The average of the potential V over the Fermi sphere is

$$-MV_{\mathrm{Av}}=-M\int_0^{kF}V(k_i)k_i^2dk_i\Big/\int_0^{kF}k_i^2dk_i$$
$$=\frac{3(\lambda_s+\lambda_t)\mu^2}{\pi^2\beta}\left[2\tan^{-1}\left(\frac{k_F}{\beta}\right)-\left(\frac{3\beta}{k_F}+\frac{2\beta^3}{k_F^3}\right)\ln\left(1+\frac{k_F^2}{\beta^2}\right)+2\frac{\beta}{k_F}\right]. \quad (7)$$

The average kinetic energy of any nucleon in our units is:

$$MT_{\mathrm{Av}}=\tfrac{3}{10}k_F^2\mu^2. \quad (8)$$

Hence the binding energy per nucleon is

$$W=\frac{3\mu^2}{10M}\left[-k_F^2+\frac{5(\lambda_s+\lambda_t)}{\pi^2\beta}\left\{2\tan^{-1}\left(\frac{k_F}{\beta}\right)-\left(3\frac{\beta}{k_F}+2\frac{\beta^3}{k_F^3}\right)\ln\left(1+\frac{k_F^2}{\beta^2}\right)+\frac{2\beta}{k_F}\right\}\right]. \quad (9)$$

An examination of the numerical value of Eq. (9) for the values $\lambda_s=0.8015$, $\lambda_t=1.1400$, and $\beta_s=\beta_t=\beta=2.0304$, shows that W never becomes positive in the region of the physically interesting value of k_F. Hence it seems that the S-state separable potential of Yamaguchi does not lead to sufficient binding energy for a nucleus when surface effects are not taken into account.

Note added in proof.—Equations (3) to (9) should involve λ_s and λ_t in the form $(\lambda_s+3\lambda_t)/4$ rather than as $(\lambda_s+\lambda_t)$. This does not, however, lead to any change in the final result.

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Nucleon Structure in the Static Theory

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The static theory of Chew and Low is used to calculate the form factors for the charge and current of the proton. The cross section obtained with these form factors is found to be in close agreement with the electron-proton scattering data of McAllister and Hofstadter up to 236 Mev. Root-mean-square radii of 6.5×10^{-14} cm for the charge and 9.7×10^{-14} cm for the current are found. The role of an extended core charge is discussed. The value of the coupling constant used is $f^2=0.08$.

1. INTRODUCTION

THE Stanford experiments on elastic electron-proton scattering[1] indicate that the proton is a structured particle which occupies a region of the order of 10^{-13} cm. Additional information on the spatial distribution of the charge and current of the proton is beginning to emerge from these experiments. Due to the strong pion-nucleon interaction, the gross features of this structure may be expected to be largely determined by the cloud of virtual pi mesons surrounding the nucleon core. Thus electrons whose wavlength is sufficiently large, so that they do not probe the proton structure in too much detail, may have an elastic cross section whose principal deviation from the Mott-Rutherford cross section is determined by the pion-nucleon interaction.[2] It is therefore of interest to see whether the Chew-Low form of a Yukawa-type theory, which has had a degree of success in explaining the P-wave part of the low-energy pion nucleon scattering,[3] and low-energy photomeson production,[4] is also in accord with these experiments.

The electric and magnetic form factors for the proton, which determine the electron proton cross section, have been calculated in the first approximation of the Chew-Low theory. The calculations of Miyazawa,[5] Zachariasen,[6] and Treiman and Sachs,[7] indicate the strong likelihood that the inclusion of higher order corrections would not qualitatively alter the form factors predicted by the first approximation of the static theory. In addition to the meson charge and current, a statically spread out nucleon core charge has been assumed, as suggested by the related problem of the neutron-electron interaction.[8] As has been shown, the first approximation of this theory can be brought into rough agreement with

[1] R. Hofstadter and R. McAllister, Phys. Rev. **98**, 217 (1955); R. Hofstadter and E. E. Chambers, Bull. Am. Phys. Soc. Ser. II, **1**, 10 (1956); R. Hofstadter, *Proceedings of the Sixth Annual Rochester Conference on High Energy Nuclear Physics* (Interscience Publishers, New York, 1956); E. E. Chambers and R. Hofstadter, Phys. Rev. **103**, 1454 (1956); R. McAllister and R. Hofstadter, Phys. Rev. **102**, 851 (1956).

[2] A 200-Mev electron has a reduced de Broglie wavelength $\lambda\!\!\!^{-}\sim10^{-13}$ cm.

[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[4] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).

[5] H. Miyazawa, Phys. Rev. **101**, 1564 (1956).

[6] F. Zachariasen, Phys. Rev. **102**, 295 (1956).

[7] S. Treiman and R. G. Sachs, Phys. Rev. **103**, 435 (1956).

[8] G. Salzman, Phys. Rev. **99**, 973 (1955).

the experimental neutron-electron interaction[9] if the nucleon core charge is assumed to be statically spread out in a region of approximately the same extent as that of the virtual pion cloud.[10] We have not attempted to calculate, in the static theory, contributions of those terms which are of purely relativistic origin. Such contributions are likely to be unreliable, as is discussed in Sec. 3. With this treatment, we find that for low energies (up to $\sim$200 Mev) the differential cross section for electron proton scattering is in close agreement with the experiments.

In Sec. 2 the invariant form factors are defined and their static limits are expressed in terms of weighted integrals of the charge and magnetic moment density of the nucleon. The charge and current of the "physical" nucleon are calculated using the Chew-Low theory in Sec. 3, and in Sec. 4 the form factors are evaluated and the results briefly discussed.

2. STATIC LIMIT OF THE INVARIANT FORM FACTORS

The most general form of the S-matrix element for the elastic scattering of an electron and a nucleon, correct to second order in the electromagnetic coupling and to all orders in the meson nucleon interaction, is

$$\begin{aligned} S_{fi}&=-i(2\pi)^4\delta_4(p+q)M_{fi},\\ M_{fi}&=-\{\bar{u}(p_2')i(\epsilon\gamma_\mu+\tfrac{1}{2}i\mu\gamma_{\mu\nu}q_\nu)u(p_2)\}\\ &\qquad\times\{\bar{w}(p_1')j^e{}_\mu w(p_1)\}/p_\sigma p_\sigma, \end{aligned}\tag{1}$$

where $p=p_1'-p_1$ and $q=p_2'-p_2$ are the four momenta transferred to the electron and nucleon, respectively, during the collision, $u(p_2)$ and $\bar{u}(p_2')[=u^*(p_2')\beta]$ are the normalized Dirac spinors for the initial and final nucleon states, u^* is the Hermitian adjoint of u, $w(p_1)$ and $\bar{w}(p_1')$ likewise for the initial and final electron states, $p_\sigma p_\sigma=p^2-p_4{}^2$, $j^e{}_\mu=i(-e)\gamma_\mu$, $-e$ is the electron charge, Greek indices have the range of values 1, 2, 3, 4, the summation convention applies to repeated indices, γ_μ are the Dirac matrices, $\gamma_{\mu\nu}=\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu$, and ϵ and μ are invariant functions of $q_\sigma q_\sigma$. Natural units ($\hbar=c=1$) are used throughout. Equation (1) may be obtained by an argument similar to that given in reference 8.

The invariant functions ϵ and μ are the effective charge and effective anomalous moment of the nucleon, and may be expanded as follows:

$$\epsilon=\sum_{n=0}^{\infty}(-1)^n\epsilon_n(q_\sigma q_\sigma)^n,\quad \mu=\sum_{n=0}^{\infty}(-1)^n\mu_n(q_\sigma q_\sigma)^n.\tag{2}$$

The coefficients in these expansions are the invariant parameters introduced by Foldy,[11] which characterize the electromagnetic properties of the nucleon and which can in principle be experimentally determined. They are independent of any details of the electromagnetic field with which the nucleon is interacting. In particular, ϵ_0 equals e for the proton, zero for the neutron; μ_0 is 1.79 nuclear magnetons for the proton and -1.91 nm for the neutron. The physical interpretation of the parameters is given by Eq. (7).

In describing the electron-proton scattering experiments it is convenient to introduce two invariant form factors,[12] one of which represents the fact that the charge is distributed in space and the other that the current (or magnetic moment) is likewise not confined to a point. The charge and current (or electric and magnetic) form factors F_1 and F_2 may be defined by the equations

$$\epsilon=\epsilon_0F_1,\quad \mu=\mu_0F_2.$$

From Eq. (1) it then follows that the differential cross section for elastic electron-proton scattering[13] is given by

$$\begin{aligned} \sigma&=\sigma_{NS}\left\{F_1{}^2+\frac{q^2}{4M^2}\left[2\left(F_1+\frac{2M}{e}\mu_0F_2\right)^2\tan^2\left(\frac{\theta}{2}\right)+\left(\frac{2M}{e}\mu_0F_2\right)^2\right]\right\},\\ \sigma_{NS}&=\left(\frac{e^2}{2E}\right)^2\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\frac{1}{1+(2E/M)\sin^2(\theta/2)}. \end{aligned}\tag{3}$$

This result is correct to second order in the electromagnetic coupling and to all orders in the meson nucleon interaction. Here E is the electron energy, assumed to be $\gg m_0$, θ is the electron scattering angle, both quantities as measured in the laboratory system, and M is the proton rest mass.

The fixed nucleon theory of Chew and Low will be used to calculate the static limits of the two form factors, and then use made of Eq. (3). For this purpose we need to identify the static limits of F_1 and F_2, or equivalently of ϵ and μ. In order to make the identification, it will be assumed that the static theory is to be considered the limit, as the nucleon mass becomes infinite, of a relativistic theory. We proceed as follows. The matrix element for the elastic scattering of an

[9] Hughes, Harvey, Goldberg, and Stafne, Phys. Rev. **90**, 497 (1953). A more recent value of -4.2 ± 0.3 kev for the effective interaction energy given by Melkonian, Rustad, and Havens, Bull. Am. Phys. Soc. Ser. II, **1**, 62 (1956) agrees with the earlier value of -3.9 ± 0.4 kev. For addition references, and discussion of the experimental situation, see Crouch, Krohn, and Ringo, Phys. Rev. **102**, 1321 (1956).

[10] This conclusion is unaffected by the fact that the currently accepted value of the pion-nucleon coupling constant, f^2, is about one and one-half times as large as the value used in reference 8. The reason for this is that both the meson contribution and the assumed core contribution are proportional to f^2, so that the extended core charge still cancels the same fraction of the pion contribution to the effective interaction energy as it did before.

[11] L. L. Foldy, Phys. Rev. **87**, 688 (1952); **87**, 693 (1952).

[12] See for example, D. R. Yennie, *Proceedings of the Fifth Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1955).

[13] M. N. Rosenbluth, Phys. Rev. **79**, 615 (1950).

electron by a fixed nucleon, calculated with the interaction energy density $-j^e_\mu(\mathbf{r})A^N_\mu(\mathbf{r})$, where $A^N_\mu(\mathbf{r})$ is the four-potential due to the nucleon, is given by

$$\left(\Psi_f,\ -\int d\mathbf{r}\, j^e_\mu(\mathbf{r})A^N_\mu(\mathbf{r})\Psi_i\right)$$

$$=-\{\bar{w}(p_1')j^e_\mu w(p_1)\}\int d\mathbf{r}\, e^{-i\mathbf{p}\cdot\mathbf{r}}\frac{1}{4\pi}\int d\mathbf{r}'\frac{j^N_\mu(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},$$

which is easily transformed to

$$-\int d\mathbf{r}\, j^N_\mu(\mathbf{r})e^{-i(\mathbf{p}_1'-\mathbf{p}_1)\cdot\mathbf{r}}\{\bar{w}(p_1')j^e_\mu w(p_1)\}/p^2. \quad (4)$$

The static limit of M_{fi}, given in Eq. (1), is now identified with expression (4), that is,

$$\lim\{\bar{u}(p_2')i(\epsilon\gamma_\mu+\tfrac{1}{2}i\mu\gamma_{\mu\nu}q_\nu)u(p_2)\}$$

$$=\int d\mathbf{r}\, j^N_\mu(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}, \quad (5)$$

where "lim" means "the limit as $M\rightarrow\infty$."

We note that

$$u(p_2)=N(p_2)\begin{bmatrix}\chi_2\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}_2}{E_2+M}\chi_2\end{bmatrix},\quad \gamma_j=\begin{pmatrix}0 & -i\sigma_j\\ i\sigma_j & 0\end{pmatrix},$$

where $N(p_2)$ is the normalization constant, χ_2 is the Pauli spinor for the initial nucleon state, Latin indices have the range of values 1, 2, 3, the σ_j are the 2×2 Pauli spin matrices, and E_2 is the initial nucleon energy. Since the small components of the nucleon spinors vanish as $M\rightarrow\infty$, we obtain from Eq. (5)

$$\lim\epsilon=\int d\mathbf{r}\,\rho^N(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}, \quad (6a)$$

$$\lim i\mu\boldsymbol{\sigma}\times\mathbf{q}=\int d\mathbf{r}\,\mathbf{j}^N(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}, \quad (6b)$$

where the Pauli spinors have been omitted. From Eqs. (2) and (6) it follows, as shown in the appendix, that

$$\lim\epsilon_n=\frac{1}{(2n+1)!}\int d\mathbf{r}\,\rho^N(r)r^{2n}, \quad (7a)$$

$$\lim\mu_n=\frac{6(n+1)}{(2n+3)!}\frac{\boldsymbol{\sigma}}{3}\cdot\int d\mathbf{r}[\tfrac{1}{2}\mathbf{r}\times\mathbf{j}^N(\mathbf{r})]r^{2n}. \quad (7b)$$

The $\rho^N(r)$ and $\mathbf{j}^N(\mathbf{r})$ which appear in Eqs. (7) should be the total charge and current density of the stationary nucleon, including core as well as meson contributions, but not including terms of purely relativistic origin.

Since from Eq. (7a),

$$\lim\epsilon_0=\int d\mathbf{r}\,\rho^N(r)\quad\text{and}\quad\lim\epsilon_1=\tfrac{1}{6}\int d\mathbf{r}\,\rho^N(r)r^2,$$

an rms radius with respect to the charge density may be defined,

$$\langle r_{\epsilon^2}\rangle=[6\lim(\epsilon_1/\epsilon_0)]^{\frac{1}{2}},$$

and likewise from Eq. (7b) for the current density,

$$\langle r_{\mu^2}\rangle=[10\lim(\mu_1/\mu_0)]^{\frac{1}{2}}.$$

The static limits of the form factors may then be written

$$\lim F_1=1-\frac{1}{6}\langle r_{\epsilon^2}\rangle^2q^2+\frac{1}{120}\langle r_{\epsilon^4}\rangle^4q^4-\cdots, \quad (8a)$$

$$\lim F_2=1-\frac{1}{10}\langle r_{\mu^2}\rangle^2q^2+\frac{1}{280}\langle r_{\mu^4}\rangle^4q^4-\cdots. \quad (8b)$$

For sufficiently low momentum transfer, only terms of order up to q^2 need be considered. Once charge and current distributions are specified for the nucleon, Eqs. 7(a), and 7(b) determine the static limits of the terms comprising the form factors.

3. NUCLEON CHARGE AND CURRENT IN THE CHEW-LOW THEORY

In this section we obtain the expectation values of the mesonic charge and current densities of a "physical" nucleon. Closely related calculations, using the techniques of Chew and Low[3,4] and Wick[14] have been carried out by Miyazawa[5] for the frequency-independent part of the nucleon magnetic moment, by Zachariasen[6] for the proton charge density, by Fubini[15] for the nucleon charge and current density, and by Treiman and Sachs[7] for the rms radius of the neutron-electron interaction.

In a static theory calculation of an electric or magnetic moment of a nucleon, there is a question as to how much of the total moment one should attempt to include in the calculation. Consider the frequency-independent part of the proton magnetic dipole moment. If terms of relativistic origin are to be calculated, then the operator which corresponds to the Dirac moment must be included, as is done by Sachs[16] and Miyazawa,[5] who in effect calculate the expectation value, in the physical proton state, of the operator

$$\int d\mathbf{r}[\tfrac{1}{2}\mathbf{r}\times\mathbf{j}_\pi(\mathbf{r})]+\frac{e}{2M}\left(\frac{1+\tau_3}{2}\right)\boldsymbol{\sigma},$$

and then subtract $e/2M$ to obtain the anomalous moment. Here the nucleon is fixed at the origin, $\mathbf{j}_\pi$ is the pion current density, and $\boldsymbol{\tau}$ is the isotopic spin operator for the nucleon.

[14] G. C. Wick, Revs. Modern Phys. 27, 339 (1955).
[15] S. Fubini, Nuovo cimento 3, 1425 (1956).
[16] R. G. Sachs, Phys. Rev. 87, 1100 (1952).

As remarked by Miyazawa, the calculation of the relativistic terms in a static cutoff theory should not be taken too seriously. These terms depend quadratically on the cutoff, and, as pointed out by Sachs,[16] and by Miyazawa,[5] they involve the assumption that the bare proton magnetic moment is one nuclear magneton. Because of the unreliability of these terms, we choose to omit the Dirac moment part of the operator, as was done by Chew,[17] Friedman,[18] and Fubini.[15] This omission is not expected to be serious, in view of the fact that Miyazawa found that this part of the operator gave a contribution to the isotopic vector part of the anomalous moment of the order of one-tenth the contribution of the meson current. Further, the isotopic scalar part of the anomalous moment, which is thus also omitted, in experimentally very small (-0.06 nm). The remainder of this section contains a derivation of the meson charge and current in the physical nucleon state.

Following Chew and Low, we take as the Hamiltonian

$$\begin{aligned} H &= H_0 + H_I, \\ H_0 &= \textstyle\sum_k \omega a_k^\dagger a_k, \\ H_I &= \textstyle\sum_k (V_k a_k + V_k^\dagger a_k^\dagger), \\ V_k &= (4\pi)^{\frac{1}{2}} \frac{f^0}{\mu} \frac{iv(k)}{(2\omega)^{\frac{1}{2}}} \tau_\kappa \boldsymbol{\sigma}\cdot\mathbf{k}, \end{aligned} \tag{9}$$

where the index k includes momentum ($\mathbf{k}$) and isotopic spin (κ), $\omega=(\mu^2+k^2)^{\frac{1}{2}}$, $a_k^\dagger$ and a_k are creation and annihilation operators, respectively, for a single meson of type k, f^0 is the unrenormalized unrationalized coupling constant, μ is the meson mass, and $v(k)$ is the cut-off function. The symbol k is also used for the magnitude of the momentum $\mathbf{k}$. In this representation the charge and current operators for the meson fields may be written as[19]

$$\mathbf{j}_\pi = -e(\phi_1 \nabla\phi_2 - \phi_2\nabla\phi_1), \quad \rho_\pi = -e(\pi_1\phi_2 - \pi_2\phi_1),$$

and the usual procedure leads to

$$\begin{aligned} \mathbf{j}_\pi &= \frac{ie}{2}\sum_{k,k'} \frac{\mathbf{k}}{(\omega\omega')^{\frac{1}{2}}} (a_{k'}^\dagger + a_{-k'})(a_k + a_{-k}^\dagger)\delta^{12}_{\kappa\kappa'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}, \\ \rho_\pi &= \frac{ie}{2}\sum_{k,k'} \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} (a_{k'}^\dagger - a_{-k'})(a_k + a_{-k}^\dagger)\delta^{12}_{\kappa\kappa'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}, \end{aligned} \tag{10}$$

where the isotopic spin indices κ,κ' are here summed only from 1 to 2, since neutral mesons don't contribute, and $\delta^{12}_{\kappa\kappa'}$ is $+1$ or -1 according as κ,κ' is an even or odd permutation, respectively, of 1, 2, and zero otherwise. Using Ψ_0 for the four single physical nucleon states (the spin and isotopic spin indices are omitted), we then want to obtain

$$(\Psi_0, \mathbf{j}_\pi \Psi_0) \quad \text{and} \quad (\Psi_0, \rho_\pi \Psi_0).$$

For this purpose we need

$$(\Psi_0,\ (a_{k'}^\dagger \pm a_{-k'})(a_k + a_{-k}^\dagger)\Psi_0),$$

which may be written as

$$\begin{aligned} &(a_{k'}\Psi_0, a_k\Psi_0) + (a_{-k}a_{k'}\Psi_0, \Psi_0) \\ &\qquad \pm (\Psi_0, a_{-k'}a_k\Psi_0) \pm (a_{-k}\Psi_0, a_{-k'}\Psi_0), \end{aligned} \tag{11}$$

where a term $(\Psi_0, \delta_{kk'}\Psi_0)$ has been omitted because it will give a vanishing contribution when it is multiplied by $\delta^{12}_{\kappa\kappa'}$ and the isotopic spin sums are performed.

The identity

$$[H, a_k] = Ha_k - a_k H = [H_0, a_k] + [H_I, a_k] = -\omega a_k - V_k^\dagger$$

may be written

$$(H+\omega)a_k = -V_k^\dagger + a_k H,$$

and, simply adding $\omega' a_k$ to each side, we also have

$$(H+\omega+\omega')a_k = -V_k^\dagger + a_k(H+\omega').$$

From these identities one gets immediately that

$$a_k\Psi_0 = \frac{-1}{H+\omega} V_k^\dagger \Psi_0,$$

and

$$\begin{aligned} a_k a_{k'}\Psi_0 &= \frac{-1}{H+\omega+\omega'} V_k^\dagger \frac{-1}{H+\omega'} V_{k'}^\dagger \Psi_0 \\ &\quad + \frac{-1}{H+\omega+\omega'} V_{k'}^\dagger \frac{-1}{H+\omega} V_k^\dagger \Psi_0, \end{aligned} \tag{12}$$

where we have taken $H\Psi_0=0$, used the fact that $[a_k, V_k^\dagger]=0$, and made use of the assumption that there are no states of this system with energy less than zero. Substitution of (12) into (11) yields

$$\begin{aligned} &\left(\Psi_0,\ V_{k'} \frac{1}{H+\omega'} \frac{1}{H+\omega} V_k^\dagger \Psi_0\right) \\ &\quad + \left(\Psi_0,\ V_{-k} \frac{1}{H+\omega} V_{k'} \frac{1}{\omega+\omega'} \Psi_0\right) \\ &\quad \pm \left(\Psi_0,\ V_{k'} \frac{1}{H+\omega'} V_{-k} \frac{1}{\omega+\omega'} \Psi_0\right) \\ &\quad \pm \left(\Psi_0,\ \frac{1}{\omega+\omega'} V_k^\dagger \frac{1}{H+\omega'} V_{-k'}^\dagger \Psi_0\right) \\ &\quad \pm \left(\Psi_0,\ \frac{1}{\omega+\omega'} V_{-k'}^\dagger \frac{1}{H+\omega} V_k^\dagger \Psi_0\right) \\ &\qquad \pm \left(\Psi_0,\ V_{-k} \frac{1}{H+\omega} \frac{1}{H+\omega'} V_{-k'}^\dagger \Psi_0\right). \end{aligned}$$

[17] G. F. Chew, Phys. Rev. **95**, 1669 (1954).
[18] M. H. Friedman, Phys. Rev. **97**, 1123 (1955).
[19] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949).

If the complete orthonormal set of "incoming" eigenstates $\Psi_n^{(-)}$ is introduced, the definition $T_k(n) = (\Psi_n^{(-)}, V_k\Psi_0)$ adopted, and the properties

$$(\Psi_0, V_k\Psi_n^{(-)}) = -T_k^\dagger(n) = T_k(n) = -T_{-k}(n) = T_{-k}^\dagger(n)$$

made use of, the above six terms are easily combined to give the following result

$$\sum_n \frac{2\omega+\omega'+E_n \pm (\omega'+E_n)}{(\omega+\omega')(\omega+E_n)(\omega'+E_n)} \times \{T_{k'}^\dagger(n)T_k(n) \pm T_k^\dagger(n)T_{k'}(n)\}.$$

Substitution of this expression into Eqs. (10), and transformation of the $T_{k'}^\dagger(n)T_k(n)$ terms by changing the summation variables ($\mathbf{k}\to -\mathbf{k}'$; $\mathbf{k}'\to -\mathbf{k}$) leads to the following:

$$\mathbf{j}_\pi = ie \sum_{k,k',n} \frac{(\omega+\omega'+E_n)(\mathbf{k}+\mathbf{k}')}{(\omega\omega')^{\frac{1}{2}}(\omega+\omega')(\omega+E_n)(\omega'+E_n)} \times T_k^\dagger(n)T_{k'}(n)\delta^{12}{}_{\kappa\kappa'}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}},$$
$$\rho_\pi = -2ie \sum_{k,k',n} \frac{(\omega\omega')^{\frac{1}{2}}}{(\omega+\omega')(\omega+E_n)(\omega'+E_n)} \times T_k^\dagger(n)T_{k'}(n)\delta^{12}{}_{\kappa\kappa'}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}. \tag{13}$$

These expressions are exact, within the framework of the static theory. To the extent that $T_k(n)$ is known, they determine the meson charge and current.

The complete set of states $\Psi_n^{(-)}$ which we have in mind consists of the four physical nucleon states, denoted Ψ_0, states with the physical nucleon and one meson, and so forth. The contribution of the physical nucleon states is easy to obtain because

$$T_k(0)[= (\Psi_0, V_k\Psi_0)]$$

is determined by invariance arguments. We proceed as follows, recalling that $E_0=0$:

$$\mathbf{j}_\pi{}^0 = ie \sum_{k,k'} \frac{\mathbf{k}+\mathbf{k}'}{(\omega\omega')^{\frac{1}{2}}} \sum_1^4 T_k^\dagger(0)T_{k'}(0)\delta^{12}{}_{\kappa\kappa'}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}},$$
$$\rho_\pi{}^0 = -2ie \sum_{k,k'} \frac{1}{(\omega\omega')^{\frac{1}{2}}(\omega+\omega')} \sum_1^4 T_k^\dagger(0) \times T_{k'}(0)\delta^{12}{}_{\kappa\kappa'}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}. \tag{14}$$

The superscripts 0 in Eqs. (14) denote that this is the contribution of the physical nucleon states. Now

$$T_k{}^{\alpha\beta}(0) = \left(\frac{2\pi}{\omega}\right)^{\frac{1}{2}} \frac{f^0}{\mu} iv(k)(\Psi_0{}^{(\alpha)}, \tau_\kappa\boldsymbol{\sigma}\cdot\mathbf{k}\Psi_0{}^{(\beta)})$$
$$= \left(\frac{2\pi}{\omega}\right)^{\frac{1}{2}} \frac{f}{\mu} iv(k)(u_\alpha, \tau_\kappa\boldsymbol{\sigma}\cdot\mathbf{k}u_\beta),$$

where f is the renormalized unrationalized coupling constant,[3] u_α and u_β are normalized Pauli spinors, and the spin and isotopic spin indices α and β are temporarily unsuppressed. It follows that

$$\sum_1^4 T_k^\dagger(0)T_{k'}(0) = 2\pi\left(\frac{f}{\mu}\right)^2 \frac{v(k)v(k')}{(\omega\omega')^{\frac{1}{2}}} \tau_\kappa\boldsymbol{\sigma}\cdot\mathbf{k}\tau_{\kappa'}\boldsymbol{\sigma}\cdot\mathbf{k}',$$

where the spinors flanking the operators have been dropped, and we shall regard our results as operators in the nucleon spin and isotopic spin variables. If this result is substituted into (14), the sum over the isotopic spin indices done, $\sum_{\mathbf{k},\mathbf{k}'}$ replaced by $(2\pi)^{-6}\int d\mathbf{k}d\mathbf{k}'$, and terms of the integrand that lead to zero on integration dropped, one obtains

$$\mathbf{j}_\pi{}^0 = \frac{-4ie}{(2\pi)^5}\left(\frac{f}{\mu}\right)^2 \tau_3 \int d\mathbf{k}d\mathbf{k}' \frac{v(k)v(k')}{\omega^2\omega'^2} \mathbf{k}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{k}')e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}},$$
$$\rho_\pi{}^0 = \frac{4e}{(2\pi)^5}\left(\frac{f}{\mu}\right)^2 \tau_3 \int d\mathbf{k}d\mathbf{k}' \frac{v(k)v(k')}{\omega\omega'(\omega+\omega')} \mathbf{k}\cdot\mathbf{k}'e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}. \tag{15}$$

These are equal to the expressions given by the lowest order perturbation calculation, except that the renormalized coupling constant appears here. This feature of the static theory, i.e., that the physical nucleon states reproduce the lowest order perturbation result with the renormalized coupling constant, has been noted previously.[4–15]

The contribution of the one-meson states is seen from Eqs. (13) to be

$$\mathbf{j}_\pi{}^1 = ie \sum_{k,k',k''} \frac{(\omega+\omega'+\omega'')(\mathbf{k}+\mathbf{k}')}{(\omega\omega')^{\frac{1}{2}}(\omega+\omega')(\omega+\omega'')(\omega'+\omega'')} \times T_k^\dagger(k'')T_{k'}(k'')\delta^{12}{}_{\kappa\kappa'}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}},$$
$$\rho_\pi{}^1 = -2ie \sum_{k,k',k''} \frac{(\omega\omega')^{\frac{1}{2}}}{(\omega+\omega')(\omega+\omega'')(\omega'+\omega'')} \times T_k^\dagger(k'')T_{k'}(k'')\delta^{12}{}_{\kappa\kappa'}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}.$$

If, following Chew and Low, we introduce the decomposition

$$T_k(k') = \frac{-2\pi v(k)v(k')}{(\omega\omega')^{\frac{1}{2}}} \{P_{11}(k',k)h_1(\omega') + (P_{13}+P_{31})h_2(\omega') + P_{33}h_3(\omega')\},$$

where

$$P_{11}(k',k) = \tfrac{1}{3}\tau_{\kappa'}\tau_\kappa\boldsymbol{\sigma}\cdot\mathbf{k}'\boldsymbol{\sigma}\cdot\mathbf{k},$$
$$P_{13}(k',k) = \tfrac{1}{3}\tau_{\kappa'}\tau_\kappa[3\mathbf{k}'\cdot\mathbf{k} - \boldsymbol{\sigma}\cdot\mathbf{k}'\boldsymbol{\sigma}\cdot\mathbf{k}],$$
$$P_{31}(k',k) = (\delta_{\kappa\kappa'} - \tfrac{1}{3}\tau_{\kappa'}\tau_\kappa)\boldsymbol{\sigma}\cdot\mathbf{k}'\boldsymbol{\sigma}\cdot\mathbf{k},$$
$$P_{33}(k',k) = (\delta_{\kappa\kappa'} - \tfrac{1}{3}\tau_{\kappa'}\tau_k)[3\mathbf{k}'\cdot\mathbf{k} - \boldsymbol{\sigma}\cdot\mathbf{k}'\boldsymbol{\sigma}\cdot\mathbf{k}],$$

then straightforward calculation leads to the following expressions:

$$\mathbf{j}_\pi{}^1=\frac{-8ie}{3(2\pi)^6}\tau_3\int d\mathbf{k}d\mathbf{k}'dk''\frac{k''^4v(k)v(k')v^2(k'')(\omega+\omega'+\omega'')}{\omega\omega'\omega''(\omega+\omega')(\omega+\omega'')(\omega'+\omega'')}$$
$$\times\mathbf{k}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{k}')\{|h_1(\omega'')|^2-2|h_2(\omega'')|^2+|h_3(\omega'')|^2\}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}},\tag{16}$$
$$\rho_\pi{}^1=\frac{8e}{3(2\pi)^6}\tau_3\int d\mathbf{k}d\mathbf{k}'dk''\frac{k''^4v(k)v(k')v^2(k'')}{\omega''(\omega+\omega')(\omega+\omega'')(\omega'+\omega'')}$$
$$\times\mathbf{k}\cdot\mathbf{k}'\{|h_1(\omega'')|^2+|h_2(\omega'')|^2-2|h_3(\omega'')|^2\}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}.$$

It may be noted that the contribution of the (3,3) state to $\rho_\pi{}^1$ has the opposite sign of $\rho_\pi{}^0$, in agreement with the conclusion of Treiman and Sachs that if the major contribution of higher meson states comes from the vicinity of the large (3,3) resonance then this will tend to cancel the contribution of the physical nucleon states to the neutron-electron interaction.

4. EVALUATION OF THE FORM FACTORS AND RESULTS

Equations (7) and (15) determine the meson current contribution of the physical nucleon states to the static limits of the electric and anomalous magnetic moments, ϵ_n and μ_n. If Eqs. (15) are substituted into Eqs. (7), the exponential $\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]$ used to enable the replacements $\mathbf{r}\rightarrow i\boldsymbol{\nabla}_{k'}$ and $r^{2n}\rightarrow(-\nabla^2{}_{k'})^n$ to be made, the differential operators transferred by integrations by parts, the $\delta(\mathbf{k}-\mathbf{k}')$ functions which remain used to perform the $\mathbf{k}'$ integrations, and several identities for the resulting differential operators made use of to perform the angular parts of the $\mathbf{k}$ integrations, then the following results are obtained:

$$\lim\epsilon_n{}^0=\tau_3\frac{(-1)^n4e}{(2n+1)!\pi}\left(\frac{f}{\mu}\right)^2\int_0^\infty dk\frac{k^4v(k)}{\omega}\times\left[\mathrm{d}'^n\frac{v(k')}{\omega'(\omega+\omega')}\right]_{k'=k},$$
$$\lim\mu_n{}^0=\tau_3\frac{(-1)^n8e(n+1)}{(2n+3)!\pi}\left(\frac{f}{\mu}\right)^2\int_0^\infty dk\frac{k^4v(k)}{\omega^2}\mathfrak{D}^n\frac{v(k)}{\omega^2},\tag{17}$$

where

$$\mathrm{d}=\left(\frac{\partial}{\partial k}+\frac{4}{k}\right)\frac{\partial}{\partial k},$$

$$\mathfrak{D}^n=(\nabla^2{}_k)^n+2\sum_{i=1}^n(\nabla^2{}_k)^{n-i}\frac{1}{k^2}\left(\nabla^2{}_k-\frac{6}{k^2}\right)^{i-1}k\frac{d}{dk}.$$

In particular, we find

$$\lim\epsilon_0{}^0=\tau_3\frac{2e}{\pi}\left(\frac{f}{\mu}\right)^2\int_0^\infty dk\frac{k^4}{\omega^3}v^2,\tag{18a}$$

$$\lim\epsilon_1{}^0=\tau_3\frac{e}{3\pi}\left(\frac{f}{\mu}\right)^2\int_0^\infty dk\left\{\frac{k^4}{\omega^3}\left(\frac{dv}{dk}\right)^2+\left(\frac{15k^4}{2\omega^5}-\frac{5k^6}{\omega^7}\right)v^2\right\},\tag{18b}$$

$$\lim\mu_0{}^0=\tau_3\frac{4e}{3\pi}\left(\frac{f}{\mu}\right)^2\int_0^\infty dk\frac{k^4}{\omega^4}v^2,\tag{18c}$$

$$\lim\mu_1{}^0=\tau_3\frac{2e}{3\cdot5\pi}\left(\frac{f}{\mu}\right)^2\int_0^\infty dk\left\{\frac{k^4}{\omega^4}\left(\frac{dv}{dk}\right)^2+\left(\frac{10k^4}{\omega^6}-\frac{8k^6}{\omega^8}\right)v^2\right\}.\tag{18d}$$

Equations (18) contain only charge and current due to the meson cloud, whereas Eqs. (7) show that the electric and anomalous magnetic moments should be calculated using the complete charge and current of the stationary nucleon. To Eq. (18a) must be added the contribution of the nucleon core charge if it is to give the result required by Eq. (7a), namely zero for the neutron and e for the proton. There is however, no compelling *a priori* reason for adding core contributions to Eqs. (18b, c, d). If one assumes, as Treiman and Sachs do, that the core charge has no appreciable extension, then (18b) gives the static theory prediction for the contribution of the physical nucleon states to the neutron electron interaction. With this assumption the first approximation of the static theory leads to severe disagreement with the experimental neutron-electron interaction, as noted by Treiman and Sachs, and Salzman.

If however, the core charge is assumed to be statically spread out in space, for example with a density proportional to the source function

$$s(r)[=(2\pi)^{-3}\int d\mathbf{k}v(k)\exp(i\mathbf{k}\cdot\mathbf{r})],$$

then reasonable agreement can be achieved. We now want to examine the effect of this assumption on the form factors for the proton.

Since no additional current is assumed, the anomalous magnetic moments (18c, d) are unaffected. However, the extended core makes positive contributions to the electric moments for both neutron and proton, and there is a tendency to cancel the meson contribution in the case of the neutron, while adding to it in the case of the proton.

Explicitly, we assume the core charge is given by

$$\rho_c(r)=Ces(r),$$

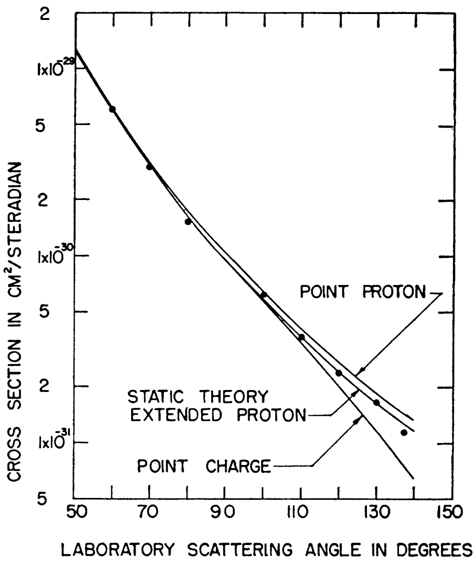

FIG. 1. Elastic scattering of 100-Mev (Lab) electrons from hydrogen. The data are those of McAllister and Hofstadter. The "point charge" curve is a plot of σ_{NS}. The "point proton" curve is a plot of σ with $\mu_0 2M/e=1.79$, and $F_1=F_2=1$. The "static theory extended proton" curve is also a plot of σ, but with $F_1=1-(1/6)(6.5\times10^{-14}\text{ cm})^2q^2$ and $F_2=1-(1/10)(9.7\times10^{-14}\text{ cm})^2q^2$. See Eqs. (3).

where the constant C is determined, for the proton, by

$$\int d\mathbf{r}\rho^N(r)=\int d\mathbf{r}[\rho_c(r)+\rho_\pi(r)]=e.$$

If now $\rho_\pi{}^0$ is used for ρ_π, and the normalization of the source function, $\int d\mathbf{r}s(r)=1$, recalled, then the core contribution to the first electric moment is, by Eq. (7a),

$$\tfrac{1}{6}C^0e[-\nabla_k{}^2v(k)]_{k=0},\quad C^0=1-\lim\epsilon_0{}^0/e. \tag{18e}$$

We then obtain the following approximation to the form factors for the proton:

$$\lim F_1{}^0=1-\left\{\frac{1}{e}\lim\epsilon_1{}^0+\frac{1}{6}\left[1-\frac{1}{e}\lim\epsilon_0{}^0\right]\times[-\nabla_k{}^2v(k)]_{k=0}\right\}q^2+\cdots,$$

$$\lim F_2{}^0=1-\{\lim(\mu_1{}^0/\mu_0{}^0)\}q^2+\cdots.$$

Numerical evaluation of these expressions, using Eqs. (18a-e), with a coupling constant $f^2=0.08$, a cutoff $k_{\max}=5.6\,\mu$ and a cut-off function $v(k)=[1+(k/k_{\max})^2]^{-1}$ leads to the following[20]

$$\begin{aligned}\lim F_1{}^0&=1-\tfrac{1}{6}(6.5\times10^{-14}\text{ cm})^2q^2+\cdots,\\ \lim F_2{}^0&=1-\tfrac{1}{10}(9.7\times10^{-14}\text{ cm})^2q^2+\cdots.\end{aligned} \tag{19}$$

[20] The ratio $\text{Lim}(\mu_1/\mu_0)$ was actually evaluated using the cutoff function $v^2=\exp(-k^2/k_{\max}{}^2)$, but this ratio is highly insensitive to the shape of the cut-off function and the quality of the fit to the experimental cross section is likewise insensitive to this choice.

A rough estimate based on the assumption that $\langle r_\epsilon{}^4\rangle$ and $\langle r_\mu{}^4\rangle$ are not appreciably larger than $\langle r_\epsilon{}^2\rangle$ and $\langle r_\mu{}^2\rangle$,[21] indicates that for the range of momentum transfer considered, inclusion of the q^4 terms can be expected to change the form factors by less than 10%. When account is taken of the fact that for a nonrelativistic proton,

$$q^2=\frac{4E^2\sin^2(\theta/2)}{1+(2E/M)\sin^2(\theta/2)},$$

then Eqs. (3) and (19) determine a cross section, which is shown labeled "static theory extended proton" in Figs. 1–3.

On each of the graphs there are two other curves, labelled "point charge" and "point proton." The point charge curve is the Mott-Rutherford cross section, corrected for recoil of the scatterer, and is given by σ_{NS} [see Eq. (3)]. The point proton curve is for scattering by a relativistic proton with an anomalous magnetic moment of 1.79 nm, but with no spatial extension, and is given by putting $F_1=F_2=1$ in Eq. (3). As indicated by Figs. 1, 2, 3, the low-energy electron proton scattering data are in closer agreement with the static theory extended proton curve than with that of the point proton. The experimental cross section is relative, and the points have in each case been normalized to the static theory extended proton curve at a low value of θ, at which the theoretical curves nearly coincide with each other.

We may then conclude that the first approximation

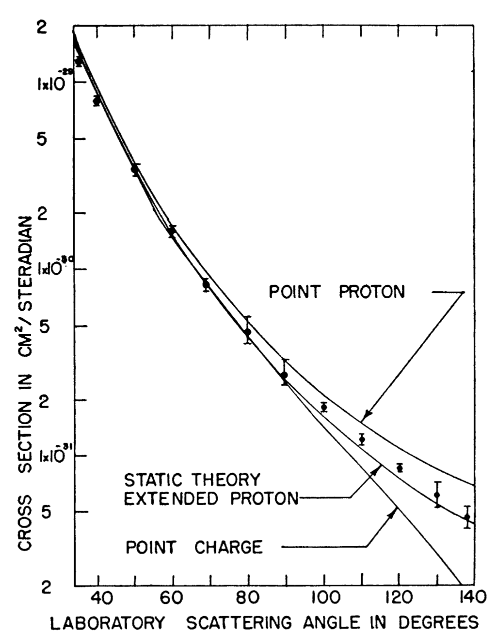

FIG. 2. Elastic scattering of 188-Mev (lab) electrons from hydrogen.

[21] This assumption is reasonable if the charge and current distributions are short-tailed, as is indicated for the charge by Zachariasen (reference 6).

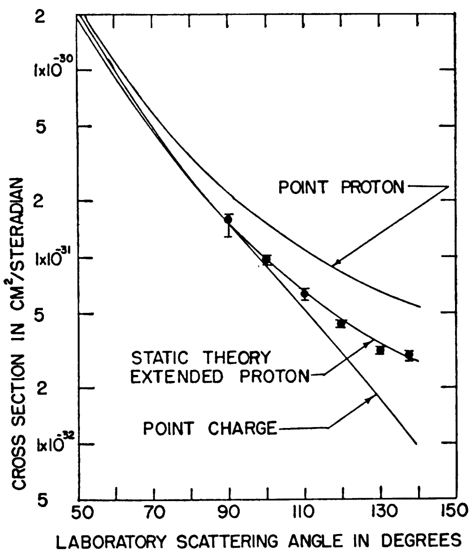

FIG. 3. Elastic scattering of 236-Mev (lab) electrons from hydrogen.

of the static theory, with an extended core charge, is in rough agreement with the neutron-electron interaction and in substantial agreement with the electron-proton scattering up to 236 Mev.[22] This agreement is achieved with the same value of the coupling constant determined by Chew and Low.

The fact that the experimental neutron-electron interaction is almost exactly accounted for by the Foldy term alone[11,23] imposes a severe test on meson theories, namely that for the neutron, the r^2 moment of the charge distribution must vanish. It seems highly unlikely that any theory which describes only the pion charge cloud associated with the neutron, ignoring spatial distribution of the charge on the residual nucleon core, will predict a vanishing r^2 moment. As mentioned, this is the case with the first approximation of the Chew-Low theory, and, as shown by Treiman and Sachs, this difficulty probably persists even if the contributions of higher order meson states are included in this theory.

APPENDIX

By making use of the identity

$$\nabla\cdot[\mathbf{j}\mathbf{r}(\mathbf{q}\cdot\mathbf{r})^k]-(\nabla\cdot\mathbf{j})\mathbf{r}(\mathbf{q}\cdot\mathbf{r})^k = (k+1)(\mathbf{q}\cdot\mathbf{r})^k\mathbf{j}+k(\mathbf{q}\cdot\mathbf{r})^{k-1}\mathbf{q}\times[\mathbf{r}\times\mathbf{j}],$$

and of the properties

$$\nabla\cdot\mathbf{j}=0 \quad\text{and}\quad \int d\mathbf{r}\nabla\cdot[\mathbf{j}\mathbf{r}(\mathbf{q}\cdot\mathbf{r})^k]=0,$$

which hold for the current of a stationary nucleon, we can transform the right hand side of (6b) as follows:

$$\int d\mathbf{r}\mathbf{j}^N(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}=\sum_{k=0}^{\infty}\frac{i^k}{k!}\int d\mathbf{r}\mathbf{j}^N(\mathbf{r})(\mathbf{q}\cdot\mathbf{r})^k$$

$$=\sum_{k=0}^{\infty}\frac{ki^k}{(k+1)!}\int d\mathbf{r}[\mathbf{r}\times\mathbf{j}^N(\mathbf{r})]\times\mathbf{q}(\mathbf{q}\cdot\mathbf{r})^{k-1}.$$

The vector character of $\mathbf{j}^N(\mathbf{r})$ and its vanishing divergence require that

$$\mathbf{j}^N(\mathbf{r})=g(r)\boldsymbol{\sigma}\times\mathbf{r}, \tag{A1}$$

where $g(r)$ is spherically symmetric. Therefore only terms with odd k can contribute in the sum. Multiplication of Eq. (6b) by $(-i/2q^2)\boldsymbol{\sigma}\times\mathbf{q}\cdot$ and use of the results just obtained enable us to write

$$\lim\mu=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+2)!}\mathbf{P}\cdot\int d\mathbf{r}[\mathbf{r}\times\mathbf{j}^N(\mathbf{r})](\mathbf{q}\cdot\mathbf{r})^{2n}, \tag{A2}$$

where

$$\mathbf{P}=\tfrac{1}{2}(\boldsymbol{\sigma}-(\boldsymbol{\sigma}\cdot\mathfrak{q})\mathfrak{q}) \quad\text{and}\quad \mathfrak{q}=\mathbf{q}/q.$$

The spherical symmetry of $\rho^N(r)$ insures that if the exponential in Eq. (6a) is expanded in powers of $i\mathbf{q}\cdot\mathbf{r}$, then only the even powers can contribute to the integral. We therefore obtain

$$\lim\epsilon=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\int d\mathbf{r}\rho^N(r)(\mathbf{q}\cdot\mathbf{r})^{2n}. \tag{A3}$$

From Eqs. (2), it follows that

$$\lim(\epsilon,\mu)=\sum_{n=0}^{\infty}(-1)^nq^{2n}\lim(\epsilon_n,\mu_n).$$

Using this, we obtain from Eqs. (A2) and (A3) the following:

$$\lim\epsilon_n=\frac{1}{(2n)!}\int d\mathbf{r}\rho^N(r)(\mathfrak{q}\cdot\mathbf{r})^{2n}, \tag{A4}$$

$$\lim\mu_n=\frac{2n+1}{(2n+2)!}\mathbf{P}\cdot\int d\mathbf{r}[\mathbf{r}\times\mathbf{j}^N(\mathbf{r})](\mathfrak{q}\cdot\mathbf{r})^{2n}. \tag{A5}$$

If use is made of the spherical symmetry of $\rho^N(r)$ and of Eq. (A1), then Eqs. (A4) and (A5) may be rewritten as Eqs. (7a) and (7b).

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[22] The electron-proton data are fit equally well with a point core as with the extended core used here. For a point core $\langle r_\epsilon^2\rangle$ is reduced from 6.5×10^{-14} cm to 5.5×10^{-14} cm. The effect of the extended core is small for the proton essentially because the core is present only a fraction C^0 of the time, which is 0.3 for the values of f^2 and $k_{\max}$ used.

[23] L. L. Foldy, Phys. Rev. **83**, 688 (1951). The Foldy term accounts for -4.1 kev of the effective interaction energy.