Pion Production in Pion-Nucleon Scattering*

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The cross section for the production of one additional pion in a pion-nucleon scattering is calculated using the Chew-Low theory of P-wave pion-nucleon scattering. The transition matrix element for the scattering of one meson into two mesons is defined in terms of exact eigenstates of the total Hamiltonian and an approximate expression is derived which expresses the one- to two-meson matrix element as a product of an elastic scattering matrix element and a meson-emission matrix element. Experimental elastic scattering phase shifts are used in calculating the two-meson cross section. The results of this calculation are used to estimate the effect of two-meson states on elastic scattering. The contribution to the effective range for elastic pion-nucleon scattering is small. Cross sections are also obtained for all possible two-meson charge states from either a π^+ or a π^- meson incident on a proton. Comparison is made with experiment and the recent theoretical work of Barshay.

I. INTRODUCTION

THE problem of elastic scattering of pions by a nucleon has been approached recently by a new method[1–3] which involves the use of exact eigenstates of the total meson-nucleon Hamiltonian taken in the static limit (fixed nucleon). In this paper this method, in the form introduced by Wick for one-meson states, is used to investigate the two-meson eigenstates of the Hamiltonian and to calculate the transition matrix for scattering from a one-meson state to a two-meson state. Barshay[4] has made a recent calculation similar to this one using Low's method, but the results are quite different from those obtained here because of different approximations introduced by Barshay in evaluating the T matrix. Barshay introduces sums over two complete sets of eigenstates and then makes the approximation of limiting one sum to include states with up to one real meson and the other to include only states with no real mesons (physical nucleon). The approximation is also made by Barshay of neglecting the energy of one of the outgoing mesons in certain energy denominators. In the present paper the procedure used corresponds to summing exactly one of Barshay's complete sets and including only the physical nucleon states in the other while all energy denominators are treated exactly.

II. TWO MESON EIGENSTATES

In the static limit the pion-nucleon Hamiltonian has the form[5]

$$H=H_0-E_0+H_I, \tag{1}$$

where

$$H_0=\sum_k a_k^\dagger a_k\omega_k, \tag{2}$$

E_0 is a constant energy subtracted to make the self-energy of a single nucleon zero,

$$H_I=\sum_k V_k^{(0)}a_k+V_k^{(0)\dagger}a_k^\dagger, \tag{3}$$

and

$$V_k^{(0)}=i(4\pi)^{\frac{1}{2}}(f^{(0)}/\mu)\boldsymbol{\sigma}\cdot\mathbf{k}\tau_k v(k)/(2\omega)^{\frac{1}{2}}. \tag{4}$$

Here $a_k^\dagger$ and a_k are, respectively, creation and annihilation operators for single, bare mesons, $\omega=(\mu^2+k^2)^{\frac{1}{2}}$, $f^{(0)}$ is the unrenormalized coupling constant, $\boldsymbol{\sigma}$ is the nucleon spin vector, τ_k is the kth component of the nucleon isotopic spin operator and $v(k)$ is a cutoff function which approaches zero for large momenta. In the notation used here the meson quantum numbers are all described by a single symbol (k) which includes the three components of momentum and the isotopic spin.

A two-meson eigenstate corresponding to one meson of type p_1 and one meson of type p_2 can be found from the assumed form

$$\Psi_{p_1,p_2}^{(\pm)}=[a_{p_1}^\dagger a_{p_2}^\dagger\Psi_0+\chi^{(\pm)\prime}]/\sqrt{2}, \tag{5}$$

which represents a state with two plane-wave mesons produced by the two creation operators acting on the eigenstate, Ψ_0, corresponding to a physical nucleon and an (outgoing/incoming)-wave scattered part, $\chi^{(\pm)\prime}$. If this form of the two-meson eigenstate were used in matrix elements, however, the two creation operators appearing in the first term would lead to two energy denominators requiring expansions in two complete sets of states. For this reason, it is better for most calculations to use a two-meson eigenstate of the form

$$\Psi_{p_1p_2}^{(\pm)}=[a_{p_1}^\dagger\Psi_{p_2}^{(\pm)}+\chi^{(\pm)}]/\sqrt{2}, \tag{6}$$

where $\Psi_{p_2}^{(\pm)}$ is a one-meson eigenstate. Equations (5) and (6) will be shown to represent identical states, but the effect of using the latter is to sum exactly one of the expansions required by the use of (5) and condense the resulting equations. Because of this, (6) will lead to a simpler and more accurate result than (5) for the same level of approximation.

The normalization of the eigenstates has been chosen

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[1] G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).
[2] F. E. Low, Phys. Rev. **97**, 1392 (1955).
[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956). The notation used in this paper follows that of Chew and Low.
[4] Saul Barshay, Phys. Rev. **103**, 1102 (1956).
[5] $\hbar$ and c have been set equal to unity.

so that[6]

$$(\Psi_{p_1p_2}{}^{(\pm)},\Psi_{k_1k_2}{}^{(\pm)})=\tfrac{1}{2}(\delta_{p_1k_1}\delta_{p_2k_2}+\delta_{p_1k_2}\delta_{p_2k_1}), \quad (7)$$

and then, in any sum over p_1 and p_2, the sum can go over all p_1 and p_2 without regard for the fact that $\Psi_{p_1p_2}{}^{(\pm)}$ and $\Psi_{p_2p_1}{}^{(\pm)}$ are the same state.

The $\chi^{(\pm)}$ appearing in (6) can be determined in a manner analogous to that used by Chew and Low. The $\Psi_{p_1p_2}{}^{(\pm)}$ are solutions of the Schrödinger equation with energy $(\omega_1+\omega_2)$, so that

$$[H-(\omega_1+\omega_2)]\Psi_{p_1p_2}{}^{(\pm)}=0, \quad (8)$$

and substitution of (6) into (8) results in

$$[H-\omega_1-\omega_2]\chi^{(\pm)}=-[H-\omega_1-\omega_2]a_{p_1}{}^{\dagger}\Psi_{p_1p_2}{}^{(\pm)}.$$

From the explicit form of the Hamiltonian, the commutation relation

$$[H,a_p{}^{\dagger}]=\omega_p a_p{}^{\dagger}+V_p{}^{(0)} \quad (9)$$

follows, and then

$$[H-\omega_1-\omega_2]\chi^{(\pm)}=-V_{p_1}{}^{(0)}\Psi_{p_2}{}^{(\pm)}.$$

Using the inverse of $(H-\omega_1-\omega_2)$ gives

$$\Psi_{p_1p_2}{}^{(\pm)}=\frac{1}{\sqrt{2}}\left[a_{p_1}{}^{\dagger}\Psi_{p_2}{}^{(\pm)}-\frac{1}{H-\omega_1-\omega_2\mp i\epsilon}V_{p_1}{}^{(0)}\Psi_{p_2}{}^{(\pm)}\right] \quad (10)$$

for the two-meson eigenstate, where the $(\mp)i\epsilon$ is inserted to produce outgoing/incoming waves in $\chi^{(\pm)}$. The $\chi^{(\pm)\prime}$ in (5) can be determined in a similar manner, with the result:

$$\Psi_{p_1p_2}{}^{(\pm)}=\frac{1}{\sqrt{2}}\left[a_{p_1}{}^{\dagger}a_{p_2}{}^{\dagger}\Psi_0-\frac{1}{H-\omega_1-\omega_2\mp i\epsilon}\times(V_{p_1}{}^{(0)}a_{p_2}{}^{\dagger}+V_{p_2}{}^{(0)}a_{p_1}{}^{\dagger})\Psi_0\right]. \quad (11)$$

Now (10) can be reduced to (11) if the one-meson eigenstate appearing in (10) is written as

$$\Psi_p{}^{(\pm)}=a_p{}^{\dagger}\Psi_0-\frac{1}{H-\omega_p\mp i\epsilon}V_p{}^{(0)}\Psi_0, \quad (12)$$

and use is made of the identity

$$a_p{}^{\dagger}\frac{1}{H-\omega_k\mp i\epsilon}=\frac{1}{H-\omega_k-\omega_p\mp i\epsilon}\times\left[a_p{}^{\dagger}+V_p{}^{(0)}\frac{1}{H-\omega_k\mp i\epsilon}\right], \quad (13)$$

which follows from the commutation relation (9). Thus either of the forms (10) or (11) can be used for the two-meson eigenstates and, for the reasons already given, (10) will be used in the evaluation of matrix elements. The reduction of (10) to (11) also shows that the symmetry between the two mesons is preserved by the use of (10) even though this symmetry is not as apparent as in (11).

It is of interest to note that the method used here can be generalized to describe an eigenstate for any number of mesons in terms of the eigenstate for one less meson. The general result is

$$\Psi_n{}^{(\pm)}=(n!)^{-\frac{1}{2}}\left[a_{p_i}{}^{\dagger}\Psi_{n-p_i}{}^{(\pm)}-\frac{1}{H-E_n\mp i\epsilon}V_{p_i}{}^{(0)}\Psi_{n-p_i}{}^{(\pm)}\right], \quad (14)$$

where $\Psi_n{}^{(\pm)}$ is a state of n mesons of types $p_1\cdots p_n$ and $\Psi_{n-p_i}{}^{(\pm)}$ is a state with $(n-1)$ mesons of types $p_1\cdots p_n$ excluding type p_i. The form (11) can also be generalized to any number of mesons, resulting in

$$\Psi_n{}^{(\pm)}=(n!)^{-\frac{1}{2}}\left[\prod_{i=1}^{n}a_{p_i}{}^{\dagger}-\frac{1}{H-E_n\mp i\epsilon}\sum_{i=1}^{n}V_{p_i}{}^{(0)}\prod_{j\neq i}^{n}a_{p_j}{}^{\dagger}\right]\Psi_0, \quad (15)$$

and repeated application of (12) and (13) should reduce (14) to the form (15).

III. MESON PRODUCTION EQUATIONS

The transition matrix element for the process of one meson of type k scattering into two mesons of types p_1 and p_2 is

$$T_k(p_1p_2)=(\Psi_{p_1p_2}{}^{(-)},V_k{}^{(0)}\Psi_0), \quad (16)$$

and an equation for this can be derived using the form of $\Psi_{p_1p_2}{}^{(-)}$ given by (10). This leads to

$$T_k(p_1p_2)=\frac{1}{\sqrt{2}}\left[(a_{p_1}{}^{\dagger}\Psi_{p_2}{}^{(-)},V_k{}^{(0)}\Psi_0)-\left(\frac{1}{H-\omega_1-\omega_2+i\epsilon}V_{p_1}{}^{(0)}\Psi_{p_2}{}^{(-)},V_k{}^{(0)}\Psi_0\right)\right],$$

and using the identity

$$a_p\Psi_0=-\frac{1}{H+\omega_p}V_p{}^{(0)\dagger}\Psi_0 \quad (17)$$

results in

$$T_k(p_1p_2)=-\frac{1}{\sqrt{2}}\left[\left(\Psi_{p_2}{}^{(-)},\ V_k{}^{(0)}\frac{1}{H+\omega_1}V_{p_1}{}^{(0)}\Psi_0\right)+\left(\Psi_{p_2}{}^{(-)},\ V_{p_1}{}^{(0)}\frac{1}{H-\omega_1-\omega_2-i\epsilon}V_k{}^{(0)}\Psi_0\right)\right]. \quad (18)$$

[6] The proof of (7) follows the method used by Wick to show the orthogonality of the one-meson states given by (12), although the proof here is more complicated. It is also possible in this way to show explicitly that the two-meson eigenstates as given by either (10) or (11) are orthogonal to the one-meson states.

Expanding in a complete set of eigenstates of the Hamiltonian gives

$$T_k(p_1p_2)=-\frac{1}{\sqrt{2}}\sum_n\left[\frac{(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_n{}^{(+)})(\Psi_n{}^{(+)},V_k{}^{(0)}\Psi_0)}{E_n-\omega_1-\omega_2-i\epsilon}+\frac{(V_k{}^{(0)\dagger}\Psi_{p2}{}^{(-)},\Psi_n{}^{(+)})(\Psi_n{}^{(+)},V_{p1}{}^{(0)\dagger}\Psi_0)}{E_n+\omega_1}\right],\quad(19)$$

where the complete set could be either outgoing- or incoming-wave solutions but the $\Psi_n{}^{(+)}$ are used because the matrix elements of the form $(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_n{}^{(+)})$ that appear can be related to T matrix elements with two mesons in the final state. Since ω_k does not appear in the denominators in (19), the dependence of $T_k(p_1p_2)$ on the energy of the initial state is a trivial one as was the case for the elastic scattering T matrix.

The "crossing symmetry" pointed out by Gell-Mann and Goldberger[7] and applied to the elastic-scattering T matrix by Chew and Low can be seen in (19). This symmetry is a consequence of the fact that, for any given Feynman diagram representing meson-nucleon interaction, there must also exist diagrams obtained by interchanging any two meson lines. Inspection of the form of (19) shows that, on the energy shell $(\omega_k=\omega_1+\omega_2)$, it possesses the crossing symmetry with respect to the incident meson and one outgoing meson and this is reflected in the occurrence of the energy of only one of the final mesons in the denominator of the "crossed" term. The other crossing symmetries are not apparent from the form of (19) because the two outgoing mesons have not been treated identically. The complete symmetry would be more evident if the T matrix were constructed from a two-meson eigenstate of the form (11),[8] but this would not be as readily related to the elastic scattering phase shifts.

Before (19) can be used as an equation for the meson-production T matrix, the matrix elements of the form $(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_n{}^{(+)})$ which appear in the sum on the right-hand side must be investigated and related to the T matrix. For the $n=0$ terms, these matrix elements reduce to the elastic-scattering T matrix which is known, either from theory or experimental phase shifts. For the $n=1$ terms, the matrix elements have the form $(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_k{}^{(+)})$ and this will be related to the meson-production T matrix.

An equation for $(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_k{}^{(+)})$ can be obtained by using (12) to expand $\Psi_k{}^{(+)}$. Then

$$(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_k{}^{(+)})=(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},a_k{}^\dagger\Psi_0)-\left(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\frac{1}{H-\omega_k-i\epsilon}V_k{}^{(0)}\Psi_0\right).\quad(20)$$

[7] M. Gell-Mann and M. L. Goldberger, in *Proceedings of the Fourth Annual Rochester Conference on High-Energy Physics* (University of Rochester Press, Rochester, 1954).

[8] Substitution of (11) into (16) leads to the form of the meson-production T matrix given by Barshay [Eq. (10) of reference 4].

In order to evaluate the first term on the right in (20), the quantity $a_k\Psi_{p2}{}^{(-)}$ has to be determined. This can be done by making use of a complete set of eigenstates of the Hamiltonian:

$$a_k\Psi_{p2}{}^{(-)}=\sum_n\Psi_n{}^{(-)}(\Psi_n{}^{(-)},a_k\Psi_{p2}{}^{(-)})=\sum_n\Psi_n{}^{(-)}(a_k{}^\dagger\Psi_n{}^{(-)},\Psi_{p2}{}^{(-)}),$$

which, by using (14), becomes

$$a_k\Psi_{p2}{}^{(-)}=\sum_n\Psi_n{}^{(-)}\left[([(n+1)!]^{\frac{1}{2}}\Psi_{n+k}{}^{(-)},\Psi_{p2}{}^{(-)})+\left(\frac{1}{H-E_n-\omega_k+i\epsilon}V_k{}^{(0)}\Psi_n{}^{(-)},\Psi_{p2}{}^{(-)}\right)\right],$$

where $\Psi_{n+k}{}^{(-)}$ is a state with $(n+1)$ mesons as defined by (14). States with different numbers of real mesons will be orthogonal, and the one-meson eigenstates form an orthonormal set so that

$$(\Psi_{n+k}{}^{(-)},\Psi_{p2}{}^{(-)})=\delta_{n,0}\delta_{k,p2},\quad(21)$$

and then

$$a_k\Psi_{p2}{}^{(-)}=\delta_{kp2}\Psi_0+\sum_n\Psi_n{}^{(-)}\times\left(\Psi_n{}^{(-)},\frac{1}{\omega_2-\omega_k-E_n-i\epsilon}V_k{}^{(0)\dagger}\Psi_{p2}{}^{(-)}\right).$$

Finally, applying closure,

$$a_k\Psi_{p2}{}^{(-)}=\delta_{kp2}\Psi_0-\frac{1}{H+\omega_k-\omega_2+i\epsilon}V_k{}^{(0)\dagger}\Psi_{p2}{}^{(-)}.\quad(22)$$

Using this result in (20) gives

$$(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_k{}^{(+)})=V_{p1}{}^\dagger\delta_{kp2}-\left(V_k{}^{(0)\dagger}\Psi_{p2}{}^{(-)},\frac{1}{H+\omega_k-\omega_2-i\epsilon}V_{p1}{}^{(0)\dagger}\Psi_0\right)-\left(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\frac{1}{H-\omega_k-i\epsilon}V_k{}^{(0)}\Psi_0\right),\quad(23)$$

and introducing the complete set of eigenstates, $\Psi_n{}^{(+)}$,

$$\frac{1}{\sqrt{2}}(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_n{}^{(+)})=\frac{1}{\sqrt{2}}V_{p1}{}^\dagger\delta_{kp2}-\frac{1}{\sqrt{2}}\sum_n\times\left[\frac{(V_k{}^{(0)\dagger}\Psi_{p2}{}^{(-)},\Psi_n{}^{(+)})(\Psi_n{}^{(+)},V_{p1}{}^{(0)\dagger}\Psi_0)}{E_n+\omega_k-\omega_2-i\epsilon}+\frac{(V_{p1}{}^{(0)}\Psi_{p2}{}^{(-)},\Psi_n{}^{(+)})(\Psi_n{}^{(+)},V_k{}^{(0)}\Psi_0)}{E_n-\omega_k-i\epsilon}\right].\quad(24)$$

The V_{p1} in (24) is now in terms of the renormalized coupling constant, i.e., $V_{p1}=(\Psi_0,V_{p1}^{(0)}\Psi_0)=(f/f^{(0)})V_{p1}^{(0)}$. Comparison of (24) with (19) shows that on the energy shell ($\omega_k=\omega_1+\omega_2$) the right-hand sides are identical, so that for a description of a real process either $(\Psi_{p1p2}^{(-)},V_k^{(0)}\Psi_0)$ or $(1/\sqrt{2})(V_{p1}^{(0)}\Psi_{p2}^{(-)},\Psi_k^{(+)})$ could be taken as the T matrix. The sums in (19) and (24) are not limited to the energy shell, however, so that the behavior of $(V_{p1}^{(0)}\Psi_{p2}^{(-)},\Psi_k^{(+)})$ off the energy shell is also important, and here the delta function in (24) leads to singular contributions. The delta function corresponds to meson production with no change in the incident meson and should not be included in the T matrix. A nonsingular T matrix can be defined by

$$\langle p_2p_1|T|k\rangle=\frac{1}{\sqrt{2}}(V_{p1}^{(0)}\Psi_{p2}^{(-)},\Psi_k^{(+)})-\frac{1}{\sqrt{2}}V_{p1}^{\dagger}\delta_{kp2}, \quad (25)$$

and then, separating out the zero-meson and one-meson terms from the sum in (24), the equation for the meson-production T matrix becomes

$$\langle p_2p_1|T|k\rangle=\frac{-1}{\sqrt{2}}\left[\frac{T_{p1}(p_2)V_k}{\omega_k}+\frac{T_k(p_2)V_{p1}^{\dagger}}{\omega_k-\omega_2-i\epsilon}-\frac{V_kT_{p1}(p_2^{+})}{\omega_k}-\frac{V_{p1}^{\dagger}T_k(p_2^{+})}{\omega_k-\omega_2+i\epsilon}\right]-\sum_q\left[\frac{\langle p_2k|T|q\rangle T_{p1}(q^{+})}{\omega_q+\omega_k-\omega_2-i\epsilon}+\frac{\langle p_2p_1|T|q\rangle T_k(q^{+})}{\omega_q-\omega_k-i\epsilon}\right]+\text{terms with } n>1. \quad (26)$$

The $T_k(p_2^{+})$ in (26) is defined by $T_k(p_2^{+})=(\Psi_{p2}^{(+)},V_k^{(0)}\Psi_0)$ and is closely related to $T_k(p_2)$, the only difference being that $T_k(p_2^{+})$ depends on $e^{-i\delta}\sin\delta$ instead of $e^{i\delta}\sin\delta$.

There are additional terms which have the appearance of one-meson terms. These come from singular two-meson terms corresponding to a state with two incident mesons, only one of which interacts with the nucleon. The contribution of these terms should be included with the one-meson terms in (24) and then, if contributions from states with more than one real meson are neglected (except for these singular terms), (24) becomes an integral equation for the meson-production T matrix. This equation is rather complicated and no attempt has been made to solve it, the numerical results of this paper being obtained from the first four terms of (24) which constitute the inhomogeneous part of the integral equation.

The procedure used in this section is not the only one by which an equation for meson production could be obtained. Starting from (25), the meson-production T matrix could be evaluated by expanding the final meson eigenstate, $\Psi_{p2}^{(-)}$. This would result in the meson production being described in terms of elastic scattering at the incident energy. It was felt that the resonance in the elastic scattering was more likely to be important for the outgoing mesons and this reasoning led to the procedure used here.

IV. APPROXIMATE T MATRIX

Keeping only the first four terms of (26) results in

$$\langle p_2p_1|T|k\rangle=\frac{1}{\sqrt{2}}\mathbf{S}_{12}\left[-\frac{T_k(p_2)V_{p1}^{\dagger}}{\omega_1}-\frac{T_{p1}(p_2)V_k}{\omega_k}+\frac{V_{p1}^{\dagger}T_k(p_2^{+})}{\omega_1}+\frac{V_kT_{p1}(p_2^{+})}{\omega_k}\right] \quad (27)$$

as an approximation to the meson-production T matrix. The symbol $\mathbf{S}_{12}$ means that the expression to the right is to be symmetrized with respect to 1 and 2 [i.e., $\mathbf{S}_{12}f(1,2)=\frac{1}{2}[f(1,2)+f(2,1)]$]. The approximate expression (27) has to be symmetrized explicitly because neglecting higher order terms in the sum over states destroys the inherent symmetry between the two outgoing mesons.

Equation (27) can be interpreted as representing two modes of meson production. One, corresponding to the first term in (27), is the production of one outgoing meson followed by the scattering, off the energy shell, of the incident meson into the other final meson. The other mode, corresponding to the third term in (27), has the scattering occurring before the production. The second and fourth terms in (27) are a consequence of the crossing symmetry of the T matrix and result from crossing one of the outgoing-meson lines with the incident-meson line in the first and third terms, respectively. The physical reasonableness of the terms in (27) and the fact that a large part of the meson-production T matrix has been treated exactly by relating it to the elastic-scattering T matrix, suggest that the approximation of neglecting nonsingular "real meson" terms in (24) does not distort the picture too greatly. The approximation used here corresponds to treating the scattering of the incident meson exactly and using the Born approximation for the production of the other meson. All f^3 terms for meson production are correctly included in (27).

Comparison of Eq. (27) with the recent calculation by Barshay [Eq. (15) or (20) of Barshay] shows that Barshay's result, aside from a factor of $1/\sqrt{2}$ resulting from different normalizations, is exactly twice the first term of (27) and Barshay does not obtain the other terms of (27). The second term of (27) vanishes for the pure $\frac{3}{2}$ state of isotopic spin (π^{+} on protons) which Barshay considers, but the rest of the discrepancy between Barshay's results and those of this paper is a consequence of the different approximations used by Barshay to identify certain parts of the meson-production T matrix with the elastic-scattering T matrix. For example Barshay reduces the quantity [the third term of Eq. (10) of Barshay]

$$2S_{12}\sum_{n,m}\frac{(\Psi_0,V_k^{(0)}\Psi_n^{(\pm)})\times(\Psi_n^{(\pm)},V_{p1}^{(0)}\Psi_m^{(\mp)})(\Psi_m^{(\mp)},V_{p2}^{(0)\dagger}\Psi_0)}{(E_n+\omega_1+\omega_2)(E_m+\omega_2)},\quad(28)$$

to

$$2S_{12}\left(\frac{V_kV_{p1}^\dagger}{\omega_1}+\sum_q\frac{(\Psi_0,V_k^{(0)}\Psi_q^{(-)})(\Psi_q^{(-)},V_{p1}^{(0)\dagger}\Psi_0)}{\omega_q+\omega_1}\right)\frac{V_{p2}^\dagger}{\omega_2}\quad(29)$$

by limiting the sum over m to zero-meson states and the sum over n to zero- and one-meson states and neglecting the ω_2 in the first factor of the denominator of (28). The factor in parentheses in (29) is then recognized as the one-meson approximation to the "crossed" term of the Chew-Low scattering matrix $[T_k(p_1)]$. This reduction can be done more exactly by using closure to sum the series in (28) and then noticing that

$$\begin{aligned}2S_{12}&\left(\Psi_0,\,V_k^{(0)}\frac{1}{H+\omega_1+\omega_2}V_{p1}^{(0)\dagger}\frac{1}{H+\omega_2}V_{p2}^{(0)\dagger}\Psi_0\right)\\&=\left(\Psi_0,\,V_k^{(0)}\frac{1}{H+\omega_1+\omega_2}\left[V_{p1}^{(0)\dagger}\frac{1}{H+\omega_2}V_{p2}^{(0)\dagger}+V_{p2}^{(0)\dagger}\frac{1}{H+\omega_1}V_{p1}^{(0)\dagger}\right]\Psi_0\right)\\&=(\Psi_0,V_k^{(0)}a_{p1}a_{p2}\Psi_0)\\&=-S_{12}\sum_m\frac{(\Psi_0,V_k^{(0)}a_{p1}\Psi_m^{(\pm)})(\Psi_m^{(\pm)},V_{p2}^{(0)\dagger}\Psi_0)}{E_m+\omega_2}.\end{aligned}\quad(30)$$

If the sum over m in (30) is now limited to zero-meson states, the result is equivalent to (29), except that the exact "crossed" term appears and the result in (30) is one-half that of (29). Extending the sum over m in (30) produces a singularity which leads to the third term of (27). The rest of the discrepancy comes from combining the second term of Eq. (10) of Barshay with the third term of that equation to obtain the final result. Actually, for a more exact identification with the elastic-scattering T matrix, only one-half of this second term should be combined with the third term, the other half combining with the first term of Barshay (10) to lead to the second and fourth terms of (27).

The T matrix for meson production is more readily applicable to various initial and final states if it is expanded in terms of eigenstates of the two-meson final state. This can be done by using projection operators for scattering from a one-meson eigenstate of isotopic spin and total angular momentum to a two-meson eigenstate. The isotopic spin projection operators for the process of one meson of type k scattering into two mesons of types p and q are

$$\begin{aligned}T_{1,0}(qp,k)&=\tfrac{1}{3}\delta_{qp}\tau_k,\\T_{1,1}(qp,k)&=(\tau_q\tau_p-\delta_{qp})\tau_k/3\sqrt{2},\\T_{3,2}(qp,k)&=(\tau_q\delta_{pk}+\tau_p\delta_{qk}-\tfrac{2}{3}\delta_{qp}\tau_k)/\sqrt{10},\\T_{3,1}(qp,k)&=[\tau_q\delta_{pk}-\tau_p\delta_{qk}-\tfrac{2}{3}(\tau_q\tau_p-\delta_{qp})\tau_k]/\sqrt{2}.\end{aligned}\quad(31)$$

The first subscript on the projection operator is twice the total isotopic spin and the second subscript is the isotopic spin of the two outgoing mesons. The states with even meson isotopic spin are symmetric with respect to interchange of the isotopic variables of the two outgoing mesons and the states with odd meson isotopic spin are antisymmetric. The projection operators for angular momentum, $J_{1,0}$, $J_{1,1}$, $J_{3,2}$, and $J_{3,1}$, have the same general properties and can be obtained from the isotopic spin projection operators by the identifications

$$\tau_q\to\sqrt{3}\boldsymbol{\sigma}\cdot\mathbf{q}\quad\text{and}\quad\delta_{qp}\to3\mathbf{q}\cdot\mathbf{p}.\quad(32)$$

The projection operators defined by (31) and (32) are exact and their application is not limited to the particular approximation used in this paper.

The elastic scattering T matrix appearing in (24) can be expressed in terms of phase shifts:

$$T_k(p)=-\frac{4\pi v(p)v(k)}{(4\omega_p\omega_k)^{\frac{1}{2}}}\sum_{\alpha=11,13,31,33}h_\alpha(p)P_\alpha(pk),\quad(33)$$

where the $P_\alpha(pk)$ are projection operators as given by Chew and Low for the four eigenstates of total angular momentum and isotopic spin of a system of one meson and one nucleon. $T_k(p^+)$ is also given by (33) except that $h_\alpha^*(p)$ appears instead of $h_\alpha(p)$. The $h_\alpha(p)$ are related to the phase shifts $\delta_\alpha(p)$, by

$$h_\alpha(p)=\exp[i\delta_\alpha(p)]\sin[\delta_\alpha(p)]/p^3v^2(p).\quad(34)$$

The phase shifts could be taken either from experimental values or could be the solutions to the Low integral equations obtained by making the "one-meson" approximation to the elastic-scattering T matrix. For the calculations in this paper experimental values, as fitted to the effective-range formula of Chew and Low by Orear,[9] have been used. Since, experimentally, h_{11}, h_{13}, and h_{31} are small compared to h_{33} and have no well defined values, the approximation will be made that only h_{33} does not equal zero.

Using (33) for the elastic-scattering T matrix and the projection operators given by (31) and (32), the meson-

[9] J. Orear, Phys. Rev. **100**, 288 (1955).

production T matrix becomes

$$\langle p_2p_1|T|k\rangle = \frac{-i(4\pi)^{\frac{3}{2}}(f/\mu)v(k)v(p_1)v(p_2)}{(48\omega_k\omega_1\omega_2)^{\frac{1}{2}}} \times \mathbf{S}_{12}\left[A(p_2p_1,k)\frac{h_{33}(p_2)}{\omega_1} - B(p_2p_1,k)\frac{h_{33}(p_2)}{\omega_k} - C(p_2p_1,k)\frac{h_{33}^*(p_2)}{\omega_1} + D(p_2p_1,k)\frac{h_{33}^*(p_2)}{\omega_k}\right], \quad (35)$$

where

$$\begin{aligned} A &= (1/36)[8T_{1,0} - 4\sqrt{2}T_{1,1} + (10)^{\frac{1}{2}}T_{3,2} - 5\sqrt{2}T_{3,1}] \\ &\quad \times [8J_{1,0} - 4\sqrt{2}J_{1,1} + (10)^{\frac{1}{2}}J_{3,2} - 5\sqrt{2}J_{3,1}], \\ B &= (2T_{1,0} - \sqrt{2}T_{1,1})(2J_{1,0} - \sqrt{2}J_{1,1}), \\ C &= \tfrac{1}{4}[(10)^{\frac{1}{2}}T_{3,2} - \sqrt{2}T_{3,1}][(10)^{\frac{1}{2}}J_{3,2} - \sqrt{2}J_{3,1}], \\ D &= (1/9)(6T_{1,0} + \sqrt{2}T_{1,1} + 2\sqrt{2}T_{3,1}) \\ &\quad \times (6J_{1,0} + \sqrt{2}J_{1,1} + 2\sqrt{2}J_{3,1}). \end{aligned} \quad (36)$$

V. CROSS SECTIONS

The differential cross section for the scattering of a meson of type k into meson of types p_1 and p_2 is

$$\frac{d\sigma(k,p_1,p_2)}{d\omega_1 d\Omega_1 d\Omega_2} = \frac{p_1p_2\omega_1\omega_2}{(2\pi)^5 v_{\rm rel}}|\langle p_2p_1|T|k\rangle|^2, \quad (\omega_1 = \omega^* - \omega_2); \quad (37)$$

where $v_{\rm rel}$ is the relative velocity between the incident meson and the nucleon and, in the static limit, $v_{\rm rel} = k/\omega_k$. No attempt has been made to correct either $v_{\rm rel}$ or the phase space factors for nucleon recoil because this would not be consistent with the no-recoil approximation used in evaluating the T matrix. The energy ω^* which is the incident-meson total energy plus the nucleon kinetic energy $[\omega^* = \omega_k + (k^2+M^2)^{\frac{1}{2}} - M]$ in the barycentric system, has been used in the energy conservation in (37) rather than ω_k because it would seem to more nearly represent the amount of energy available to the two outgoing mesons. In substituting for $\langle p_1p_2|T|k\rangle$ in (37), the approximate result given by (35) will be used.

1. Total Cross Sections for Incident Eigenstates of Total Isotopic Spin and Angular Momentum

For the extension of the Low equations to include two-meson states, the total meson-production cross sections for the "11," "13," "31," and "33" eigenstates of total isotopic spin and total angular momentum are needed. These are obtained by taking the absolute square of the T matrix as given by (35) and summing (or integrating) over all p_1 and p_2, making use of the properties of the projection operators, and keeping separate all those terms corresponding to the same initial eigenstate. The result is

$$\begin{pmatrix}\sigma_1(\omega^*)\\ \sigma_2(\omega^*)\\ \sigma_3(\omega^*)\end{pmatrix} = (1/243)\left\{\begin{pmatrix}576\\360\\954\end{pmatrix}I_1 + \begin{pmatrix}64\\-80\\424\end{pmatrix}I_2 + \begin{pmatrix}4360\\304\\64\end{pmatrix}I_3 + \begin{pmatrix}1480\\-272\\64\end{pmatrix}I_4 + \begin{pmatrix}-2592\\0\\-72\end{pmatrix}I_5 + \begin{pmatrix}-288\\0\\-72\end{pmatrix}I_6 + \begin{pmatrix}0\\0\\-450\end{pmatrix}I_7 + \begin{pmatrix}-1800\\0\\0\end{pmatrix}I_8 + \begin{pmatrix}-3528\\0\\0\end{pmatrix}I_9 + \begin{pmatrix}800\\-400\\200\end{pmatrix}I_{10} + \begin{pmatrix}1568\\560\\200\end{pmatrix}I_{11}\right\}, \quad (38)$$

where

$$I_\alpha = kv^2(k)(f/\mu)^2\int_\mu^{\omega^*-\mu} v^2(p_1)v^2(p_2)p_1^3p_2^3 d\omega_1 F_\alpha, \quad (\omega_2 = \omega^* - \omega_1) \quad (39)$$

and

$$\begin{aligned} F_1 &= |h_{33}(p_2)|^2/\omega_1^2, \\ F_2 &= h_{33}^*(p_2)h_{33}(p_1)/\omega_1\omega_2, \\ F_3 &= |h_{33}(p_2)|^2/\omega^{*2}, \\ F_4 &= h_{33}^*(p_2)h_{33}(p_1)/\omega^{*2}, \\ F_5 &= |h_{33}(p_2)|^2/\omega_1\omega^*, \\ F_6 &= h_{33}^*(p_2)h_{33}(p_1)/\omega_1\omega^*, \\ F_7 &= \mathrm{Re}[h_{33}^2(p_2)]/\omega_1^2, \\ F_8 &= \mathrm{Re}[h_{33}^2(p_2)]/\omega^{*2}, \\ F_9 &= \mathrm{Re}[h_{33}(p_2)h_{33}(p_1)]/\omega^{*2}, \\ F_{10} &= \mathrm{Re}[h_{33}^2(p_2)]/\omega_1\omega^*, \\ F_{11} &= \mathrm{Re}[h_{33}(p_2)h_{33}(p_1)]/\omega_1\omega^*. \end{aligned} \quad (40)$$

The (1,3) and (3,1) cross sections are equal and so the three cross sections $\sigma_1 = \sigma_{11}$, $\sigma_2 = \sigma_{13} = \sigma_{31}$, and $\sigma_3 = \sigma_{33}$ have been used in (38). The energy dependence of h_{33} has been obtained by using the effective-range approximation of Chew and Low,

$$\cot\delta_{33}(k) = \omega^*(8.05 - 3.8\omega^*)/k^3, \quad (41)$$

which Orear has used to fit the experimental phase shifts up to at least 300 Mev (about $\omega^* = 2.5\mu$ in the barycentric system). The energies at which h_{33} is needed are those of one of the outgoing mesons and this will usually be within the range of validity of (41). This fact coupled with the dominance, in the integrations, of the resonance in h_{33} makes it sensible to use (39) at all energies. Equation (39) corresponds to $f^2 = 0.093$ and this value has been used in calculations. None of the integrations is critically dependent on the cutoff since they all have finite upper limits and the cutoff function has been set equal to unity throughout.

The integrations have been done numerically and the results for the total meson-production cross sections for the various initial eigenstates are plotted in Fig. 1. Limiting the sum in (18) to include only those states with no real mesons, has destroyed the unitarity of the meson-production S matrix and therefore the total cross sections are not limited by the geometrical cross section ($\pi\lambda^2$) in this approximation. This is not too serious a defect because the energies at which the total cross sections approach $\pi\lambda^2$ are above those at which the static approximation should be expected to have validity anyway. However, to investigate the contribution of two-meson states to the elastic scattering equations the total cross sections are required at all energies; what has been done here is to use the cross sections plotted in Fig. 1 up to energies at which they approach $\pi\lambda^2$ and then to use $\pi\lambda^2$ beyond that energy. This procedure probably overestimates the contribution of the two-meson states since the actual cross sections must turn over smoothly and always remain below the geometrical cross section.

2. Contribution of Two-Meson States to the "Effective Range" for Elastic Scattering

The effect of two-meson states on Low's equations for elastic scattering will be investigated by looking at their contribution to the effective range defined by Chew and Low. The effective-range approximation results from expanding $\lambda_\alpha k^3 \cot[\delta_\alpha(\omega^*)]/\omega^*$ in powers of the incident energy, ω^*. The coefficient of the linear term in the expansion is the effective range, r_α, and is given by

$$r_\alpha = \frac{1}{4\pi^2\lambda_\alpha}\int_\mu^\infty \frac{d\omega^*}{kv^2(k)\omega^*}\left[\sigma_\alpha(\omega^*)+\sum_{\beta=1}^{3} A_{\alpha\beta}\sigma_\beta(\omega^*)\right], \quad (42)$$

where $\sigma_\alpha(\omega^*)$ is the total cross section for the state α and energy ω^* and

$$\lambda = \tfrac{2}{3}(f/\mu)^2\begin{pmatrix}-4\\-1\\2\end{pmatrix}, \quad A=(1/9)\begin{pmatrix}1 & -8 & 16\\ -2 & 7 & 4\\ 4 & 4 & 1\end{pmatrix}, \quad (43)$$

as given by Chew and Low. The contribution, $r_\alpha^{2\pi}$, of the two-meson states to the effective range has been calculated using the meson-production cross sections in Fig. 1 and these results are compared in Table I to the effective range, r_α^{el}, calculated from the elastic scattering cross sections obtained by using (41) with the approximation that only δ_{33} is not zero.

3. Partial Cross Sections for $p+\pi^+$ and $p+\pi^-$

The projection operators in (35) can be used to give partial cross sections for various charge states. The procedure is as follows: The initial state of a nucleon and a meson of type k is described by $Y_{1,m}(k)X_{\frac{1}{2},s}$, where $Y_{1,m}$ is a spherical harmonic in isotopic space and $X_{\frac{1}{2},s}$ is a nucleon isotopic spin function; m and s are the projections on the z axis in isotopic space; $m=+1, 0, -1$ corresponds to π^+, π^0, π^-, respectively and $s=(\pm)\frac{1}{2}$ to a (proton/neutron). This initial pion-nucleon state is then expressed in terms of the total isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$ states by using Clebsch-Gordan coefficients. The projection operators defined by (31) project out eigenstates of total isotopic spin and a given meson isotopic spin and these states are expanded in charge states for each outgoing particle using the appropriate Clebsch-Gordan coefficients. This procedure gives the T matrix for the production of any particular charge state from any given initial charge state and the cross section for each process will depend on $|T|^2$. The angular integrations are easily done when one makes use of the properties of the angular momentum projection operators.

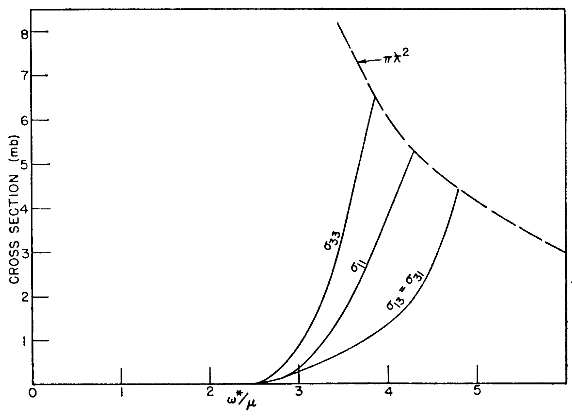

FIG. 1. Total meson production cross sections for incident eigenstates of total angular momentum and isotopic spin. The dashed curve represents the geometrical cross section ($\pi\lambda^2$) which is an upper limit for meson production.

The procedure outlined above has been applied to the case of a π^+ or π^- meson incident on a proton, and the cross sections for the various possible reactions are

$$\sigma(\omega^*)\begin{pmatrix}\pi^+\to\pi^++\pi^0\\ \pi^+\to\pi^++\pi^+\\ \pi^-\to\pi^-+\pi^0\\ \pi^-\to\pi^0+\pi^0\\ \pi^-\to\pi^-+\pi^+\end{pmatrix} = (1/81)\left\{\begin{pmatrix}396\\360\\268\\136\\136\end{pmatrix}I_1+\begin{pmatrix}48\\208\\752/9\\16/9\\-21\frac{1}{3}\end{pmatrix}I_2+\begin{pmatrix}144\\0\\208\\288\\656\end{pmatrix}I_3\right.$$

$$+\begin{pmatrix}-48\\0\\-69\frac{1}{3}\\96\\165\frac{1}{3}\end{pmatrix}I_4+\begin{pmatrix}-48\\0\\-218\frac{2}{3}\\-128\\-245\frac{1}{3}\end{pmatrix}I_5+\begin{pmatrix}-48\\0\\58\frac{2}{3}\\-85\frac{1}{3}\\-53\frac{1}{3}\end{pmatrix}I_6$$

$$+\begin{pmatrix}-180\\-120\\-73\frac{1}{3}\\-66\frac{2}{3}\\40\end{pmatrix}I_7+\begin{pmatrix}0\\0\\106\frac{2}{3}\\-160\\-346\frac{2}{3}\end{pmatrix}I_8$$

$$\left.+\begin{pmatrix}0\\0\\-149\frac{1}{3}\\-224\\-410\frac{2}{3}\end{pmatrix}I_9+\begin{pmatrix}320\\0\\1664/9\\213\frac{1}{3}\\2752/9\end{pmatrix}\right\}, \quad (44)$$

TABLE I. Contributions to the effective range.[a]

State	r_α^{el}	$r_\alpha^{2\pi}$	$r_\alpha^{2\pi}/r_\alpha^{el}$
11	−0.293	−0.00470	1.6%
13 or 31	−0.293	−0.0123	4.2%
33	0.366	0.00778	2.1%

[a] The effective ranges are in units of $1/\mu$.

TABLE II. Partial cross sections for meson production.

Center-of-mass energy (ω^*)	2.5μ	3μ	4μ
Laboratory kinetic energy	260 Mev	360 Mev	570 Mev
Process	Cross section (mb)		
$p+\pi^+\rightarrow p+\pi^++\pi^0$	0.037	1.04	6.54
$p+\pi^+\rightarrow n+\pi^++\pi^+$	0.041	0.97	9.25
$p+\pi^-\rightarrow p+\pi^-+\pi^0$	0.029	0.66	4.20
$p+\pi^-\rightarrow n+\pi^0+\pi^0$	0.007	0.30	2.47
$p+\pi^-\rightarrow n+\pi^++\pi^-$	0.016	0.39	2.94

where the I_α are given by (39) and (40). The cross sections given by (44) at several energies are listed in Table II. By charge independence the cross sections for $\pi^\pm$ on protons also apply to $\pi^\mp$ on neutrons.

VI. DISCUSSION

The contribution of two-meson states to the effective range for elastic scattering is small in the approximation used in this paper as can be seen from Table I. The smallness of the ratio $r^{2\pi}/r^{el}$ seems to be another instance of the dominance of the "33" elastic scattering resonance in integrals involving meson-nucleon cross sections; including two-meson states in a solution of the cutoff-model elastic-scattering equations will not be worth while if they contribute as little to the effective range as indicated in Table I.

The only experimental information in the range of energies at which the partial cross sections in Table II apply comes from an investigation by Blau and Caulton[10] of meson production by 500 Mev ($\omega^*=3.7\mu$) π^- mesons on emulsion nuclei. On the basis of their findings, they make a rough estimate of 3.5−10 mb for the cross section for the production of charged mesons by free nucleons. This cross section corresponds to the average (assuming equal numbers of neutrons and protons in the emulsion) of the cross sections for the processes $p+\pi^-\rightarrow n+\pi^-+\pi^+$ and $n+\pi^-\rightarrow p+\pi^-+\pi^-$, for which the theoretical prediction from Table II is about 4.5 mb.

The $p+\pi^+$ cross sections found in this paper are generally smaller than those of Barshay for the reasons given earlier. Also, the large percentage of π^0 production which Barshay found does not hold for the cross sections found here because of the added terms in the T matrix.

A strong limitation of the approach used here or by Barshay is the neglect of nucleon recoil which, among other things, restricts the treatment to p-wave mesons. To include nucleon recoil in the theory, the most promising possibility at present would seem to be a relativistic "dispersion" relation of the type which has been found for elastic scattering[11] and photoproduction.[12] Experience so far has shown that for every Low-Wick type equation in the cutoff model there is a corresponding dispersion relation. Thus one should expect to find a relativistic equation of the same general form as Eq. (24).

I wish to express my sincere appreciation to Professor Geoffrey F. Chew for suggesting this problem and for his continued assistance. This work would not have been possible without his guidance and encouragement.

[10] M. Blau and Martin Caulton, Phys. Rev. **96**, 150 (1954).

[11] M. L. Goldberger, Phys. Rev. **99**, 979 (1955).

[12] F. E. Low, *Sixth Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1956).