gives no indication of what secular changes may occur in radiating matter. Suppose for example that a Schwarzschild particle is disturbed from static spherical symmetry by an internal agency, radiates for some time, and finally is restored to static spherical symmetry. Is its total mass necessarily the same as before? This and similar problems required investigation. Also the status of the scalar invariants of the Riemann tensor in the Einstein, Infeld, and Hoffmann approximation theory deserves clarification, and may be hoped to assist in resolving the annoying ambiguities of interpretation which beset that theory.

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Application of Formal Scattering Theory to Many-Body Problems*

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It is pointed out that the Lippmann-Schwinger integral equation, as it is usually written, does not necessarily have a unique solution when applied to the motion of three or more bodies. It follows that a certain amount of caution must be exercised in using the Lippmann-Schwinger equation as the basis for an approximation procedure.

FORMAL scattering theory[1] has been used extensively in the recent literature to deal with rearrangement and inelastic collisions as well as elastic scattering. Nevertheless it is not widely recognized that in applying the formulas of formal scattering theory to reactions involving more than two particles a certain amount of caution must be exercised. This is a consequence of the fact that the Lippmann-Schwinger (L-S) integral equation for the wave function, as it is usually written, does not necessarily have a unique solution when three or more bodies interact while for the two-body case the solution is unique.

In formal scattering theory one constructs from the Schrödinger equation an integral equation for the wave function. The solution of this integral equation, besides being a solution of the original Schrödinger equation, presumably satisfies the required asymptotic boundary conditions. In view of the fact that certain forms of the Lippmann-Schwinger integral equation do not have a unique solution, evidently they fail to incorporate these conditions fully.

The fact that the solution to the Lippmann-Schwinger equation is not unique means that care must be used in applying any scheme of successive approximations to generate a solution to the L-S equation. One must make sure that the solution generated is the particular one desired.

The problem of formal scattering theory is to construct the solution of a given Schrödinger equation

$$(E-H_0-V)\Psi=0, \tag{1}$$

which satisfies certain asymptotic boundary conditions. This problem is solved in a formal way by the Lippmann-Schwinger integral equation.

$$\begin{aligned}\Psi&=\lim_{\epsilon\to 0}\Psi^{(\epsilon)},\\ \Psi^{(\epsilon)}&=i\epsilon(E-H_0-V+i\epsilon)^{-1}\Phi\\ &=\Phi+(E-H_0+i\epsilon)^{-1}V\Psi^{(\epsilon)}.\end{aligned} \tag{2}$$

The solution obtained in this way is unique. However, the Lippmann-Schwinger equation is more often written in the following way.

$$\Psi=\Phi+\lim_{\epsilon\to 0}(E-H_0+i\epsilon)^{-1}V\Psi. \tag{3}$$

While any solution obtained by the procedure of Eq. (2) will satisfy Eq. (3), Eq. (3) has solutions which cannot be obtained by the procedure of Eq. (2).

We can show that the solution of Eq. (3) is not unique by showing the existence of a solution to its homogeneous counterpart,

$$\Psi=\lim_{\epsilon\to 0}(E-H_0+i\epsilon)^{-1}V\Psi. \tag{4}$$

Then, clearly, by adding an arbitrary multiple of a solution of Eq. (4) to the solution of Eq. (3), we get a second solution of Eq. (3).

The simplest case where this occurs is the problem of the mutual scattering of two particles by a potential V acting between them provided a bound state of the two particles in the potential exists. For convenience, let us

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[1] C. Moller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945); B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950); M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953); S. Sunakawa, Progr. Theoret. Phys. Japan **14**, 175 (1955).

refer to the two particles as neutron and proton and the bound state as a deuteron, although this particular reference is quite unnecessary. In this case a plane wave of deuterons of the correct total energy E is a rigorous solution of Eq. (4) with $H_0=T=$the total kinetic energy operator. This can be verified by direct substitution. Of course, the difficulty can be easily circumvented in this case if one specifies the total momentum as well as the total energy of the solution, but such considerations are inadequate for dealing with more complicated problems.

As an example of a case in which the solution to Eq. (3) cannot be made unique by fixing the total momentum of the system, consider a system consisting of three particles—N (a neutron), P (a proton), and C (a carbon nucleus). Let V_{NP} be the interaction potential between N and P, and let V_{CP} be the interaction between C and P. Assume no interaction between C and N. Then the Schrödinger equation reads

$$(E-T-V_{PC}-V_{PN})\Psi=0, \tag{5}$$

where T is the total kinetic energy operator.

Suppose we want to describe the situation where we have a beam of neutrons incident on bound $C+P$ systems. Then the wave function, Ψ_1, is a solution of

$$\Psi_1=\Phi+\lim_{\epsilon\to 0}G_0^{(\epsilon)}V_{NP}\Psi_1, \tag{6}$$

where

$$(E-T-V_{CP})\Phi=0,$$
$$G_0^{(\epsilon)}=(E-T-V_{CP}+i\epsilon)^{-1},$$

and Φ is the wave function for the incident beam. If, on the other hand, the incident beam consists of carbon nuclei incident on deuterons, the Lippmann-Schwinger equation reads

$$\Psi_2=\chi+\lim_{\epsilon\to 0}\mathcal{G}_0^{(\epsilon)}V_{CP}\Psi_2, \tag{7}$$

where

$$(E-T-V_{NP})\chi=0,$$
$$\mathcal{G}_0^{(\epsilon)}=(E-T-V_{NP}+i\epsilon)^{-1}.$$

Suppose now that our system is partly in state Ψ_1 and partly in state Ψ_2. Then the wave function would be given by

$$\Psi=a\Psi_1+b\Psi_2=\lim_{\epsilon\to 0}\Psi^{(\epsilon)},$$
$$\Psi^{(\epsilon)}=i\epsilon(E-T-V_{CP}-V_{NP}+i\epsilon)^{-1}(a\Phi+b\chi). \tag{8}$$

Equation (8) can be rewritten as follows:

$$aG_0^{(\epsilon)-1}\Phi+b\,\mathcal{G}_0^{(\epsilon)-1}\chi=(G_0^{(\epsilon)-1}-V_{NP})\Psi^{(\epsilon)}$$
$$=(\mathcal{G}_0^{(\epsilon)-1}-V_{CP})\Psi^{(\epsilon)}. \tag{9}$$

Operating through by $G_0^{(\epsilon)}$ or $\mathcal{G}_0^{(\epsilon)}$ and taking the limit $\epsilon\to 0$ gives

$$\Psi=a\Phi+\lim_{\epsilon\to 0}G_0^{(\epsilon)}V_{NP}\Psi^{(\epsilon)}$$
$$+b\lim_{\epsilon\to 0}i\epsilon(E-T-V_{CP}+i\epsilon)^{-1}\chi \tag{10A}$$

or

$$\Psi=b\chi+\lim_{\epsilon\to 0}\mathcal{G}_0^{(\epsilon)}V_{CP}\Psi^{(\epsilon)}$$
$$+a\lim_{\epsilon\to 0}i\epsilon(E-T-V_{NP}+i\epsilon)^{-1}\Phi. \tag{10B}$$

The last terms on the right of Eqs. (10A) and (10B) vanish[2] and the $\Psi^{(\epsilon)}$ which occurs in the second terms on the right can be replaced by Ψ.[3] Thus

$$\Psi=a\Phi+\lim_{\epsilon\to 0}G_0^{(\epsilon)}V_{NP}\Psi \tag{11A}$$

$$=b\chi+\lim_{\epsilon\to 0}\mathcal{G}_0^{(\epsilon)}V_{CP}\Psi. \tag{11B}$$

We see that if we set $a=1$, then

$$\Psi=\Phi+\lim_{\epsilon\to 0}G_0^{(\epsilon)}V_{NP}\Psi \tag{12}$$

has infinitely many different solutions corresponding to all the possible choices for the constant b.

If one wishes to generate a particular eigenstate of $H=T+V_{CP}+V_{NP}$ by some scheme of successive approximations, the safest procedure would be based on Eq. (8) whose solution is unique rather than on Eq. (12) whose solution is not. Ultimately one would take the limit $\epsilon\to 0$. However, in basing an approximation scheme on Eq. (8) one must be certain that this scheme converges uniformly in ϵ in a finite region containing $\epsilon=0$ if one is to insure that the resultant approximate solution will yield a good approximation to the particular solution desired. This is a consequence of the fact that the undesired solution is an approximate solution to the homogeneous equation to order ϵ. Hence if the convergence is not uniform, one may find a residuum of the undesired solution in the limit $\epsilon\to 0$.

An approximation scheme which makes use of both Eq. (11A) and Eq. (11B) would appear to be safe from the difficulties under discussion. However, although it is certainly plausible that a simultaneous solution of Eq. (11A) and Eq. (11B) is unique, a rigorous proof to this effect is lacking.

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[3] W. Tobocman and L. L. Foldy (unpublished).