation of state of gases and liquids. Applications of this method can be made to calculate correlation functions such as $g(r)$. When the physical medium is not in a pure state (i.e., has a finite temperature), an ensemble average must be performed over its spectrum of states.

V. ENERGY OF A PARTICLE INTERACTING WITH A SYSTEM—LEVEL SHIFT OF THE π-MESONIC ATOM

In reference 9, a perturbation theory for the energy of a system of N interacting particles was described in a manner formally identical with the expression for the optical model potential. The energy of a single particle interacting with a system of particles was also obtained.[9] This was identical with our expression for $h+\mathcal{V}_C$ except for the absence of the $i\eta$ [see Eq. (10)] in the propagators. Indeed, we can write Eq. (35) as

$$\epsilon_q=h(q)+\langle q\gamma_0|v_C(\epsilon_q)|q\gamma_0\rangle. \tag{87}$$

The solution of this equation for ϵ_q gives the energy of the particle. The first approximation (Eq. (66)) for v_C gives the expression used, for instance, in theories of the level shift of the π-mesonic atom.[25,26]

The fact that the level shift is essentially given by the optical model potential has perhaps been insufficiently emphasized. This point is worth making since the approximations used for the level shift have seemed somewhat crude when applied to the calculation of the

[23] Such an expansion requires a cut-off distance for all but the lowest order term. This expansion need not be made, of course, if one performs the angular integration in some other manner in Eq. (79).

[24] See, for instance, L. Rosenfeld, *Theory of Electrons* (Interscience Publishers, Inc., New York, 1951), p. 80. Compare also M. Lax, Phys. Rev. **85**, 622 (1952).

[25] Deser, Goldberger, Baumann, and Thirring, Phys. Rev. **96**, 774 (1954).

[26] K. A. Brueckner, Phys. Rev. **98**, 769 (1955).

optical model potential.[27] Indeed, neither the "self-consistency" correction associated with the presence of $\mathcal{V}_C$ in the propagator defining the t_α nor the "correlation corrections" discussed in the last section is likely to be negligible in a quantitative theory.

Brueckner[26] has observed that the level shift of the π-mesonic atom should include a correction for true absorption of the meson. This correction also occurs in the calculation of the pion-nucleus optical model potential—and was indeed given in reference 1 in the form used by Brueckner.

When the absorption can occur via an "absorption operator" R, then Eq. (87) is corrected as follows [see Eq. (72) of reference 1]:

$$\epsilon_q = h(q) + \left\langle q\gamma_0 \left| R\frac{1}{d}R \right| q\gamma_0 \right\rangle + \langle q\gamma_0 | \mathcal{V}_C(\epsilon_q) | q\gamma_0 \rangle. \quad (88)$$

The propagator d occurring in $\mathcal{V}_C$ should also be replaced by[1]

$$D \equiv d - R\frac{1}{d}R. \quad (89)$$

Karplus[28] has observed that the level shift $R(1/a)R$ does not include all the terms which arise from meson field theory. Using field theory, the "absorption correction" must be calculated by the method of potential construction in quantum field theories.[29]

VI. SCATTERING FROM A SMALL NUMBER OF PARTICLES

When the number of particles in the scattering medium is small, it is often convenient to take $\mathcal{O}=0$, so $\Omega_C=1$, $F=\Omega$ and $P=1$ in Eqs. (12) and (16). Let us again use the model of heavy scatterers of Secs. III and IV, so

$$d = a = \epsilon_{q0} - (k^2/2M).$$

If we let the second of Eqs. (16) operate on $e^{i\mathbf{q}_0\cdot\mathbf{x}}$, we obtain

$$[\Psi_\alpha = F_\alpha e^{i\mathbf{q}_0\cdot\mathbf{x}}], \quad \Psi_\alpha = e^{i\mathbf{q}_0\cdot\mathbf{x}} + \frac{1}{a}\sum_{\beta\neq\alpha} t_\beta \Psi_\beta. \quad (90)$$

To solve this equation, we define

$$S_\beta = \frac{1}{a} t_\beta \Psi_\beta, \quad (91)$$

and write (as before $\mathbf{R}_\gamma = \mathbf{x} - \mathbf{z}_\gamma$)

$$\Psi_\beta = e^{i\mathbf{q}_0\cdot\mathbf{x}} + \sum_{\gamma\neq\beta} \frac{\exp(iq_0 R_\gamma)}{R_\gamma} Q_{\beta\gamma}, \quad (92)$$

where $Q_{\beta\gamma}$ is *independent* of $\mathbf{x}$. For $\mathbf{x}-\mathbf{z}_\beta$, we may rewrite this as

$$\Psi_\beta(\mathbf{x}\simeq\mathbf{z}_\beta) \simeq e^{i\mathbf{q}_0\cdot\mathbf{x}} + \sum_{\gamma\neq\beta} \frac{\exp(iq_0 R_{\beta\gamma})}{R_{\beta\gamma}} \times \exp[i\mathbf{k}_{\beta\gamma}\cdot(\mathbf{x}-\mathbf{z}_\beta)] Q_{\beta\gamma}. \quad (93)$$

Here $\mathbf{k}_{\beta\gamma} = q_0 \hat{n}_{\beta\gamma}$ and

$$\hat{n}_{\beta\gamma} = (\mathbf{z}_\beta - \mathbf{z}_\gamma)/|\mathbf{z}_\beta - \mathbf{z}_\gamma|. \quad (94)$$

Also, $R_{\beta\gamma} = |\mathbf{z}_\beta - \mathbf{z}_\gamma|$.

If we substitute Eq. (93) into Eq. (91), we obtain (our model supposes the scattering mean free path to be much greater than q_0^{-1})

$$S_\beta = \frac{\exp(iq_0 R_\beta)}{R_\beta} \left\{ f(\hat{n}_{R\beta}, \hat{n}_0) \exp(i\mathbf{q}_0\cdot\mathbf{z}_\beta) + \sum_{\gamma\neq\beta} \frac{\exp(iq_0 R_{\beta\gamma})}{R_{\beta\gamma}} f(\hat{n}_{R\beta}, \hat{n}_{\beta\gamma}) Q_{\beta\gamma} \right\}, \quad (95)$$

as in Eq. (57). Here f is the scattering amplitude, as defined by Eq. (60) (now $M^*=M$, of course, since $\mathcal{V}_C=0$).

Now, substituting Eqs. (95) and (92) into Eq. (90), there results (with $\mathbf{x}\simeq\mathbf{z}_\alpha$)

$$Q_{\alpha\beta} = f(\hat{n}_{\alpha\beta}, \hat{n}_0) \exp(i\mathbf{q}_0\cdot\mathbf{z}_\beta) + \sum_{\gamma\neq\beta} f(\hat{n}_{\alpha\beta}, \hat{n}_{\beta\gamma}) \frac{\exp(iq_0 R_{\beta\gamma})}{R_{\beta\gamma}} Q_{\beta\gamma}. \quad (96)$$

This represents a set of coupled *algebraic* equations for the $Q_{\alpha\beta}$'s. The complete wave function Ψ is then

$$\Psi = \{e^{i\mathbf{q}_0\cdot\mathbf{x}} + \textstyle\sum_\alpha S_\alpha\} g_{\gamma_0}(\xi). \quad (97)$$

For this problem, the multiple-scattering equations are solved algebraically.

APPENDIX

In Sec. II we described the effect on the optical model potential of a finite "relaxation time" of the medium following inelastic scatterings. We here develop this in more detail.

The incident particle with energy ϵ_{q0} scatters from particle "α" at point $\mathbf{z}_\alpha$. The particle "α" is ejected suddenly from the medium. We suppose the medium to normally have particle-particle correlations extending over a distance D. When particle "α" is suddenly removed, the particles within a distance D of $\mathbf{z}_\alpha$ are expected to readjust themselves with a relaxation time τ. The excitation energy of the medium associated with the removal of particle α we suppose to be of the order of ΔE_W.

Now if $\hbar/D \gg q$ the momentum of the scattered particle, this particle will leave the disturbed region

[27] Frank, Gammel, and Watson, Phys. Rev. **101**, 891 (1956).

[28] R. Karplus, reported at the *Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956).

[29] K. A. Brueckner and K. M. Watson, Phys. Rev. **90**, 699 (1953).

around the point $\mathbf{z}_\alpha$ before it has a definite energy and will consequently interact with an optical potential which is that of the undisturbed medium. On the other hand, if $\hbar/D \ll q$ the particle will interact until it has traveled the distance D via an optical potential which is peculiar to the locally excited region near $\mathbf{z}_\alpha$. This assumes, of course, that $D/v \ll \tau$ (where v is the velocity of the scattered particle), a condition which will be met if $\epsilon_q \gg \Delta E_M$, the spacing of levels of the medium.

To develop this from our scattering formula, we write

$$t_\alpha = \exp[-i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{z}_\alpha](\mathbf{q}|t|\mathbf{q}_0). \tag{A-1}$$

Now

$$\begin{aligned} S_\alpha &\equiv t_\alpha e^{i\mathbf{q}_0\cdot\mathbf{x}} g_{\gamma_0} \\ &= \sum_{\gamma,q} e^{i\mathbf{q}\cdot\mathbf{x}} g_\gamma (\mathbf{q}|t|\mathbf{q}_0) \\ &\qquad \times (g_\gamma, \exp[-i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{z}_\alpha{}'] g_{\gamma_0}). \end{aligned} \tag{A-2}$$

Since we assume that particle α is ejected, we may write

$$g_\gamma = g_{\gamma'}{}' \exp(i\mathbf{p}\cdot\mathbf{z}_\alpha)/(2\pi)^3, \tag{A-3}$$

where $\mathbf{p}$ is the momentum of the recoil particle α and $g_{\gamma'}{}'$ is an eigenfunction of the residual medium (with particle α gone).

We must suppose $\mathbf{p}$ to be large compared to the average momentum in g_{γ_0}, so $\mathbf{p} \simeq (\mathbf{q}_0 - \mathbf{q})$. More specifically,

$$\begin{aligned} \sum_{\mathbf{p}} \frac{\exp(i\mathbf{p}\cdot\mathbf{z}_\alpha)}{(2\pi)^{\frac{3}{2}}} &(g_\gamma, \exp[-i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{z}_\alpha] g_{\gamma_0}) \\ &= \exp[-i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{z}_\alpha](g_{\gamma'}{}', g_{\gamma_0}), \end{aligned} \tag{A-4}$$

where *no* integration is performed over $\mathbf{z}_\alpha$ in

$$C_{\gamma'}(\mathbf{z}_\alpha) \equiv (g_{\gamma'}{}', g_{\gamma_0}). \tag{A-5}$$

Then

$$\begin{aligned} \frac{1}{d} S_\alpha = \sum_{q,\gamma'} \frac{\exp[i\mathbf{q}\cdot(\mathbf{x}-\mathbf{z}_\alpha)]}{d} & g_{\gamma'}{}' C_{\gamma'}(\mathbf{z}_\alpha) \\ & \times (\mathbf{q}|t|\mathbf{q}_0) \exp(i\mathbf{q}_0\cdot\mathbf{z}_\alpha). \end{aligned} \tag{A-6}$$

The physical interpretation of this is quite apparent. The phase of the wave at $\mathbf{z}_\alpha$ is just $\exp(i\mathbf{q}_0\cdot\mathbf{z}_\alpha)$. The additional phase change in traveling to point $\mathbf{x}$ is $\exp[i\mathbf{q}\cdot(\mathbf{x}-\mathbf{z}_\alpha)]$. Energy conservation is determined by the pole of d when $R \equiv |\mathbf{x}-\mathbf{z}_\alpha|$ is very large. When this distance is much larger than $\hbar/q$, the uncertainty in ϵ_q is $\simeq \Delta E = v\hbar/R$. Even when $\Delta E/\epsilon_q \ll 1$, $\Delta E/\Delta E_M$ may be $\gg 1$. This condition will obtain for some range of R if ϵ_q is large (as we have assumed) and then

$$\sum\nolimits_{\gamma'} g_{\gamma'}{}' C_{\gamma'}(\mathbf{z}_\alpha) = g_{\gamma_0}, \tag{A-7}$$

which follows from the completeness of the $g_{\gamma'}{}'$. This means that the medium is "frozen" as it was before α was ejected. As the scattered particle travels farther, R becomes large and eventually $\Delta E/\Delta E_M \ll 1$. Then the medium has settled into a definite eigenstate.

Now Eq. (A-7) describes a "wave-packet" state of the medium having an uncertainty in energy $\simeq \Delta E_W$. The relaxation time of the medium is determined by the time required for ΔE to become comparable to ΔE_W. As discussed in Sec. II, it is in terms of the "wave-packet state" that it is desirable to define P_{ND}. Then the "elastic scatterings" which make up the "optical potential" are "elastic" with respect to the wave-packet state.

We emphasize that this change in the definition of P_{ND} is only important if the medium has strong correlations between particles. Had we used the original definition of P_{ND} in terms of γ states, it would have been necessary to sum over many "slightly elastic" scatterings to obtain this result.