Pion Production in Nucleon-Nucleon Collisions at Energies near Threshold*

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A calculation using the Chew theory indicates that nucleon-nucleon and pion-nucleon final-state interactions both play a role in contributing to the pion production cross sections in nucleon-nucleon collisions at energies near threshold. In particular, it is found that the cross section will be enhanced if (1) tensor forces lead to a D-wave nucleon-nucleon final state interaction or if (2) the pion and one of the nucleons is in a state of isotopic spin 3/2 and angular momentum 3/2. Both of these effects are present if the isotopic spin of the initial two nucleons is unity and the isotopic spin of the final nucleons is zero. In the case of deuteron formation, it is found that the two effects are of the same order of magnitude, but depend differently on energy.

I. INTRODUCTION

In a previous work[1] we made a calculation of the differential cross section for the reaction $p+p\rightarrow\pi^++d$ at energies just above threshold. Using the Low-Wick formulation of field theory,[2] we derived an expression for the matrix element for this reaction in terms of the wave functions of a physical deuteron and diproton and an interaction given by the Chew theory.[3] We were able to evaluate this matrix element by making several approximations, including the replacement of the physical two-nucleon wave functions by bare wave functions. Starting with an expression similar to the one we derived, an independent calculation was carried out by Geffen.[4] Closest agreement with experiment was obtained in these calculations when two conditions were satisfied: first, nuclear potentials with repulsive cores were used to obtain the two-nucleon wave functions; and second, the contribution from the D-state part of the deuteron wave function was not neglected.

In Sec. II of this paper we apply the method of A to the general case

$$N+N\rightarrow\pi+N+N, \tag{1}$$

where N can be either a proton or a neutron. We discuss qualitative features of the various cross sections near threshold and compare our results with the phenomenological model of Brueckner and Watson[5] as discussed by Rosenfeld.[6] In Sec. III we treat the question of final-state scattering of the emitted pion by the two nucleons and discuss how this scattering modifies our previous results. A number of authors have considered the role of pion-nucleon final-state scattering, especially Aitken *et al.*[7] However, in these previous treatments an approximation was used which makes the cross section proportional to the square of the final two-nucleon wave function at a particular distance from the origin. The magnitude of the wave function is very sensitive to the choice of this "characteristic distance for pion production," especially if the nucleon-nucleon potential has a repulsive core. We do not make an approximation of this type and are thereby able to estimate how pion-nucleon scattering affects the energy dependence as well as the magnitude of the pion-production cross section.

II. QUALITATIVE FEATURES OF THRESHOLD PRODUCTION

The transition matrix T for the reaction of Eq. (1) can be written[8]

$$T=(\psi_{p'}{}^{(-)},V_q{}^*\psi_p{}^{(+)}), \tag{2}$$

where $\psi_p{}^{(+)}$ is the (outgoing) wave function of the initial two nucleons with relative momentum p, and $\psi_{p'}{}^{(-)}$ is the (incoming) wave function of the final two nucleons with relative momentum p'. The interaction V_q is given by

$$V_q=V_{1q}+V_{2q}, \tag{3}$$

where V_{iq} $(i=1, 2)$ expresses the interaction of a pion with the ith nucleon as given by the Chew theory. If we restrict ourselves to consideration of the problem at threshold, V_q does not depend on the positions of the nucleons.

Because of charge independence, the matrix elements for all the reactions given by Eq. (1) can be written in terms of three isotopic spin matrix elements[9] T_{11}, T_{10}, and T_{01} where the first subscript refers to the isotopic spin of the initial two nucleons and the second to that of the final two nucleons. It is convenient to classify these matrix elements further, according to whether the initial nucleons are in a triplet ^3T or singlet ^1T spin state. There is no interference between these two cases.

The isotopic spin variables can be eliminated from the matrix elements immediately. If this is done, the interaction appearing in any single transition matrix will be either symmetric or antisymmetric in the nucleon

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[1] D. B. Lichtenberg, Phys. Rev. 99, 618 (1955); 100, 303 (1955). This latter work will be denoted by A.
[2] F. E. Low, Phys. Rev. 97, 1392 (1955); G. C. Wick, Revs. Modern Phys. 27, 339 (1955).
[3] G. F. Chew, Phys. Rev. 95, 1669 (1954).
[4] D. A. Geffen, Phys. Rev. 99, 1534 (1955).
[5] K. M. Watson and K. A. Brueckner, Phys. Rev. 83, 1 (1951).
[6] A. H. Rosenfeld, Phys. Rev. 96, 139 (1954).
[7] Aitken, Mahmoud, Henley, Ruderman, and Watson, Phys. Rev. 93, 1349 (1954).
[8] In A, we actually derived the expression for the transition matrix of the inverse reaction.
[9] Van Hove, Marshak, and Pais, Phys. Rev. 88, 1211 (1952).

spin operators and will be of the form (with $\hbar=c=1$)

$$V_q=i(2\pi/\omega_q)^{\frac{1}{2}}(f/\mu)(\boldsymbol{\sigma}_1\cdot\mathbf{q}\pm\boldsymbol{\sigma}_2\cdot\mathbf{q}),$$

where f is a coupling constant determined from pion-nucleon scattering, ω_q is the energy of a pion of momentum $\mathbf{q}$, μ is the pion mass, and $\boldsymbol{\sigma}$ is the usual Pauli spin operator. It is convenient to write the two-nucleon wave functions as a product of space and spin functions

$$\psi_p=\varphi_p(\text{space})\chi(\text{spin}),$$

where it is understood that if χ is symmetric, φ_p contains spin operators. With the assumptions we have made, only the following matrix elements contribute to the cross sections:

$$\begin{aligned} {}^1T_{10}&=(\varphi_{p'}{}^s\chi^s,V_q{}^a\varphi_p{}^s\chi^a),\\ {}^3T_{10}&=(\varphi_{p'}{}^a\chi^a,V_q{}^a\varphi_p{}^a\chi^s),\\ {}^1T_{01}&=(\varphi_{p'}{}^a\chi^s,V_q{}^a\varphi_p{}^a\chi^a),\\ {}^3T_{01}&=(\varphi_{p'}{}^s\chi^a,V_q{}^a\varphi_p{}^s\chi^s),\\ {}^3T_{11}&=(\varphi_{p'}{}^a\chi^s,V_q{}^s\varphi_p{}^a\chi^s),\end{aligned}\tag{4}$$

where the superscripts s and a refer to symmetric and antisymmetric respectively. The matrix element ${}^1T_{11}$ is seen to be zero just from the spin matrix multiplication. In all cases given by Eq. (4), the pion is emitted in a p-state with respect to the two-nucleon system. We have not considered the emission of s-wave pions.

Since the interaction V_q does not depend on nucleon spatial variables, we see from Eq. (4) that the magnitudes of the various cross sections are essentially determined by the spatial integrals $\int\varphi_{p'}{}^*\varphi_p d\mathbf{r}$. If we expand φ_p and $\varphi_{p'}$ in partial waves, we can carry out the angular integration. We are then left with radial integrals of the form

$$F_l=\int u_{lp'}u_{lp}dr,\tag{5}$$

where u_{lp}/r is a radial two-nucleon wave function corresponding to momentum p and orbital angular momentum l. We have suppressed an index referring to the total angular momentum of the two nucleons.

The magnitudes of the integrals F_l will of course depend strongly on the explicit form assumed for the two-nucleon initial and final wave functions. The more u_{lp} and $u_{lp'}$ differ from regular spherical Bessel functions (plane wave), the larger F_l will be in general. Since the final nucleons have low energy, the $u_{lp'}$ will differ from a plane wave only in the states of low angular momentum. We would expect the contributions from the S-, P-, and D-state overlap integrals to decrease in that order. This is not necessarily the case. In the first place, F_0 is depressed because the repulsive core in the nuclear potential reduces the $l=0$ wave functions near the origin—a region from which an important contribution to F_0 is expected to come. States of higher angular momentum are not affected so much by the core. In the second place, if the final nucleons are in a triplet spin state, the tensor force will couple the D-state wave function to the S-state function. This has the effect of enhancing F_2. In A, using wave functions obtained from Gartenhaus potentials[10] and a deuteron final state, we found that the contribution to F_0 from the region within the range of the nuclear potential (reduced by the repulsive core) was very nearly canceled by the contribution from the outside region. This had the effect of making F_0 negligibly small when compared to F_2.

The enhancement of F_2, due to tensor forces, occurs only if the final two-nucleon wave function is symmetric both in space and spin. From Eq. (4) we see that enhancement occurs only in the case of ${}^1T_{10}$. According to the model of Brueckner and Watson, the matrix element T_{10} is also the largest. However, the contribution from F_2 is neglected, and the enhancement is attributed to a strong pion-nucleon final state interaction in the state of isotopic spin 3/2 and angular momentum 3/2 [(3-3) state]. In the next section, we shall see that both these effects are present.

Of the matrix elements given by Eq. (4), ${}^3T_{11}$ is the smallest. This can be seen by noting that the initial and final two-nucleon states have the same spin and isotopic spin. Therefore, the space wave functions $\varphi_p{}^a$ and $\varphi_{p'}{}^a$ are eigenfunctions of the same potential belonging to different energies. Thus, only to the extent that the nuclear potential is velocity-dependent will the overlap integrals F_l differ from zero. The nuclear potential is not known in sufficient detail at the present time to enable us to get a good estimate of the size of ${}^3T_{11}$. We can only say that it is small. It is observed experimentally that the reaction $p+p\to\pi^0+p+p$, which arises only from T_{11}, has the smallest cross section.[6] According to the Brueckner-Watson model, this reaction is enhanced by the strong pion-nucleon (3-3) state interaction. However, the Pauli principle forbids the final nucleons from being emitted in an S state if the pion is in a p state, a fact which reduces the cross section.

III. PION FINAL STATE SCATTERING

In obtaining the matrix element of Eq. (2), we have made the approximation of replacing the complete two-nucleon wave functions by their zero meson parts. In so doing, the effect of scattering of the emitted pion by the two nucleons is neglected. In the following, we shall not try to obtain an expression for the complete two-nucleon wave functions but shall instead modify the interaction. As in A, it proves convenient to write the matrix element for the inverse reaction. We write

$$T=(\psi_p{}^{(-)},H_1'\psi_2{}^{(+)}\psi_{p'}{}^{(+)})+(\psi_p{}^{(-)},H_2'\psi_1{}^{(+)}\psi_{p'}{}^{(+)}),\tag{6}$$

where

$$H_i'=\Sigma_k(V_{ik}a_k+V_{ik}{}^*a_k{}^*),\quad i=1,2$$

[10] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).

is the interaction of the pion with the ith nucleon, a_k and a_k^* are annihilation and creation operators, and $\psi_j(j\neq i)$ is the wave function of the pion modified by by scattering from the jth nucleon. If the pion can be described by a plane wave, the matrix element of Eq. (6) reduces to that of Eq. (2) (with initial and final states reversed). The justification for writing Eq. (6) is that it seems reasonable to describe the pion by a wave function more nearly resembling its actual wave function than a plane wave. This treatment neglects multiple scattering of the pion. An approximation of this type, in which a pion emitted by one nucleon is allowed to be scattered by the other, was suggested by Gammel.[11]

The wave function which describes the scattering of a pion by the jth nucleon can be written

$$\psi_j^{(+)}=\delta(\mathbf{q}-\mathbf{k})+\frac{(k|T_j|q)}{\omega_q-\omega_k+i\epsilon}\exp(-i\mathbf{k}\cdot\mathbf{r}_j), \qquad (7)$$

where $(k|T_j|q)$ is the off-the-energy-shell transition matrix describing the scattering and r_j is the position of the jth nucleon. We have neglected the pion-nucleon center of mass motion and have made our usual restriction $q\ll\mu$ so that $\exp(i\mathbf{q}\cdot\mathbf{r}_j)\simeq 1$. Substituting Eq. (7) in Eq. (6), we obtain

$$T=(\psi_p^{(-)},\{V_q+\Sigma_k(\omega_q-\omega_k+i\epsilon)^{-1}[V_{1k}(k|T_2|q)e^{i\mathbf{k}\cdot\mathbf{r}} + V_{2k}(k|T_1|q)e^{-i\mathbf{k}\cdot\mathbf{r}}]\}\psi_{p'}^{(+)}), \qquad (8)$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$. If we neglect scattering in states other than the (3-3) state, the pion-nucleon scattering matrix becomes

$$(k|T_j|q)=P_{33}^{(j)}t_{kq}, \qquad (9)$$

where $P_{33}^{(j)}$ is a projection operator for the (3-3) state and t_{kq} depends only on the magnitude of k and q. The operator $P_{33}^{(j)}$ is given by

$$P_{33}^{(j)}=(4\pi kq)^{-1}[2\mathbf{k}\cdot\mathbf{q}-i\boldsymbol{\sigma}_j\cdot(\mathbf{k}\times\mathbf{q})].$$

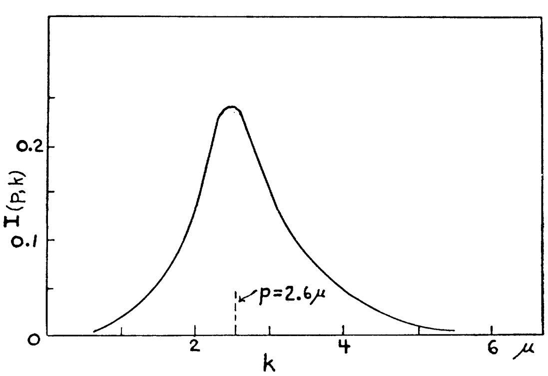

FIG. 1. The integral $I(p,k)$ plotted as a function of k for $p=2.6\,\mu$. The abscissa is in units of the pion mass μ and the ordinate in units of μ^{-1}.

[11] J. Gammel, Phys. Rev. **95**, 209 (1954).

In Eq. (8), we shall call the term in T containing V_q the *direct production* term and the remainder of T the *scattered* term.

We shall again restrict ourselves for simplicity to deuteron formation (the reaction $p+p\rightarrow\pi^++d$). Then the only contribution due to the scattered term in the matrix element will come from the D-state part of the initial diproton wave function. This is true because the pion cannot be in a (3-3) state with respect to one of the final nucleons if the total angular momentum of the system is zero (initial S state). Other angular momentum states are forbidden by the requirement that the pion be emitted in a p state. As in the direct production term, there will be a contribution from the

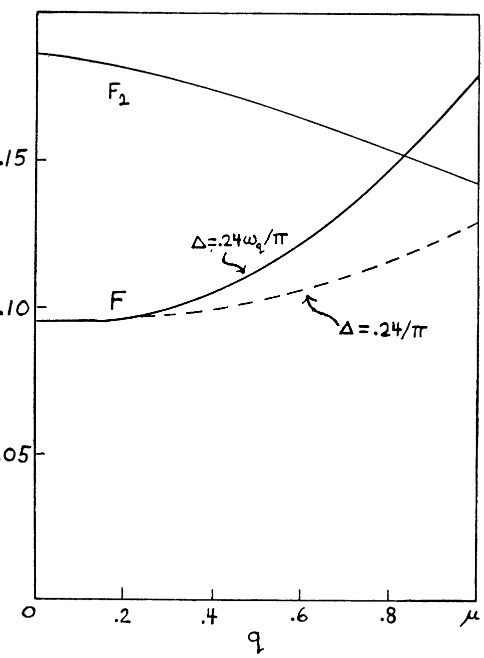

FIG. 2. The behavior of the direct production term in the matrix element F_2 and the scattered production term F as a function of the momentum q of the emitted pion. The units are the same as in Fig. 1.

deuteron S-state and D-state wave functions. However we can neglect the D-state contribution because of the small D-state probability of the deuteron. The peculiar cancellation which depresses the S-state integral F_0 is not important here because of the additional r dependence in the scattered term of the factor $\exp(\pm i\mathbf{k}\cdot\mathbf{r})$.

If we now use Eq. (9) in Eq. (8) and carry out the angular part of the integration, square, and sum and average over spin states, we obtain

$$\Sigma|T|^2=(4\pi)^2(f/\mu)^2(q/p)^2\omega_q^{-1} \times\{F_0^2+2F_0\,\mathrm{Re}(F_2/\sqrt{2}+F)\cos(\delta_0-\delta_2)(3\cos^2\theta-1) +|F_2/\sqrt{2}+F|^2(3\cos^2\theta+1)\}, \qquad (10)$$

where θ is the angle between the incident proton and emitted pion, δ_0 and δ_2 are the diproton S and D phase shifts, Σ indicates the appropriate spin summation, and Re means the real part. This expression differs from that obtained in A by the additional factor F, which arises from the scattered term, and is given by

$$F=\frac{\omega_q^{\frac{1}{2}}}{q}\int dk\frac{k^3 t_{kq} I(p,k)}{(\omega_q-\omega_k+i\epsilon)\omega_k^{\frac{1}{2}}}, \tag{11}$$

where

$$I(p,k)=\int dr u_{2p}(r)u_{0d}(r)j_2(kr).$$

Here u_{0d} is the deuteron radial S-wave function and j_2 is a spherical Bessel function of order 2. The diproton function u_{2p} is normalized to $u_{2p}\rightarrow\sin(pr-\pi+\delta_2)$.

It is apparent from Eq. (10) that $F_2/\sqrt{2}$ and F appear symmetrically so far as their contribution to the angular distribution is concerned. However, there is no reason to expect F_2 and F to depend in the same way on energy. In order to get an estimate of the magnitude and energy dependence of F, we need to specify t_{kq}. For simplicity, we choose a form for t_{kq} suggested by Aitken *et al.*[7] These authors assume

$$t_{kq}=-\Delta\frac{kq}{(\omega_k\omega_q)^{\frac{1}{2}}}\times\begin{cases}1/\omega_q & \text{for} \quad \omega_q>\omega_k\\ 1/\omega_k & \text{for} \quad \omega_q<\omega_k,\end{cases}$$

with Δ a constant. However, we do not take Δ to be constant, but choose it so that t_{kq} is normalized on the energy shell; that is, we choose Δ so as to satisfy the equation

$$t_{kq}\rightarrow t_{qq}=-(\pi q\omega_q)^{-1}e^{i\delta}\sin\delta,$$

where δ is the (3-3) state scattering phase shift. For $q\leq\mu$, δ can be approximated quite closely by

$$e^{i\delta}\sin\delta\simeq\delta=0.24(q/\mu)^3.$$

We then obtain

$$\Delta=0.24\omega_q/(\pi\mu^3). \tag{12a}$$

If we had followed Aitken *et al.*, we would have

$$\Delta=0.24/(\pi\mu^2). \tag{12b}$$

The choice of Δ has an increasing effect on the magnitude of the cross section as the energy increases. However, our calculation has its greatest reliability near threshold, where the difference is unimportant.

The double integral for F must be performed numerically. If Gartenhaus wave functions are used, the integral $I(p,k)$ is given by Fig. 1. This integral was computed only at threshold, but is not very sensitive to small changes in the momentum p of the initial diproton wave function. For higher momentum, the maximum value of $I(p,k)$ will be smaller and shifted to the right. The important feature of $I(p,k)$ is that it is very small for high values of k, and therefore makes the integral F insensitive to the behavior of t_{kq} near the cut-off momentum $k_c=6\,\mu$. The imaginary part of F is negligible in the energy range we are considering.

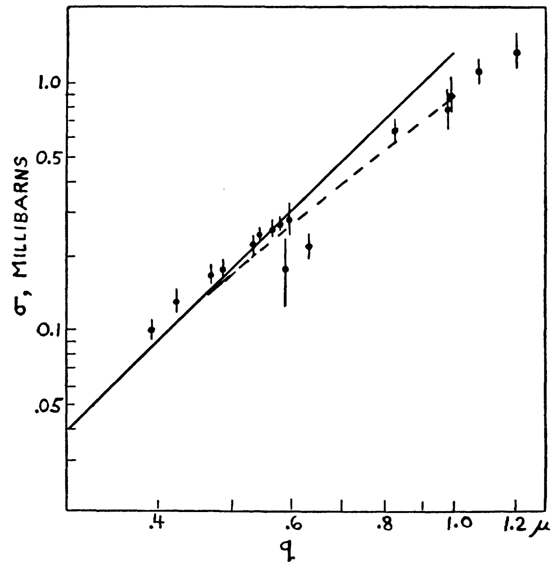

FIG. 3. The calculated cross section for the reaction $p+p\rightarrow\pi^++d$ (solid line) and the experimental points as a function of the pion momentum q. The dashed line indicates the calculated cross section when one uses the Δ of Eq. (12b).

A plot of F as a function of q is given in Fig. 2. For comparison, F_2 is also shown. The principal reason that F_2 decreases with increasing energy is that u_{2p} oscillates more rapidly with increasing p, causing the integrand to average more nearly to zero.

We now consider the cross section. Since F_0 is negligibly small, the additional term F does not affect the angular distribution, which remains $3\cos^2\theta+1$. Using Eq. (10) and setting $F_0=0$, we get for the total cross section σ:

$$\sigma=16\pi M(f/\mu)^2(q/p)^3(F_2/\sqrt{2}+F)^2, \tag{13}$$

where M is the nucleon mass. In Fig. 3 this cross section is plotted against q for $f^2=0.08$. The experimental points (from Fig. 3, reference 6) are also shown. If we had used the value of Δ given by Aitken *et al.* instead of Δ as given by Eq. (12a), the calculated cross section would be reduced in the region $q\sim\mu$ and would be closer to the experimental points. It should be pointed out that we have not normalized the theoretical curve to the experimental points. The magnitude of the cross section is determined by our choice of $f^2=0.08$, a value determined by pion-nucleon scattering data. However, the actual curve given in Fig. 3 should not be taken too seriously in view of the approximations made in evaluating the matrix element. In addition a two-nucleon phase-shift calculation by Gammel and Thaler[12] indicates that the Gartenhaus potentials are not valid at high energy.

Nevertheless, there are several features of the calculation that have a somewhat more general validity. Of particular interest is the fact that the contribution from

[12] J. Gammel and R. Thaler, Phys. Rev. **103**, 1874 (1956).

tensor forces and the contribution from pion nucleon final state interactions are of the same order of magnitude. Previous calculations have neglected one or the other of these effects. In our particular calculation, neither of these effects is large enough to account for the experimental results by itself.

According to Eq. (10), three important contributions to the cross section come from the terms F_0, F_2, and F. The F_0 term arises from a transition from an initial two-nucleon S state to a final two-nucleon S state. This term is expected to be rather small, but the fact that it is negligibly small is an accident of the Gartenhaus wave functions. The principal effect of F_0 is to alter the angular distribution. If the diproton S phase shift δ_0 is negative (at 300-Mev proton laboratory energy), F_0 will be positive and will make the angular distribution more anisotropic than $3\cos^2\theta+1$ and more in agreement with experiment. The F_2 term arises from a D- to D-state transition. The magnitude of F_2 is sensitive to the details of the D-state functions, but the fact that F_2 decreases with increasing energy is rather insensitive to the particular choice of wave functions. If the D-wave phase shift δ_2 is large and negative, however, F_2 may not decrease with increasing energy. The scattered term F arises from a D- to S-state transition. The magnitude of F depends on the two-nucleon wave functions and on the off-the-energy-shell behavior of t_{kq}. However, for several different choices of t_{kq} suggested by meson theory, F will increase as the energy increases.[13]

A rather good fit to the experimental data on pion production with deuteron formation has been obtained with the semiempirical formula

$$\sigma = aq+bq^3, \tag{14}$$

where the first term is supposed to arise from s-wave pions and the second from p-wave pions. Here a and b are constant parameters determined from experiment. However, we have seen that the production of p-wave pions arises from several terms, none of which is expected to remain constant as q increases. Furthermore, a factor p^{-3} appears in the expression of Eq. (13) for the cross section. One factor of p^{-1} arises simply from the fact that the relative velocity of the incoming protons appears in the denominator of the formula for the cross section. The factor p^{-2} appears because we have taken a factor p^{-1} out of the matrix element by normalizing the proton wave functions to $\sin(pr-\frac{1}{2}l\pi+\delta_l)$. This factor becomes squared in the expression for the cross section. The factor p^{-3} has the effect of making the p-wave production rise less rapidly than q^3 and thus to seem like a mixture of s- and p-waves if analyzed according to Eq. (14). If a phenomenological analysis of the energy dependence of the cross section is wanted, it seems more reasonable to use instead of Eq. (14) the formula

$$\sigma = (a'q+b'q^3)(p_0/p)^3, \tag{15}$$

where p_0 is the relative proton momentum at threshold and a' and b' are new constants. However it should be re-emphasized that there is no theoretical reason for a' and b' to be constant; in fact our calculation indicates that b' at first slightly decreases and then increases with energy.

If the constants a and b of Eq. (14) are fitted by experiment, it turns out that $a/b=0.14$. If Eq. (15) is used, a rough estimate indicates that $a'/b'\sim 0.03$. The data are not sufficiently accurate to determine this ratio precisely. The important question of how much of the cross section is due to s-wave pions needs to be answered by (increasingly difficult) experiments at energies very close to threshold.

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[13] For example, this statement is true for the (3-3) state transition matrix found by Gammel, reference 11.