# Invariant Formulation of Gravitational Radiation Theory

F. A. E. Pirani
*Department of Mathematics, King's College, Strand, London, England*

(Received October 18, 1956)

In this paper, gravitational radiation is defined invariantly within the framework of general relativity theory. The definition is arrived at by assuming (*a*) that gravitational radiation is characterized by the Riemann tensor, and (*b*) that it is propagated with the fundamental velocity. Therefore a gravitational wave front should appear as a discontinuity in the Riemann tensor across a null 3-surface; the possible form of this discontinuity is here calculated from Lichnerowicz's continuity conditions.

The concept of an observer who follows the gravitational field is defined in terms of the eigenbivectors of the Riemann tensor. It is shown that the 4-velocity of this observer is timelike for one of Petrov's three canonical types of Riemann tensor, but null for the other two types. The first type is identified with the absence of radiation, the other two with its presence. This constitutes the definition. It is shown that the difference between the no-radiation type and one of the radiation types can be made to correspond to the discontinuity possible across a null 3-surface; this demonstrates the consistency of the wave front and following-the-field concepts.

A covariant approximation to the canonical energy-momentum pseudo-tensor is defined, using normal coordinates, which are given a physical interpretation. It is shown that when gravitational radiation is present, the approximate gravitational energy-flux cannot be removed by a local Lorentz transformation, which supports the definition of radiation.

It is proved that, as would be demanded of a sensible definition, there can be no gravitational radiation present in a region of empty space-time where the metric is static.

## 1. INTRODUCTION

The investigation of gravitational radiation in general relativity theory is hampered by the lack of an invariant definition of that concept. The presence of gravitational radiation must be distinguishable, mathematically, from a peculiar choice of the coordinate system, and physically, from a peculiar motion of the observer. In a covariant, nonlinear theory, the definition should not, if the concept of radiation has any real validity, depend on the weakness of fields or on special coordinate conditions. An invariant definition is proposed in this paper.

This definition is given in terms of the Riemann tensor. Just as it is the Riemann tensor which indicates a genuine gravitational field in the first place, so

(*A*) It is the Riemann tensor which characterizes the presence of radiation.

Physically, this is because the Riemann tensor describes the variations in the gravitational field from event to event in space-time. In accordance with the principle of equivalence, only the variations in the field, and not the field itself, can produce any real physical effects. The question is: what sort of variations in the field should be classified as gravitational radiation?

To answer this question, one must first of all decide which attributes of radiation, a concept until now familiar largely through electromagnetic theory, may be assumed to apply also to the gravitational case. In making the present definition, it will be assumed that an essential attribute is:

(*B*) In empty space-time, gravitational radiation is propagated with the fundamental velocity.

The two assumptions (*A*) and (*B*) serve to characterize gravitational radiation completely. Two main arguments are developed in the following sections to support the definition; these arguments depend respectively on the following consequences of (*A*) and (*B*):

(*C*) A gravitational wave-front manifests itself as a discontinuity in the Riemann tensor across a null 3-surface.

(*D*) The motion of an observer following the gravitational field is determined by the Riemann tensor. In the presence of gravitational radiation, such an observer would have to move with the fundamental velocity in order to keep up with the field.

These ideas will now be developed in more detail. In connection with (*A*), one may investigate the variations in the gravitational field directly by writing down the equation of geodesic deviation. This equation gives the variation in the field between neighboring space-time events in terms of the Riemann tensor.[1] The physical effects so represented are set out in detail in Sec. 2.

Assumption (*B*) is supported by very general considerations, as well as some specific ones, like the result of Lichnerowicz[2] that the characteristic surfaces of Einstein's equations are null 3-surfaces. Lichnerowicz starts from continuity conditions which are sufficient to ensure that the equations have physically unique solutions in empty space-time.

In Sec. 2, Lichnerowicz's conditions will be used to determine what discontinuity in the Riemann tensor is permissible across a null 3-surface. In accordance with statement (*C*) above, one would expect to find such a discontinuity whenever a source of gravitational radiation was switched on or off.

The idea of an observer following the field, introduced

[1] F. A. E. Pirani, Helv. Phys. Acta (to be published); and Acta Phys. Polon. (to be published).
[2] A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme* (Masson et Cie, Paris, 1955), p. 33.

in statement ($D$), is one already well known in the ordinary Maxwell-Lorentz electromagnetic theory, and not difficult to generalize to gravitational field theory. In the Maxwell-Lorentz theory, an observer is said to be following the field if he moves so that in his rest-frame the Poynting vector vanishes. He therefore observes no flux of field energy. If this idea is restated covariantly (but still in the Maxwell-Lorentz theory) in terms of the energy-tensor of the field, it is found that an observer may always follow the field by acquiring a suitable 4-velocity, unless it is a null field (self-conjugate field), in which case it would be necessary to acquire the fundamental velocity in order to make the energy flux vanish. This is because in a null field **E** and **H** are perpendicular and of equal magnitude in every Lorentz frame. Plane waves and spherical waves are common examples of null fields.[3] From the point of view adopted here, only null fields will be counted as radiation.

The idea of following the field, expressed in such terms, does not admit an immediate covariant generalization to the case of the gravitational field, for, because of the principle of equivalence, there is no covariant gravitational field energy-tensor. The generalization will be achieved by considering the geometrical properties of the two fields. It will be found that in each case certain eigenvectors of the field can be defined. In the electromagnetic case it turns out that an observer following the field has the timelike eigenvector for 4-velocity; when the field is a null field this timelike vector collapses onto the null cone,[3] and it is this which is characteristic of the presence of radiation.

In the same way, a timelike eigenvector may in general be defined, in terms of the Riemann tensor, for the gravitational field in empty space-time. This vector is interpreted as the 4-velocity of an observer following the field, and in some fields this vector collapses onto the null cone. As in the electromagnetic case, this situation is identified with the presence of radiation.

These timelike eigenvectors can be given a further physical significance in both gravitational and electromagnetic fields. For example, in a non-null electromagnetic field, the 4-velocity which follows the field yields, for a given field, an extreme magnitude for the Lorentz force. Similarly, the (more complicated) physical effects of the gravitational field also reach extreme values for an observer having the 4-velocity which follows the field. This will be discussed in detail in Sec. 4.

The eigenvectors of the gravitational field are defined in Sec. 3, with the aid of Petrov's classification[4] of empty space-time Riemann tensors into canonical types. The elegant geometrical-algebraic techniques developed by Ruse[5] and others supply the basis for this classification, which yields two types with radiation present and one with no radiation. It will be shown that the difference between the no-radiation type and one of the radiation types can be made, by a suitable alignment of axes, to correspond exactly to the discontinuity in the Riemann tensor across a wave front permitted by Lichnerowicz's conditions. This will demonstrate the consistency of the idea (Sec. 2) of a gravitational wave front with the idea (Sec. 3) that an observer following the gravitational radiation field must have the fundamental velocity.

This geometrical approach is necessary because there is no covariant gravitational energy-momentum tensor in Einstein's theory. This lack is to be expected, because of the principle of equivalence. In Lorentz-invariant field theories the energy-momentum tensor depends on the field strengths, but these have locally no absolute significance for the gravitational field. The canonical energy-momentum pseudotensor[6] $t_\mu{}^\nu$ is a quadratic function of the field strengths (Christoffel symbols) and satisfies a conservation law, but it is not covariant—it can in fact be made to vanish entirely along an arbitrary open curve in space-time. Nevertheless, $t_\mu{}^\nu$ has been used in definitions of gravitational radiation by various authors,[7] but always in weak field approximations and under physically obscure coordinate conditions.

It might be possible to construct covariant, and therefore physically significant, expressions out of $t_\mu{}^\nu$ over extended regions of space-time, into which only the variations in the gravitational field, not the field itself, could enter, but no one seems to have succeeded in doing this, or even to have attempted it. The difficulties could be resolved if one could reformulate the weak-field approximation method in a covariant way valid in an extended region.

The alternative to this, which is adopted in Sec. 4, is to develop a covariant local approximation method valid in arbitrarily strong fields but only in small regions of space-time. This is done by introducing normal coordinates,[8] which are given a physical interpretation. The energy-momentum pseudotensor $t_\mu{}^\nu$ vanishes at the origin of normal coordinates but not in a finite neighborhood of the origin, and may be expanded in a power series with tensor coefficients which are functions of the Riemann tensor and its covariant derivatives. By averaging over a small 2-region, one may construct a covariant approximation $\bar{t}_\mu{}^\nu$ to the mean energy-momentum pseudotensor in the region. It

[3] For a detailed discussion of the geometrical and algebraic properties of the electromagnetic field, see J. L. Synge, *Relativity: the Special Theory* (North-Holland Publishing Company, Amsterdam, 1956), Chap. IX.

[4] A. Z. Petrov, Sci. Not. Kazan State Univ. **114**, 55 (1954).

[5] H. S. Ruse, Proc. Roy. Soc. (Edinburgh) **62**, 64 (1944); Quart. J. Math. (Oxford) **17**, 1 (1946); Proc. London Math. Soc. **50**, 75 (1948).

[6] Range and summation conventions: lower case Greek indices 0, 1, 2, 3; lower case Latin indices 1, 2, 3.

[7] E.g., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Cambridge, 1951); J. N. Goldberg, Phys. Rev. **99**, 1873 (1955).

[8] B. Riemann, Göttingen Abhandl. **13**, 1 (1862); see O. Veblen, *Invariants of Quadratic Differential Forms* (Cambridge University Press, Cambridge, 1927), for a lucid exposition.

is found that when no gravitational radiation is present, there exist observers (with suitable 4-velocities) who observe no gravitational energy flux, but that when gravitational radiation is present, such observers cannot be found. This corresponds exactly to the electromagnetic field case, and supports the definition of gravitational radiation.

Various examples are discussed in Sec. 5, and some deficiencies of the present approach are mentioned in Sec. 6.

## 2. NATURE OF A GRAVITATIONAL WAVE FRONT

The nature of a gravitational wave front will now be investigated, by finding the discontinuity in the Riemann tensor permissible across a null 3-surface. The calculation is based on Lichnerowicz's continuity conditions,[9] which are conditions on the metric tensor and its derivatives sufficient to ensure that Einstein's equations *in vacuo*,[10]

$$G_{\mu\nu}=0, \tag{2.1}$$

are physically unique. These conditions are essentially the same as those found by O'Brien and Synge[11] from assumptions about the finiteness of certain quantities in the boundary layer between two regions of a continuous medium.

Lichnerowicz postulates that space-time can be divided up into overlapping regions, in each of which there exists a coordinate system such that (i) the metric tensor $g_{\mu\nu}$ is continuous, (ii) the first partial derivatives[12] $g_{\mu\nu,\rho}$ are continuous, (iii) the second and third partial derivatives of $g_{\mu\nu}$ are piecewise continuous. Space-time is assumed to be a Riemannian manifold with certain differentiability properties which do not affect the present argument.

Lichernerowicz's analysis is developed by taking in one of the regions a coordinate system such that, say, the surface $S$: $x^0=0$ is a surface of discontinuity of the gravitational field. Then according to the postulates, the coordinate system can be chosen so that all the $g_{\mu\nu}$, all the $g_{\mu\nu,\rho}$, and all the $g_{\mu\nu,\rho\sigma}$ with the possible exception of $g_{\mu\nu,00}$ are continuous across $S$. Now the covariant components of the Riemann tensor are[13]

$$R_{\rho\sigma\mu\nu}=[\sigma\nu,\rho]_{,\mu}-[\sigma\mu,\rho]_{,\nu}+\Gamma^{\pi}{}_{\sigma\mu}[\rho\nu,\pi]-\Gamma^{\pi}{}_{\sigma\nu}[\rho\mu,\pi]. \tag{2.2}$$

If the first two terms of this are written out in full, it may be seen that the only components of $R_{\rho\sigma\mu\nu}$ which admit discontinuities across $S$ are those with just two indices 0, one in each of the pairs $\rho\sigma$, $\mu\nu$. This is the same as a result of O'Brien and Synge.[14]

Results of this kind may be put into covariant form by transforming to a local Minkowskian coordinate system and then introducing a *tetrad* (orthonormal frame, quadruped, Vierbein, 4-nuple) of orthogonal unit vectors directed along the axes of the local Minkowskian system. Tensor equations may then be rewritten as scalar equations by contracting with tetrad vectors, and the special coordinate system discarded.

Physically, the timelike tetrad vector is identified as the 4-velocity of an observer, making measurements at the event in question, who uses space axes having the directions of the three spacelike tetrad vectors. The scalars formed by contracting any tensor with tetrad vectors are just the physical components of the tensor, measured by this observer.[15]

For the present purpose, the 3-surface $S$ is to be a null 3-surface. It is not convenient to have any of the $x^\mu$ as a null coordinate; therefore one writes, say, $\xi=0$ in place of $x^0=0$ for the equation of $S$. One may by a linear transformation introduce at any point $P$ of $S$ local Minkowskian coordinates such that $ds^2=\eta_{\mu\nu}dx^\mu dx^\nu$, where[16]

$$\eta_{\mu\nu}=\mathrm{diag}(1,\,-1,\,-1,\,-1),$$

and in a finite neighborhood of $P$, one may take $\xi=2^{-\frac{1}{2}}(x^0-x^1)$. If also $\zeta=2^{-\frac{1}{2}}(x^0+x^1)$ in this neighborhood, then at $P$,

$$ds^2=2d\xi d\zeta-(dx^2)^2-(dx^3)^2.$$

Then if $\Delta$ denotes amount of discontinuity across $S$, Lichnerowicz's conditions require that at $P$,

$$\Delta(g_{\mu\nu})=0,$$
$$\Delta(g_{\mu\nu,\sigma})=\Delta(\partial g_{\mu\nu}/\partial\xi)=\Delta(\partial g_{\mu\nu}/\partial\zeta)=0,$$
$$\Delta(\partial^2 g_{\mu\nu}/\partial\xi\partial\zeta)=\Delta(\partial^2 g_{\mu\nu}/\partial\zeta^2)=0,$$

but

$$\Delta(\partial^2 g_{\mu\nu}/\partial\xi^2)=a_{\mu\nu},$$

where $a_{\mu\nu}$ are any numbers. The possible discontinuities in $R_{\mu\nu\rho\sigma}$ are now easily found from (2.2). A straightforward calculation shows that the only $a$'s contributing to the Riemann tensor in empty space-time are

$$-a_{22}=a_{33}=\sigma \quad\text{and}\quad a_{23}=\phi, \tag{2.3}$$

where $\sigma$ and $\phi$ are arbitrary. These correspond exactly to the two types of "transverse-transverse" waves found in the linear approximation theory of gravitational radiation.[17]

All the terms in $\phi$ may be reduced to zero by a rotation of axes through angle $\tan^{-1}(\phi/\sigma)$ in the 23-plane.

[9] Reference 2, Chap. I.

[10] The cosmological term is disregarded throughout this paper. It could be restored to the work without any difficulty, but at some cost in conciseness.

[11] S. O'Brien and J. L. Synge, Comm. Dublin Inst. A, No. 9 (1952).

[12] A comma denotes partial differentiation: $g_{\mu\nu,\rho}\equiv\partial g_{\mu\nu}/\partial x^\rho$.

[13] $[\sigma\nu,\rho]\equiv\frac{1}{2}(g_{\nu\rho,\sigma}+g_{\rho\sigma,\nu}-g_{\sigma\nu,\rho})$ and $\Gamma^{\pi}{}_{\sigma\nu}\equiv g^{\pi\rho}[\sigma\nu,\rho]$ are Christoffel symbols of the first and second kinds respectively.

[14] Reference 11, Eq. (5.12).

[15] For details, see the second paper of reference 1.

[16] Units are chosen, throughout, so that $c=1$.

[17] A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge University Press, New York, 1924), second edition, p. 247.

The resulting discontinuity in the Riemann tensor will now be written in covariant form by introducing a a tetrad of unit vectors[18] $\lambda_\alpha{}^\mu$, which at $P$ are directed along the coordinate axes, so that at $P$, $\lambda_\alpha{}^\mu=\delta_\alpha{}^\mu$. On account of the orthonormality, it is true everywhere that

$$g_{\mu\nu}\lambda_\alpha{}^\mu\lambda_\beta{}^\nu=\eta_{\alpha\beta}.$$

Now define

$$\lambda^{\alpha\mu}\equiv\eta^{\alpha\beta}\lambda_\beta{}^\mu,$$

so that $\lambda^{0\mu}=\lambda_0{}^\mu$, $\lambda^{a\mu}=-\lambda_a{}^\mu$. Then it is not difficult to prove that

$$\eta_{\alpha\beta}\lambda^{\alpha\mu}\lambda^{\beta\nu}=\lambda^{\alpha\mu}\lambda_\alpha{}^\nu=g^{\mu\nu},\quad \lambda_\mu{}^\alpha\lambda_\alpha{}^\nu=\delta_\mu{}^\nu,$$

and so forth. It is convenient to abbreviate

$$\lambda_0{}^\mu=\lambda^{0\mu}\equiv\lambda^\mu.$$

This notation differs slightly from that of Eisenhart,[19] who uses indicators instead of $\eta_{\alpha\beta}$.

It is convenient to introduce a simple 6-dimensional formalism for discussing the Riemann tensor and other bivector-tensors. The 6-dimensional pseudo-Euclidean space (the Klein space) is introduced whose vectors are just the bivectors (skew tensors) in the local tangent Minkowski space defined by the tetrad $\lambda_\alpha{}^\mu$. The rule for going over to the 6-dimensional formalism is the following:

If $H_{\alpha\beta}$ are the physical components, with respect to a given tetrad at a given space-time event $P$, of any skew tensor $H_{\mu\nu}$, then the corresponding 6-vector in the 6-space at $P$ is[20] $H_A$, got by relabeling the suffixes $\alpha\beta$ according to the scheme

$$\begin{array}{rcccccc} \alpha\beta: & 23 & 31 & 12 & 10 & 20 & 30, \\ A: & 1 & 2 & 3 & 4 & 5 & 6. \end{array} \qquad (2.4)$$

Accordingly, any bivector-tensor corresponds to a symmetric tensor in the 6-space. The physical components $R_{\alpha\beta\gamma\delta}$ of the Riemann tensor, for example, go over to the components of a symmetric 6-tensor $R_{AB}$, each of the suffix pairs $\alpha\beta$, $\gamma\delta$ being relabeled according to the scheme (2.4).

In order that the raising and lowering of indices in the 6-space should correspond to the raising and lowering of index pairs in the 4-space of physical components, the metric tensor of the 6-space must be chosen to be

$$\eta_{AB}=\mathrm{diag}(1,1,1,-1,-1,-1) \qquad (2.5)$$

which corresponds to the bivector-tensor $\eta_{\alpha\gamma}\eta_{\beta\delta}-\eta_{\alpha\delta}\eta_{\beta\gamma}$.

The discontinuity in the Riemann tensor at any event on the null 3-surface $S$ in empty space-time may be calculated straightforwardly from (2.2) and (2.3) and written in terms of $R_{AB}$. It turns out to be

$$\Delta R_{AB}=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\sigma & -\phi & \cdot & -\phi & \sigma \\ \cdot & -\phi & \sigma & \cdot & \sigma & \phi \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\phi & \sigma & \cdot & \sigma & \phi \\ \cdot & \sigma & \phi & \cdot & \phi & -\sigma \end{bmatrix} \qquad (2.6)$$

Here $\sigma$ and $\phi$ are arbitrary numbers, but the terms in $\phi$ may be eliminated by a rotation, as stated above. It will be shown in Sec. 3 that the difference between the no-radiation and one of the radiation canonical types of Riemann tensor, referred to a suitably oriented tetrad, is precisely the array of $\sigma$'s in (2.6), which supports the interpretation of (2.6) as the discontinuity across a gravitational wave front.

The physical effects of the discontinuities (2.6) may be studied in terms of the equation of geodesic deviation[21]

$$\delta^2\eta^\mu/\delta\tau^2+R^\mu{}_{\nu\rho\sigma}\lambda^\nu\eta^\rho\lambda^\sigma=0, \qquad (2.7)$$

which describes the relative acceleration of two neighboring (spherically symmetric) test particles.[1] In Eq. (2.7), $\lambda^\mu=dx^\mu/d\tau$ is the unit tangent vector to the geodesic world-line $\mathcal{C}$ of one of the particles, $\tau$ is proper time along $\mathcal{C}$, and $\eta^\mu$ is the orthogonal displacement vector to the (neighboring) world-line of the other particle. To reach this physical interpretation directly, one has only to refer (2.7) to a tetrad comprising $\lambda^\mu$, which is the 4-velocity of the particle with world-line $\mathcal{C}$, and three spacelike vectors $\lambda_a{}^\mu$ orthogonal to and parallelly propagated along $\mathcal{C}$. Then (2.7) becomes

$$d^2X^a/d\tau^2+K^a{}_b(\tau)X^b=0, \qquad (2.8)$$

where $X^a=\eta^\mu\lambda_\mu{}^a$ are the physical components of the displacement vector ($X^0$ vanishes) and

$$K^a{}_b=R^a{}_{0b0} \qquad (2.9)$$

are some of the physical components of the Riemann tensor. In the Newtonian equation corresponding to (2.8), $X^a$ are the coordinates of the second particle relative to the first, and $K^a{}_b=\partial^2V/\partial x^a\partial x^b$, where $V$ is the ordinary Newtonian gravitational potential. Thus $-K^a{}_bX^b$ is to be identified as the relative acceleration of two particles with relative coordinates $X^b$, arising from the difference in gravitational field between the particles.

It follows that as the gravitational wave front described by (2.6) passes the pair of test particles,

[18] Here $\mu$ is a vector index and $\alpha$ a label distinguishing the four vectors. The Greek letters $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and Latin letters $a$, $b$, $c$, $d$, $e$ will be used only for labels, but shall satisfy the same range and summation conventions (Greek 0, 1, 2, 3; Latin 1, 2, 3) as ordinary vector and tensor indices $\mu$, $\nu$, $\rho$, $\sigma$, $\tau$, $\cdots$ and $m$, $n$, $p$, $r$, $\cdots$. From now on, an index given a particular value will be understood to be a label index, not a tensor index. This notation obviates the necessity of bracketing indices.

[19] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1949), Chap. 3.

[20] Upper case letters $A$, $B$, $C$ range and sum over 1, 2, $\cdots$, 6.

[21] J. L. Synge and A. Schild, *Tensor Calculus* (University of Toronto Press, Toronto, 1949), p. 93.

there will be a discontinuity

$$\Delta K^a{}_b = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & -\sigma & \phi \\ \cdot & \phi & \sigma \end{bmatrix} \qquad (2.10)$$

across the wave front. The tetrads to which (2.6) and (2.8) are referred have been (and always may be) chosen to coincide at the space-time event where the wave front passes the particles. It can be seen that the discontinuity in the relative acceleration depends on the relative position of the particles, and in particular that there is no discontinuity if the two particles are aligned in the direction of propagation of the wave front, which is the 1-direction.

This result represents in an invariant manner the transverse character of gravitational radiation. Two particles lying in the 23-plane (which is perpendicular to the direction of propagation) will suffer a discontinuous change in relative acceleration. If, for example, the 2-axis is chosen so that $\phi=0$, and the line joining the particles makes an angle $\theta$ with this axis, then according to (2.10) the change in the relative acceleration will take place in a direction making an angle $-\theta$ with the same axis.

### 3. CANONICAL FORMS FOR THE RIEMANN TENSOR

In this section, the idea of *following the gravitational field* is made precise. This is done by fairly straightforward generalization from the case of the electromagnetic field. In that case, eigenvectors for the field are defined by the equations

$$T_\mu{}^\nu \xi_\nu = \lambda \xi_\mu, \qquad (3.1)$$

where $T_\mu{}^\nu$ is the electromagnetic energy tensor. It is found[3] that in a general field both timelike, spacelike, and null eigenvectors exist; these lie in two orthogonal 2-spaces but are otherwise undetermined. In a null field, on the other hand, there is no timelike eigenvector; all the eigenvectors are spacelike, except for one which is null, and all lie in a 3-space tangent to the null cone along the direction of the null eigenvector.

The sense in which an observer follows the electromagnetic field comes most easily out of consideration of the Poynting vector. Rather than introduce a particular Lorentz frame, one may define a Poynting 4-vector in a covariant way, to be

$$P_\rho = (\delta_\rho{}^\mu - v_\rho v^\mu) T_{\mu\nu} v^\nu, \qquad (3.2)$$

where $v^\mu$ is the 4-velocity of the observer measuring the field. This is easily seen to reduce to the usual definition when $v^\mu$ lies along the time axis in a local Lorentz frame. Since

$$P_\rho v^\rho = 0, \qquad (3.3)$$

$P_\rho$ must be spacelike. If $n_\rho$ is the 4-normal to any small 2-surface $\Sigma$ carried along by the observer with velocity $v^\rho$, then the electromagnetic energy flux across $\Sigma$ is

$$P_\rho n^\rho = T_{\mu\nu} v^\mu n^\nu. \qquad (3.4)$$

Now one says that such and such an observer is following the electromagnetic field if he measures zero energy flux across all 2-surfaces which he carried along, however oriented. By (3.3) and (3.4), this can occur only if

$$P_\rho = 0, \qquad (3.5)$$

which implies that

$$T_{\rho\nu} v^\nu = (T_{\mu\nu} v^\mu v^\nu) v_\rho, \qquad (3.6)$$

so that $v_\rho$ must be an eigenvector of $T_{\mu\nu}$. This establishes the connection between the concept of following the electromagnetic field and the eigenvectors of the electromagnetic energy tensor. As mentioned above, a null field has *no* timelike eigenvector, so that the Poynting vector will not vanish for any observer with a finite velocity. The energy flow in a null field cannot be abolished by a Lorentz transformation. A null field has one null eigenvector, say $\xi^\mu$, belonging to the eigenvalue zero, so that

$$T_{\mu\nu} \xi^\nu = 0. \qquad (3.7)$$

Thus an "observer" moving with the speed of light in the direction of $\xi^\nu$ (which is essentially the propagation vector) would observe no energy flux past him.

In the gravitational case, there is no energy-momentum tensor of the gravitational field itself (the pseudotensor is discussed in Sec. 4), but in accord with the arguments developed in Sec. 1, one may seek in the geometrical structure of the Riemann tensor a definition of "following the field" analogous to that developed in the electromagnetic case. The definition is naturally more complicated, because the Riemann tensor is a more complicated object than the Maxwell energy tensor. The definition is made in two stages. First of all, eigenbivectors (skew tensors) are defined for the Riemann tensor. By using Petrov's canonical forms,[4] these eigenbivectors may be written down explicitly for the three algebraically distinct types of Riemann tensor in empty space-time. The eigenbivectors correspond geometrically to 2-spaces, or pairs of 2-spaces, in space-time. The intersections of these 2-spaces with one another define a number of 4-vectors (assumed normalized if they are not null), which will be referred to as *Riemann principal vectors.*

> An observer with a timelike Riemann principal vector as 4-velocity is said to be following the gravitational field.

It turns out that for two of the three types of Riemann tensor, this timelike principal vector collapses onto the null cone. The occurrence of these types of

Riemann tensor is identified with the presence of gravitational radiation.

The eigenbivectors $P_{\mu\nu}$ of the Riemann tensor are defined by the equation

$$R_{\mu\nu\rho\sigma}P^{\rho\sigma}=\lambda P_{\mu\nu}, \tag{3.8}$$

or

$$R_{AB}P^{B}=\lambda P_{A}$$

in the 6-dimensional formalism introduced in Sec. 2.

Now Petrov[4] has shown that by a suitable choice of the reference tetrad at any event in empty space-time, one may reduce the Riemann tensor to a canonical form of one of the following three types:

Type I:

$$R_{AB}=\begin{bmatrix} \alpha_1 & \cdot & \cdot & \beta_1 & \cdot & \cdot \\ \cdot & \alpha_2 & \cdot & \cdot & \beta_2 & \cdot \\ \cdot & \cdot & \alpha_3 & \cdot & \cdot & \beta_3 \\ \beta_1 & \cdot & \cdot & -\alpha_1 & \cdot & \cdot \\ \cdot & \beta_2 & \cdot & \cdot & -\alpha_2 & \cdot \\ \cdot & \cdot & \beta_3 & \cdot & \cdot & -\alpha_3 \end{bmatrix}. \tag{3.9}$$

$$\left[\begin{matrix} \sum_{k=1}^{3}\alpha_k=0 \\ \\ \sum_{k=1}^{3}\beta_k=0 \end{matrix}\right]$$

Type II:

$$R_{AB}=\begin{bmatrix} -2\alpha & \cdot & \cdot & -2\beta & \cdot & \cdot \\ \cdot & \alpha-\sigma & \cdot & \cdot & \beta & \sigma \\ \cdot & \cdot & \alpha+\sigma & \cdot & \sigma & \beta \\ -2\beta & \cdot & \cdot & +2\alpha & \cdot & \cdot \\ \cdot & \beta & \sigma & \cdot & -(\alpha-\sigma) & \cdot \\ \cdot & \sigma & \beta & \cdot & \cdot & -(\alpha+\sigma) \end{bmatrix}. \tag{3.10}$$

Type III:

$$\begin{bmatrix} \cdot & -\sigma & \cdot & \cdot & \cdot & \sigma \\ -\sigma & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sigma & \cdot & \cdot \\ \cdot & \cdot & \sigma & \cdot & \sigma & \cdot \\ \cdot & \cdot & \cdot & \sigma & \cdot & \cdot \\ \sigma & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{3.11}$$

In Type I, the reference tetrad yielding the canonical form is in general fully determined; accidental equality between different $\alpha$'s or $\beta$'s may introduce some freedom. in Types II and III, the reference tetrad is determined only up to a Lorentz rotation in the 10-plane and a spatial rotation in the 23-plane. The $\alpha$'s and $\beta$'s are scalar invariants of the Riemann tensor, but the value of $\sigma$ depends on the choice of axes in the 10-plane.

These forms of $R_{AB}$ are determined, first, by the limitation of transformations in the 6-space to those generated by changes of tetrad (i.e., by real Lorentz transformations, including rotations) in space-time, and secondly, by the nonsymmetry of $R_A{}^B$ (equivalently, by the indefinite character of the metric $\eta_{AB}$). As a result, the elementary divisors of $R_A{}^B$ need not be simple, and Types II and III result when they are not.

The eigenbivectors of $R_{AB}$, defined by (3.8), are easily found from (3.9)–(3.11); they are either simple bivectors, dual in pairs or of the form $P_A=S_A\pm i\,{}^0S_A$, where $S_A$ is simple and ${}^0S_A$ is its dual.[22]

[22] A simple bivector may be characterized by $\det(S_{\mu\nu})=0$; it may always be written in the form $S_{\mu\nu}=X_\mu Y_\nu-X_\nu Y_\mu$, and defines a 2-space in space-time. The dual of any bivector $P_{\mu\nu}$ is

$${}^0P_{\mu\nu}=\tfrac{1}{2}g_{\mu\rho}g_{\nu\sigma}e^{\rho\sigma\tau\pi}P_{\tau\pi}\ \ ({}^0P_A=\tfrac{1}{2}g_{AB}e^{BC}P_C),$$

where $e^{\rho\sigma\tau\pi}=\pm\sqrt{(-g)}$ is the alternating tensor, which will be understood to take the negative sign when $\rho\sigma\tau\pi$ are in the order 0123. The dual of a simple behavior $S_{\mu\nu}$ defines a 2-space orthogonal to that defined by $S_{\mu\nu}$ itself.

Thus each eigenbivector $P^A$ of $R_{AB}$ defines a pair of orthogonal 2-spaces. The $P^A$ are readily found from (3.9)–(3.11) to be (conveniently normalized) the following:

Type I: Six independent eigenbivectors:

If $\beta_1=0$, $P^A=\delta_1{}^A$ and $P^A=\delta_4{}^A$ (dual pair); if $\beta_1\neq 0$, $P^A=\delta_1{}^A\pm i\delta_4{}^A$,
If $\beta_2=0$, $P^A=\delta_2{}^A$ and $P^A=\delta_5{}^A$ (dual pair); if $\beta_2\neq 0$, $P^A=\delta_2{}^A\pm i\delta_5{}^A$,
If $\beta_3=0$, $P^A=\delta_3{}^A$ and $P^A=\delta_6{}^A$ (dual pair); if $\beta_3\neq 0$, $P^A=\delta_3{}^A\pm i\delta_6{}^A$.

Type II: Four independent eigenbivectors:

If $\beta=0$, $P^A=\delta_1{}^A$ and $P^A=\delta_4{}^A$ (dual pair), and $P^A=\delta_2{}^A-\delta_6{}^A$ and $P^A=\delta_3{}^A+\delta_5{}^A$ (dual pair).

If $\beta\neq 0$, $P^A=\delta_1{}^A\pm i\delta_4{}^A$, and $P^A=\delta_2{}^A-\delta_6{}^A\pm i(\delta_3{}^A+\delta_5{}^A)$.

Type III: Two independent eigenbivectors:

$$P^A=\delta_2{}^A-\delta_6{}^A,\quad \text{and}\quad P^A=\delta_3{}^A+\delta_5{}^A.$$

The different pairs of 2-spaces represented by these simple bivectors are not orthogonal. Their intersections yield the Riemann principal vectors $r^\alpha$, which are (conveniently normalized) as follows:

Type I: $r^\alpha=\delta_1{}^\alpha$, $\delta_2{}^\alpha$, $\delta_3{}^\alpha$, $\delta_0{}^\alpha$. The Riemann principal vectors are just the vectors of the reference tetrad. One is timelike, three spacelike.

Type II: $r^\alpha=\delta_0{}^\alpha-\delta_1{}^\alpha$, $\delta_2{}^\alpha$, $\delta_3{}^\alpha$. The first is null, the others spacelike. Because of the freedom of rotation in the 23-plane, the last two may be replaced by any linear combination of themselves.

Type III: $r^\alpha=\delta_0{}^\alpha-\delta_1{}^\alpha$. There is only one Riemann principal vector, and it is a null vector.

According to the criteria set out earlier, gravitational radiation is now defined as follows:

> At any event in empty space-time, gravitational radiation is present if the Riemann tensor is of Type II or Type III, but not if it is of Type I.

Now it will be noticed that the $\sigma$'s in (3.10) appear in the same positions and with the same signs as the $\sigma$'s in (2.6), which exhibits the discontinuities permissible across a null 3-surface. This correspondence has of course been achieved in part by orienting the two reference tetrads so that the 1-direction is picked out for asymmetrical treatment in each case. However, it has also the physical significance that the discontinuities possible across a gravitational wave front, according to Lichnerowicz's conditions, are just what are required for the transition from a space-time region without gravitational radiation to one with gravitational radiation, according to the definition just proposed. This is of course precisely what one would wish, to show the compatibility of the two approaches to the problem.

It will be noticed also that the transition from Type I to Type II reduces the number of independent $\alpha$'s and $\beta$'s from two each to one each. This implies some additional symmetry in the radiation field, which may at first sight be surprising. However, a physical interpretation which at once suggests itself is that because of the nonlinearity of the field (that is, because the gravitational field effectively enters its own source-function), gravitational waves without any kind of symmetry would interfere with themselves to the extent of destruction.[22a]

The physical effects of gravitational waves may be investigated by using the equation of relative acceleration (2.8), in exactly the same way as the effects of discontinuities were investigated in Sec. 2. The difference between Type I and Type II space-times shows up clearly if one examines the behavior of test particles moving with velocities different from that specified by the timelike tetrad vector. As an example, consider the effect on $K_b{}^a$ [defined by (2.9)] of the local Lorentz transformation defined at an event by

$$\begin{aligned}\bar{\lambda}^0{}_\mu&=\lambda^0{}_\mu\cosh\theta+\lambda^1{}_\mu\sinh\theta,\\ \bar{\lambda}^1{}_\mu&=\lambda^0{}_\mu\sinh\theta+\lambda^1{}_\mu\cosh\theta,\\ \bar{\lambda}^2{}_\mu&=\lambda^2{}_\mu,\quad \bar{\lambda}^3{}_\mu=\lambda^3{}_\mu,\end{aligned}\tag{3.12}$$

where the unbarred tetrad is that to which the canonical forms (3.9) and (3.10) are referred. Then the comparative values of $K_{ab}$ (omitting $K_{1b}$, which are unaltered) are

Type I:

$$\begin{aligned}K_{22}&=-(\alpha-\sigma), & \bar{K}_{22}&=-(\alpha-\sigma\cosh2\theta),\\ K_{23}&=0, & \bar{K}_{23}&=\tfrac{1}{2}(\beta_3-\beta_2)\sinh2\theta,\\ K_{33}&=-(\alpha+\sigma), & \bar{K}_{33}&=-(\alpha+\sigma\cosh2\theta).\end{aligned}$$

Type II:

$$\begin{aligned}K_{22}&=-(\alpha-\sigma), & \bar{K}_{22}&=-(\alpha-\sigma e^{-2\theta}),\\ K_{23}&=0, & \bar{K}_{23}&=0,\\ K_{33}&=-(\alpha+\sigma), & \bar{K}_{33}&=-(\alpha+\sigma e^{-2\theta}).\end{aligned}$$

Here the barred $K$'s are those referred to the tetrad $\bar{\lambda}^\alpha{}_\mu$, and in Type I, $\alpha_2$ and $\alpha_3$ have been replaced by $\alpha=\frac{1}{2}(\alpha_2+\alpha_3)$ and $\sigma=\frac{1}{2}(\alpha_3-\alpha_2)$, for ready comparison with Type II.

An essential difference between the types is represented by the fact that in Type I, the changes in $K_{22}$ and $K_{33}$ go as $\theta^2$, for small $\theta$, while in Type II, they go as $\theta$. The Type I changes are essentially a special-relativistic effect, in the sense of "being of the same kind as familiar effects such as the Lorentz contraction," but the Type II changes are characteristic of a non-Lorentzian phenomenon. The first-order change in $K_{23}$, another instance of non-Lorentzian behavior, suggests that the $\beta$'s may be connected with the rotational properties of the field.

For strong Lorentz transformations (large $\theta$), the Type I $K$'s become large in absolute value for both signs of $\theta$, but the Type II $K$'s approach finite limits for large positive $\theta$, so that in Type I the $K$'s have extreme values for $\theta=0$, while in Type II the extreme values are approached only as $\theta\to\infty$, that is, as the observer's velocity approaches the fundamental velocity in the direction of propagation of the radiation.

## 4. REDUCTION OF THE ENERGY-MOMENTUM PSEUDOTENSOR

As is well known, one may convert the covariant conservation equations[23]

$$T_\mu{}^\nu{}_{;\nu}=0$$

into the form

$$\{(-g)^{\frac{1}{2}}T_\mu{}^\nu+(1/\kappa)t_\mu{}^\nu\}_{,\nu}=0$$

by introducing the canonical energy-momentum pseudotensor $t_\mu{}^\nu$. This fact, and the canonical origin of $t_\mu{}^\nu$, lead one to identify it as the "energy-momentum pseudotensor of the gravitational field." The physical argument is, roughly, that the deviations of space-time from flatness introduce additional terms into the conservation equations, and that these deviations are consequencies of the existence of the gravitational field. All would be well, were it not that $t_\mu{}^\nu$ depends on idiosyncrasies in the choice of coordinates as well as on actual physical phenomena. The nonhomogeneous transformation properties of $t_\mu{}^\nu$ make it impossible to

[22a] *Note added in proof.*—The remaining discussion, where it refers to particular canonical types, is restricted to Types I and II. The absence of scalar invariants in Type III suggests that spacetimes of this type would represent radiation without sources, but the interpretation of this type is not obvious, and further consideration of it is left to a subsequent paper. The writer knows of no example of a Type III spacetime; he would be grateful if new examples of empty spacetime metrics of any type were sent to him for study.

[23] A semicolon denotes a covariant derivative.

construct any scalar quantities out of it, at least in a direct way, and so its physical interpretation must be suspect, because of this essential dependence on the coordinate system. It is hard to see how one can attach any physical meaning to $t_\mu{}^\nu$ unless one can first attach a physical meaning to the coordinate system. The same difficulty would arise with vectors and tensors, except that one can construct scalars (e.g., physical components) out of them by contracting with other vectors or tensors, and these scalars are of course independent of the coordinate system.

The usual procedure in dealing with $t_\mu{}^\nu$ is to make weak-field approximations and to assume mathematically convenient coordinate conditions. These methods are controversial and their physical significance obscure. Difficulties in dealing with $t_\mu{}^\nu$ might anyhow have been expected from consideration of the principle of equivalence. Since the gravitational field can be abolished at an event by a coordinate transformation (in the sense that the $\Gamma^\rho{}_{\mu\nu}$ can be made to vanish), the gravitational energy, momentum and stress at an event can readily be understood to be as ephemeral as the coordinate system. The energy of the field resides not in its value at a single event, but in its variation from one event to another. It is not surprising that one cannot abolish $t_\mu{}^\nu$ throughout any finite 2-surface in a general space-time. However, a mean value $\bar{t}_\mu{}^\nu$ may be defined over the 2-surface of a small 3-volume, and by a suitable physical prescription of the coordinate system, such a definition can be made covariant.

The coordinates to be prescribed are well known to mathematicians under the name normal coordinates,[8] and have been used in general relativity theory before,[24,25] but it is nevertheless desirable to give some physical justification for this choice.

The choice of coordinate system depends on the physical situation involved. For many purposes it is enough to specify at an event a tetrad of unit vectors, or the corresponding local Minkowskian coordinates axes, representing the 4-velocity of an observer and rectangular Cartesian axes in his local instantaneous 3-space. The essential thing is that it should be possible in principle to identify the chosen system with one which could be used by an observer in the given physical situation. Recently,[1] the writer compared the behavior of test particles in a gravitational field in the general relativity theory and in the Newtonian theory (see Sec. 2 above). In that case it was appropriate to introduce local Cartesian coordinate systems in the instantaneous 3-spaces along one of the particle world-lines, and the coordinate systems at different events were related by parallelly propagating along the world-line the tetrad vectors representing the coordinate axes. As might be expected, it was found that this mode of propagation led to a description of gravitational phenomena most closely resembling that obtained in the Newtonian theory from the use of ordinary Newtonian inertial frames. However the whole formalism, being designed for a comparison with the Newtonian theory, was essentially nonrelativistic.

The present case is rather different. The formalism just described is appropriate to the discussion of dynamical effects, as in the discussion following Eq. (2.8) above, but the whole idea of the energy-momentum tensor is essentially a relativistic one, developed largely within the framework of a relativistic theory—the Maxwell theory—and it would be inappropriate to develop the same idea in general relativity theory, regarded now specifically as a field theory of gravitation, except in a relativistic manner. Therefore what is required is a convenient 4-dimensional analog of the Minkowskian inertial systems of special relativity, but one defined more completely than by a tetrad of unit vectors. Some loss of general covariance is inevitable, and the whole aim is anyhow a little artificial, the idea being to relate the novel concept of gravitational radiation developed here to a conventional idea of radiation developed specifically for electromagnetic theory—although it must be admitted that the discussion of the Poynting vector at the beginning of Sec. 3 applies also to flows of other sorts of energy.

Having, then, the aim of investigating the energy-momentum pseudotensor by analogy with Lorentz-invariant field theories, it is appropriate to choose a coordinate system which approximates to a Minkowskian system. In the weak-field approximation method this is done by considering a metric which deviates slightly from the Minkowski metric at sufficiently large distances from material particles. The conceptual difficulties which arise in the use of that method can be ascribed to the lack of a covariant formulation of the weak-field approximation. The alternative adopted here is a local approximation method capable of invariant formulation: the introduction, in the neighborhood of any chosen space-time event, of a normal coordinate system, which approximates to Minkowskian inertial system in a mathematically and physically well-defined way.

The physical interpretation of normal coordinates comes out of their exact correspondence to Minkowskian coordinates in one particular respect, namely the measurement of interval. This is best explained by summarizing the relevant properties of such a coordinate system, which are the following:

Normal coordinates $x^\mu$ can always be chosen so that at any chosen space-time event $O$,

(i) $$x^\mu=0, \tag{4.1}$$

(ii) $$g_{\mu\nu}=\eta_{\mu\nu}, \tag{4.2}$$

(iii) $$\Gamma_{\mu\nu}{}^\rho=g_{\mu\nu,\rho}=0, \tag{4.3}$$

(iv) $$g_{\mu\nu,\rho\sigma}=\tfrac{1}{3}(R_{\rho\mu\nu\sigma}+R_{\rho\nu\mu\sigma}), \tag{4.4}$$

[24] G. D. Birkhoff, *Relativity and Modern Physics* (Harvard University Press, Cambridge, 1923).

[25] T. Y. Thomas, Phil. Mag. 48, 1056 (1924).

and

(v) at every point $P$ in the neighborhood of $O$,

$$x^\mu = u p_0{}^\mu, \tag{4.5}$$

where

$$p_0{}^\mu = dx^\mu/du \tag{4.6}$$

is a vector tangent at $O$ to the geodesic $OP$, and

(a) if $OP$ is timelike, $u$ is the proper time $\tau$ from $O$ to $P$;

(b) if $OP$ is spacelike, $u$ is the proper distance $s$ from $O$ to $P$;

(c) if $OP$ is null, $u$ is a preferred parameter in terms of which the equation of the null geodesic $OP$ takes the form

$$\frac{d^2x^\mu}{du^2} + \Gamma_{\nu\rho}{}^\mu \frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0,$$

(this defines $u$ up to a linear transformation on the null geodesics through $O$. The origin of $u$ is chosen to be at $O$, and $x^\mu$ defined by (4.5) does not depend on the scale of $u$).

(vi) the normal coordinate systems at $O$ are connected to one another by homogeneous Lorentz transformations at $O$.

It is clear from examination of these properties that an observer who assigns coordinates in the neighborhood of a given event $O$ by theodolite measurements at $O$ and interval measurements from $O$ as if space-time were flat, will assign normal coordinates. Thus the employment of normal coordinates exploits to the full the locally Minkowskian properties of a Riemannian space. In order to connect this to previous work, it is convenient again to introduce a tetrad of unit vectors directed along the coordinate axes.

It is property (iv) which supplies the key to a covariant expression for the energy-momentum pseudotensor $t_\mu{}^\nu$. The latter is homogeneous quadratic in the $g_{\mu\nu,\sigma}$, and so if it is expanded in a power series about the origin of normal coordinates, the first nonvanishing term has an invariant coefficient, a function of $R_{\mu\nu\rho\sigma}$. By taking an average over a small 2-sphere, an invariant average expression is obtained.[25a] The details of the calculation are as follows: The energy-momentum pseudotensor is defined by

$$t_\mu{}^\nu = L\delta_\mu{}^\nu - g_{\rho\sigma,\mu}\frac{\partial L}{\partial g_{\rho\sigma,\nu}}, \tag{4.7}$$

where

$$L = (-g)^{\frac{1}{2}} g^{\mu\nu}[\Gamma_{\mu\nu}{}^\rho \Gamma_{\rho\sigma}{}^\sigma - \Gamma_{\mu\sigma}{}^\rho \Gamma_{\nu\rho}{}^\sigma] \tag{4.8}$$

is the first-order Lagrangian for the field. A straightforward calculation yields for $L$ this explicit expression in terms of the $g_{\mu\nu,\rho}$:

$$L = \tfrac{1}{8}(-g)^{\frac{1}{2}} S^{\pi\tau\mu\nu\rho\sigma} g_{\pi\tau,\sigma} g_{\mu\nu,\rho}, \tag{4.9}$$

where

$$S^{\pi\tau\mu\nu\rho\sigma} = S^{\mu\nu\pi\tau\sigma\rho} = g^{\pi\tau}U^{\mu\nu\rho\sigma} + g^{\rho\sigma}U^{\mu\nu\tau\pi} - g^{\pi\rho}U^{\mu\nu\tau\sigma} + g^{\tau\rho}U^{\mu\nu\pi\sigma}, \tag{4.10}$$

$$U^{\mu\nu\rho\sigma} = g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}.$$

From (4.7), (4.8), and (4.9) one may write $t_\mu{}^\nu$ explicitly in terms of the $g_{\mu\nu,\rho}$:

$$t_\psi{}^\phi = \tfrac{1}{8}(\delta_\tau{}^\rho\delta_\psi{}^\phi - 2\delta_\psi{}^\rho\delta_\tau{}^\phi)(-g)^{\frac{1}{2}} S^{\pi\chi\mu\nu\tau\sigma} g_{\pi\chi,\sigma} g_{\mu\nu,\rho}. \tag{4.11}$$

Differentiating (4.11) twice with respect to $x^\xi$ and setting $g_{\mu\nu,\rho}=0$ in agreement with (4.3), one obtains

$$t_\psi{}^\phi = 0, \quad t_\psi{}^\phi{}_{,\xi} = 0, \tag{4.12}$$

$$t_\psi{}^\phi{}_{,\xi\eta} = \tfrac{1}{8}(\delta_\tau{}^\rho\delta_\psi{}^\phi - 2\delta_\psi{}^\rho\delta_\tau{}^\phi)(-g)^{\frac{1}{2}} S^{\pi\chi\mu\nu\tau\sigma} \times (g_{\pi\chi,\sigma\xi} g_{\mu\nu,\rho\eta} + g_{\pi\chi,\sigma\eta} g_{\mu\nu,\rho\xi}). \tag{4.13}$$

Now making use of (4.4) and the field equations for empty space-time (2.1), one finds after a straightforward calculation

$$t_\psi{}^\phi{}_{,\xi\eta} = \tfrac{1}{9}(\delta_\tau{}^\rho\delta_\psi{}^\phi - 2\delta_\psi{}^\rho\delta_\tau{}^\phi)(\delta_\xi{}^\kappa\delta_\eta{}^\lambda + \delta_\eta{}^\kappa\delta_\xi{}^\lambda) \times (R^{\tau\mu\nu}{}_\kappa + R^{\tau\nu\mu}{}_\kappa) R_{\rho\mu\nu\lambda}. \tag{4.14}$$

It follows from (4.9) that the mean value of $t_\psi{}^\phi$ over the surface of a small sphere about $O$ in the 3-space $t=0$ will be

$$\bar{t}_\psi{}^\phi = \tfrac{1}{6} t_\psi{}^\phi{}_{,ss}. \tag{4.15}$$

Substituting from (4.14) into (4.15) and introducing the unit vector $\lambda^\rho = \delta_0{}^\rho$ directed along the time axis, one obtains

$$\bar{t}_\psi{}^\phi = (1/27)(\delta_\tau{}^\rho\delta_\psi{}^\phi - 2\delta_\psi{}^\rho\delta_\tau{}^\phi)(\lambda^\kappa\lambda^\lambda - g^{\kappa\lambda}) \times (R^{\tau\mu\nu}{}_\kappa + R^{\tau\nu\mu}{}_\kappa) R_{\rho\mu\nu\lambda}. \tag{4.16}$$

This covariant expression is to be interpreted as the approximate mean gravitational energy-momentum tensor determined by an observer with 4-velocity $\lambda^\kappa$ by measurements in his instantaneous 3-space. It will be observed that $\bar{t}_{\psi\chi} = g_{\chi\phi}\bar{t}_\psi{}^\phi$ is a symmetric tensor.

A straightforward but lengthy calculation yields the value of $\bar{t}_\psi{}^\phi$ for the Riemann tensor in canonical form. It is of importance to compare the physical components for Types I and II. One finds for Type I

$$\bar{t}_\alpha{}^\beta = (1/27)[2\{\sum_{k=1}^{3} \alpha_k{}^2(\delta_\alpha{}^0\delta_0{}^\beta + 2\delta_\alpha{}^k\delta_k{}^\beta + 3\delta_\alpha{}^{k+1}\delta_{k+1}{}^\beta + 3\delta_\alpha{}^{k+2}\delta_{k+2}{}^\beta)\} - 9\delta_\alpha{}^\beta \sum_{k=1}^{3}\alpha_k{}^2]. \tag{4.17}$$

It will be noticed that all the off-diagonal terms vanish. It follows that an observer measuring these physical components in his rest frame observes no gravitational energy flow. On the other hand, for Type II one finds

$$\bar{t}_\alpha{}^\beta = (1/27)[\alpha^2(-42\delta_\alpha{}^\beta + 16\delta_\alpha{}^1\delta_1{}^\beta + 22\delta_\alpha{}^2\delta_2{}^\beta + 22\delta_\alpha{}^3\delta_3{}^\beta) + 4\alpha\sigma(\delta_\alpha{}^2\delta_2{}^\beta - \delta_\alpha{}^3\delta_3{}^\beta) + 8\sigma^2(\delta_\alpha{}^0 + \delta_\alpha{}') (\delta_0{}^\beta - \delta_1{}^\beta)]. \tag{4.18}$$

[25a] *Note added in proof.*—This is perhaps rather an unusual definition of average, being in effect

$$\bar{t}_\psi{}^\phi = \lim_{r\to 0} (4\pi r^4)^{-1} \int t_\psi{}^\phi d^2S.$$

This is of the form

$$t_\alpha{}^\beta = \text{a diagonal part } + (8/27)\sigma^2\xi_\alpha\xi^\beta,$$

where $\xi_\alpha = \delta_\alpha{}^0 + \delta_\alpha{}'$ is a null vector in the direction of propagation of the radiation represented by Type II. This part of $t_\alpha{}^\beta$ is of exactly the same form as the energy tensor of an electromagnetic null field, and so should be identified as that part arising purely from gravitational radiation. The terms in $\alpha^2$, on the other hand, are to be associated with the nonradiative part of the field, and the terms in $\alpha\sigma$ with the interaction between the two parts of the field. An observer measuring a Type II field will, according to this definition of $t_\mu{}^\nu$, observe a gravitational energy flow in the 1-direction.

These results lend plausibility to the definition of gravitational radiation proposed in Sec. 3. If one accepts the energy-momentum pseudotensor as a respectable part of Einstein's theory, then the calculations in this section show that when, according to the proposed definition, gravitational radiation is present, there must be an energy flux through a small 2-surface.

## 5. EXAMPLES

It would not be satisfactory if empty static space-time regions could admit the presence of radiation; that they cannot is shown by the following rather clumsy proof.

A static space-time region, rigorously defined, is one in which there is an everywhere-timelike group of motions of the region into itself (apart from boundaries) whose generators form a normal congruence.

It follows that if the timelike tetrad vector $\lambda^\mu$ is taken to be tangent to the generators, then

$$\gamma_{0ab} = 0, \tag{5.1}$$

where $\gamma_{0ab} = \lambda_{\mu;\nu}\lambda_a{}^\mu\lambda_b{}^\nu$ are some of the Ricci rotation coefficients.[19] A standard formula[19] then at once gives

$$R_{0abc} = 0. \tag{5.2}$$

Then a rotation of the spacelike tetrad vectors will diagonalize the symmetric 3-tensor $R_{0a0b}$, and it follows from the field equations (2.1) that $R_{abcd}$ must be simultaneously diagonalized. Hence the Riemann tensor is now in Type I canonical form, and so no gravitational radiation is present. It follows from a result of Taub[26] that there can be no plane gravitational waves filling all space-time.

The simplest empty space-time gravitational field is the Schwarzschild field. Taking the metric in the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.3}$$

and labeling $r\theta\phi t$ in the order 1230, one finds, with tetrad vectors directed along the coordinate axes, a Type I Riemann tensor already in canonical form with

$$-\tfrac{1}{2}\alpha_1 = \alpha_2 = \alpha_3 = m/r^3, \quad \beta_k = 0. \tag{5.4}$$

The Riemann principal vectors are not fully determined, however, because of the symmetries of the field, which show up in the equality of $\alpha_2$ and $\alpha_3$. According to Birkhoff's theorem,[27] there can be no spherical waves, since the Schwarzschild field is the only spherically symmetric empty space-time solution of Einstein's Eq. (2.1).

The cylindrically symmetric metric introduced by Rosen,[28] in discussing cylindrical waves.

$$ds^2 = e^{2\gamma - 2\psi}(dt^2 - d\rho^2) - e^{-2\psi}\rho^2 d\phi^2 - e^{2\psi}dz^2, \quad \psi = \psi(\rho,t), \quad \gamma = \gamma(\rho,t), \tag{5.5}$$

is of Type II, with

$$\sigma = \frac{\partial^2\psi}{\partial\rho\partial t} + 5\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial t}\frac{\partial\gamma}{\partial\rho} - 3\frac{\partial\psi}{\partial\rho}\frac{\partial\gamma}{\partial t}, \tag{5.6}$$

as may readily be found by taking tetrad vectors along the coordinate axes. Radiation will be present unless the above expression for $\sigma$ vanishes.

## 6. DISCUSSION

The definition proposed in this paper provides an unambiguous local criterion of the presence of gravitational radiation, but it suffers from several defects. In the first place it counts as radiation only those gravitational disturbances which are propagated with the fundamental velocity. If it should turn out to be desirable that phenomena propagated with lower velocities be classified as radiation, then they would not be included under this definition. In particular, standing waves are not included. However, the analysis points to a new and powerful tool for the investigation of gravitational fields in general, namely the scalar invariants $\alpha_k$ and $\beta_k$.[29]

In the second place, the definition is a local geometric-algebraic one, and does not reveal at all how the properties of the radiation may vary along the path of propagation. This hiatus can be filled, at least formally, by introducing Petrov's canonical forms (3.9)–(3.11) into the conservation law for the matter-free gravitational field:

$$R^\mu{}_{\nu\rho\sigma;\mu} = 0, \tag{6.1}$$

which may readily be deduced from the Bianchi identities and the field equations (2.1). The resulting equations, which bear a striking similarity to the ordinary conservation laws for a medium with density and pressures, will be discussed in a subsequent paper.

Another defect of the present discussion is that it

[26] A. H. Taub, Ann. Math. 53, 472 (1951).

[27] Reference 24, p. 253.

[28] N. Rosen, Bull. Research Council Israel 3, 328 (1954).

[29] That these might become significant had already been emphasized by several workers in the formal talks and the discussion at the Berne Conference on Relativity Theory, July, 1955 (unpublished).

gives no indication of what secular changes may occur in radiating matter. Suppose for example that a Schwarzschild particle is disturbed from static spherical symmetry by an internal agency, radiates for some time, and finally is restored to static spherical symmetry. Is its total mass necessarily the same as before? This and similar problems required investigation. Also the status of the scalar invariants of the Riemann tensor in the Einstein, Infeld, and Hoffmann approximation theory deserves clarification, and may be hoped to assist in resolving the annoying ambiguities of interpretation which beset that theory.

I am much indebted to H. Bondi for a remark which stimulated this research, and for many discussions, and to L. Bass for suggesting a valuable improvement in presentation.

PHYSICAL REVIEW VOLUME 105, NUMBER 3 FEBRUARY 1, 1957

# Application of Formal Scattering Theory to Many-Body Problems*

L. L. Foldy and W. Tobocman†
*Case Institute of Technology, Cleveland, Ohio*
(Received October 22, 1956)

It is pointed out that the Lippmann-Schwinger integral equation, as it is usually written, does not necessarily have a unique solution when applied to the motion of three or more bodies. It follows that a certain amount of caution must be exercised in using the Lippmann-Schwinger equation as the basis for an approximation procedure.

Formal scattering theory[1] has been used extensively in the recent literature to deal with rearrangement and inelastic collisions as well as elastic scattering. Nevertheless it is not widely recognized that in applying the formulas of formal scattering theory to reactions involving more than two particles a certain amount of caution must be exercised. This is a consequence of the fact that the Lippmann-Schwinger (L-S) integral equation for the wave function, as it is usually written, does not necessarily have a unique solution when three or more bodies interact while for the two-body case the solution is unique.

In formal scattering theory one constructs from the Schrödinger equation an integral equation for the wave function. The solution of this integral equation, besides being a solution of the original Schrödinger equation, presumably satisfies the required asymptotic boundary conditions. In view of the fact that certain forms of the Lippmann-Schwinger integral equation do not have a unique solution, evidently they fail to incorporate these conditions fully.

The fact that the solution to the Lippmann-Schwinger equation is not unique means that care must be used in applying any scheme of successive approximations to generate a solution to the L-S equation. One must make sure that the solution generated is the particular one desired.

The problem of formal scattering theory is to construct the solution of a given Schrödinger equation

$$(E-H_0-V)\Psi=0, \tag{1}$$

which satisfies certain asymptotic boundary conditions. This problem is solved in a formal way by the Lippmann-Schwinger integral equation.

$$\begin{aligned}\Psi&=\lim_{\epsilon\to 0}\Psi^{(\epsilon)},\\ \Psi^{(\epsilon)}&=i\epsilon(E-H_0-V+i\epsilon)^{-1}\Phi\\ &=\Phi+(E-H_0+i\epsilon)^{-1}V\Psi^{(\epsilon)}.\end{aligned} \tag{2}$$

The solution obtained in this way is unique. However, the Lippmann-Schwinger equation is more often written in the following way.

$$\Psi=\Phi+\lim_{\epsilon\to 0}(E-H_0+i\epsilon)^{-1}V\Psi. \tag{3}$$

While any solution obtained by the procedure of Eq. (2) will satisfy Eq. (3), Eq. (3) has solutions which cannot be obtained by the procedure of Eq. (2).

We can show that the solution of Eq. (3) is not unique by showing the existence of a solution to its homogeneous counterpart,

$$\Psi=\lim_{\epsilon\to 0}(E-H_0+i\epsilon)^{-1}V\Psi. \tag{4}$$

Then, clearly, by adding an arbitrary multiple of a solution of Eq. (4) to the solution of Eq. (3), we get a second solution of Eq. (3).

The simplest case where this occurs is the problem of the mutual scattering of two particles by a potential $V$ acting between them provided a bound state of the two particles in the potential exists. For convenience, let us

* Supported in part by the U. S. Atomic Energy Commission.
† Now at the University of Birmingham, Birmingham, England.
[1] C. Moller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945); B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950); M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953); S. Sunakawa, Progr. Theoret. Phys. Japan **14**, 175 (1955).