quite likely that similar relationships can be found for particles of higher spin.

APPENDIX A

Here, we will derive Eq. (3.8) from Eq. (3.7). Let $\epsilon_l=\eta_{l+1}-\eta_l$. Then we can write Eq. (3.8) as

$$e^{2i(\delta_l-\eta_l)}=\tfrac{1}{2}(1+e^{2i\epsilon_l}). \quad \text{(A.1)}$$

Now $\epsilon_l\to 0$ when $k\to\infty$, probably like V/k. To see this, we must note that[3] $\eta_0\simeq Va$ when $k\to\infty$, and the number of η_l which are appreciably different from zero is of the order of ka. Thus ϵ_l is probably of the order of $(Va/ka)=V/k$. It follows from Eq. (A.1) that $\delta_l-\eta_l$ is small, of the same order as ϵ_l. If we expand the exponentials and drop the square terms, we get

$$\delta_l=\tfrac{1}{2}(\eta_l+\eta_{l+1}),$$

with an error of the order of ϵ_l^2 which is probably of the order $(V/k)^2$. The exact power of $1/k$ involved in the error here is not entirely certain, but that is not essential in our derivation, as the error in our cross-section relationship is already uncertain for other reasons.

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Adiabatic Approximation for Scattering Processes*†

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The adiabatic approximation, frequently utilized in treating bound states of quantum-mechanical systems, is shown to be applicable also to scattering processes in cases for which the number of excited states of the target contributing significantly to the wave function is limited, and for which the time of traversal of the scattered particle through the region of interaction is small compared with the period of the target motion excitable in the collision. In this approximation, calculation of partial inelastic cross sections is reduced to two steps: (1) determination of the elastic scattering amplitude for fixed target coordinates and (2) subsequent evaluation of scalar products in the space of the target. The method is illustrated by means of a simple one-dimensional example. Calculation of cross sections for nuclear rotational excitation by neutrons is formulated.

1. INTRODUCTION

THE use of the adiabatic approximation in treating bound states of quantum-mechanical systems in which the characteristic periods of certain different degrees of freedom are highly disparate has long been well known.[1,2] The adiabatic point of view has been of great value in molecular physics[3] and, with the development of the collective or unified model, also in nuclear physics.[4] It does not appear to be generally known, however, that the adiabatic approximation is applicable also to scattering processes in some cases where the time required for the scattered particle to cross the region of interaction is small compared with the period of the target motion which may be excited in the collision. The approximation reduces the problem of obtaining the various partial cross sections for inelastic and elastic scattering to one of finding the elastic scattering amplitude as a function of the target coordinates, regarded as fixed, and subsequently evaluating appropriate scalar products in the space of the target.

In Sec. 2 below is given a formal derivation of the adiabatic approximation for scattering amplitudes with a discussion of its limitations. In Sec. 3, by way of example, this method is formulated for a one-dimensional scattering problem which will be treated more fully in a subsequent paper.[2] In Sec. 4 the calculation of cross sections for rotational excitation of strongly deformed nuclei by neutron impact is formulated in the adiabatic approximation.

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† This paper is based on part of the doctoral dissertation of the author [Princeton University, 1955 (unpublished)].

‡ Holder during much of this work of National Science Foundation Predoctoral and Peter B. Putnam, Jr., Fellowships, which are gratefully acknowledged.

[1] M. Born and J. R. Oppenheimer, Ann. Physik **84**, 457 (1927). The term "adiabatic approximation" is to be understood here as that approximation which (1) regards certain relatively slowly-varying coordinates of the system as fixed for the purpose of defining a "partial adiabatic wave function" for the remaining degrees of freedom which depends parametrically upon the slowly-varying coordinates, then (2) assumes as an approximate wave function for the entire system (which may be called the "complete adiabatic wave function") the product of the partial adiabatic wave function and a function of the slowly-varying coordinates only. The latter function is determined from the Schrödinger equation for the system by application of the Ritz variational principle with the parametric dependence of the partial adiabatic wave function neglected.[2]

[2] David M. Chase (to be published). This paper will be referred to hereafter as B.

[3] See, for example, G. Herzberg, *Spectra of Diatomic Molecules* (D. Van Nostrand Company, Inc., New York, 1950).

[4] A. Bohr, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **26**, No. 14 (1952) and, with B. R. Mottelson, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **27**, No. 16 (1953); D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).

2. DERIVATION OF THE ADIABATIC APPROXIMATION FOR TRANSITION AMPLITUDES

The Hamiltonian for the system of incident particle and target is assumed to be of the form

$$H=T_p+V(\mathbf{r},\xi)+\lambda H_T(\xi), \tag{1}$$

in which T_p is the kinetic energy operator for the relative motion of particle and target, $V(\mathbf{r},\xi)$ is an interaction potential, and $\lambda H_T(\xi)$ is the target Hamiltonian, with λ a perturbation parameter to be regarded as small.[5] The relative coordinates of the particle and the internal coordinates of the target are denoted, respectively, by $\mathbf{r}[=(r,\theta,\phi)]$ and ξ. H_T defines a series of eigenenergies and wave functions for the target:

$$H_T\Psi_\Gamma(\xi)=\epsilon_\Gamma\Psi_\Gamma(\xi), \tag{2}$$

where Γ is regarded as specifying the eigenvalues of a complete set of commuting observables including H_T.

A partial adiabatic wave function corresponding to a particle of energy $\mathcal{E}$ with incident wave vector $\mathbf{k}$ is defined by

$$[T_p+V(\mathbf{r},\xi)]w_\mathbf{k}^{(+)}(\mathbf{r},\xi)=\mathcal{E}w_\mathbf{k}^{(+)}(\mathbf{r},\xi), \tag{3}$$

just as in the case of a bound state except that here, of course, $\mathcal{E}$ is independent of ξ. The approximate, adiabatic wave function, $\chi_{\mathbf{k}\Gamma}^{(+)}(\mathbf{r},\xi)$, for the process initiated by the incident particle with wave vector $\mathbf{k}$ in channel Γ is then given by

$$\chi_{\mathbf{k}\Gamma}^{(+)}(\mathbf{r},\xi)=w_\mathbf{k}^{(+)}(\mathbf{r},\xi)\Psi_\Gamma(\xi). \tag{4}$$

The asymptotic form of $w_\mathbf{k}^{(+)}$ defines a ξ-modulated *adiabatic elastic scattering amplitude* $f(\theta,\phi,\xi)$:

$$w_\mathbf{k}^{(+)}(\mathbf{r},\xi)\underset{r\to\infty}{\longrightarrow}e^{i\mathbf{k}\cdot\mathbf{r}}+f(\theta,\phi,\xi)r^{-1}e^{ikr}.$$

If $\psi_{\mathbf{k}\Gamma}^{(+)}(\mathbf{r},\xi)$ is the exact wave function for the process, therefore satisfying $H\psi_{\mathbf{k}\Gamma}^{(+)}=(\mathcal{E}+\lambda\epsilon_\Gamma)\psi_{\mathbf{k}\Gamma}^{(+)}$, then the amplitude for scattering from channel Γ to Γ', $f_{\Gamma'\Gamma}(\theta,\phi)$, is given exactly by[6]

$$f_{\Gamma'\Gamma}=(m/2\pi\hbar^2)\langle\phi_{\mathbf{k}'\Gamma'}|\,V\,|\psi_{\mathbf{k}\Gamma}^{(+)}\rangle, \tag{5}$$

where $\phi_{\mathbf{k}\Gamma}(\mathbf{r},\xi)\equiv e^{i\mathbf{k}\cdot\mathbf{r}}\Psi_\Gamma(\xi)$ in the $\mathbf{r},\xi$ representation and conservation of energy requires $\mathcal{E}'=\mathcal{E}+\lambda(\epsilon_\Gamma-\epsilon_{\Gamma'})$. The corresponding differential cross section is then

$$d\sigma_{\Gamma'\Gamma}=k_{\Gamma'}k_\Gamma^{-1}|f_{\Gamma'\Gamma}(\theta,\phi)|^2d\Omega,$$

where k_Γ and $k_{\Gamma'}$ are the channel wave numbers denoted also by k and k'.

If one sets

$$\psi_{\mathbf{k}\Gamma}^{(+)}=\chi_{\mathbf{k}\Gamma}^{(+)}+\lambda\Delta_{\mathbf{k}\Gamma}^{(+)}, \tag{6}$$

then the correction $\Delta_{\mathbf{k}\Gamma}^{(+)}$ satisfies the equation

$$(\mathcal{E}+\lambda\epsilon_\Gamma-H)\Delta_{\mathbf{k}\Gamma}^{(+)}=H_T(w_\mathbf{k}^{(+)}\Psi_\Gamma)-w_\mathbf{k}^{(+)}H_T\Psi_\Gamma. \tag{7}$$

[5] In the present derivation the incident particle is considered to possess no internal degrees of freedom and in particular, for simplicity, no spin.

[6] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

The inhomogeneous source function which constitutes the right member of Eq. (7) will be designated by $U_{\mathbf{k}\Gamma}(\mathbf{r},\xi)$. Since the $\Psi_\Gamma(\xi)$ form a complete set of functions with respect to ξ, $\Delta_{\mathbf{k}\Gamma}^{(+)}$ can be expanded as

$$\Delta_{\mathbf{k}\Gamma}^{(+)}(\mathbf{r},\xi)=\textstyle\sum_{\Gamma'}B_{\Gamma\Gamma'}(\mathbf{r})\Psi_{\Gamma'}(\xi), \tag{8}$$

in which an index $\mathbf{k}$ on $B_{\Gamma\Gamma'}$ is suppressed. Substitution of (8) and the additional expansions

$$V(\mathbf{r},\xi)\Psi_{\Gamma'}(\xi)=\textstyle\sum_{\Gamma''}v_{\Gamma'\Gamma''}(\mathbf{r})\Psi_{\Gamma''}(\xi),$$
$$U_{\mathbf{k}\Gamma}(\mathbf{r},\xi)=\textstyle\sum_{\Gamma'}u_{\Gamma\Gamma'}(\mathbf{r})\Psi_{\Gamma'}(\xi),$$

into Eq. (7) and use of (1) and (2) yields the following set of coupled differential equations for the $B_{\Gamma\Gamma'}$:

$$[\mathcal{E}+\lambda(\epsilon_\Gamma-\epsilon_{\Gamma'})-T_p]B_{\Gamma\Gamma'}(\mathbf{r})-\textstyle\sum_{\Gamma''}v_{\Gamma''\Gamma'}(\mathbf{r})B_{\Gamma\Gamma''}(\mathbf{r})=u_{\Gamma\Gamma'}(\mathbf{r}). \tag{9}$$

Equations (9) show clearly that the $B_{\Gamma\Gamma'}$ are of order zero in λ. It does not follow, however, that $\Delta_{\mathbf{k}\Gamma}^{(+)}$ is of order zero, because the sum in (8) contains an infinite number of terms. In fact, if it is necessary to include in that sum all component states up to some Γ' for which $\lambda\epsilon_{\Gamma'}$ differs from $\lambda\epsilon_\Gamma$ by a fixed energy, sensibly independent of λ, then the number of terms in the sum will be of order λ^{-1}, and the correction term $\lambda\Delta_{\mathbf{k}\Gamma}^{(+)}$ in (6) of order λ^0. In this event the true values of the quantities in which we are interested will not, as $\lambda\to0$, approach the corresponding values calculated from the adiabatic wave function $\chi_{\mathbf{k}\Gamma}^{(+)}$. However, there do exist problems such that the number of terms which need be included in (8) as $\lambda\to0$ is not the number falling within a fixed energy interval, but rather some fixed, finite number, effectively independent of λ. This will be the case, for example, if the orbital angular momentum required to excite a state Γ' increases indefinitely as the quantum number of excitation (included in the specification Γ') increases indefinitely (see Sec. 4). If consideration is restricted to such cases, $\Delta_{\mathbf{k}\Gamma}^{(+)}$ can be regarded as of order λ^0. From (5) and (6) one then obtains

$$f_{\Gamma'\Gamma}(\theta,\phi)=\frac{m}{2\pi\hbar^2}\langle\phi_{\mathbf{k}'\Gamma'}|\,V\,|\chi_{\mathbf{k}\Gamma}^{(+)}\rangle+O_1(\lambda) \tag{10}$$

$$=\frac{m}{2\pi\hbar^2}\int d\xi\Psi_{\Gamma'}^*(\xi)\times\left[\int d\mathbf{r}e^{-i\mathbf{k}'\cdot\mathbf{r}}V(\mathbf{r},\xi)w_\mathbf{k}^{(+)}(\mathbf{r},\xi)\right]\Psi_\Gamma(\xi)+O_1(\lambda)$$

$$=\int d\xi\Psi_{\Gamma'}^*(\xi)f(\theta,\phi,\xi)\Psi_\Gamma(\xi)+O_2(\lambda), \tag{11}$$

where $\int d\xi$ denotes a scalar product in the space of the coordinates ξ and $O(\lambda)$ a quantity of the order of λ. The adiabatic approximation consists in neglecting the term $O_2(\lambda)$ in (11).

In the passage from (10) to (11) k_Γ and $k_{\Gamma'}$, which differ by $O(\lambda)$, have been set equal in the first term;

hence a requirement for validity of the approximation is that

$$(\Delta k)R=\frac{\lambda(\epsilon_{\Gamma'}-\epsilon_\Gamma)}{D_0(k_{\Gamma'}+k_\Gamma)R}\ll 1, \tag{12}$$

where $D_0=\hbar^2/2mR^2$ and R is an effective radius of the region of interaction. This condition, for processes involving exchange of only a small number of quanta with the target, is equivalent to the condition that the average level spacing of the target be much less than the single-particle width at the average (of incident and emergent) particle energy or, classically stated, that the period of the target motion be much greater than the time required for the particle to cross the region of interaction at its average outside velocity. When $k_{\Gamma'}R\ll 1$, a more stringent requirement than (12), namely,

$$\frac{(\Delta k)R}{k_{\Gamma'}R}=\frac{\lambda(\epsilon_{\Gamma'}-\epsilon_\Gamma)}{D_0(k_{\Gamma'}+k_\Gamma)R\cdot k_{\Gamma'}R}\ll 1,$$

must be satisfied for the approximation to be valid. Precisely at threshold ($k_{\Gamma'}R=0$), therefore, the approximation fails; in particular, if some kind of average of k_Γ and $k_{\Gamma'}$ is used in calculating $f(\theta,\phi,\xi)$, then the cross section calculated from (11) does not vanish as it should when $k_{\Gamma'}R\to 0$. That calculated from (10) [with omission of $O_1(\lambda)$], however, does vanish as $k_{\Gamma'}R\to 0$, provided that the true $k_{\Gamma'}$ is used in $\phi_{k'\Gamma'}$.

In certain problems may occur resonances due to the particle-target coupling and corresponding to temporary capture of the particle into a quasi-bound state by the target. Such resonances evidently will not be given by the adiabatic method, and to the extent that these resonances remain important for small λ the condition on the number of terms which must be retained in the sum in (8), a condition for validity of the adiabatic approximation, is necessarily violated; a resonance through an intermediate state in which the particle is captured into a quasi-bound state of (negative) particle-energy $\simeq -B$, say, can appear only via terms Γ' in (8) for which $\lambda(\epsilon_{\Gamma'}-\epsilon_\Gamma)\gtrsim \mathcal{E}+B$, an energy difference independent of λ.

3. FORMULATION FOR ONE-DIMENSIONAL MODEL FOR SCATTERING OF NUCLEONS BY A NUCLEAR WALL

For the purpose of fixing ideas, the adiabatic formulation will be given here for a simple idealized example, even though the restriction on the sum in (8) required for validity of the approximation is unjustified in the example to be considered. The present problem will be studied in another context in B.

The one-dimensional system of interacting "nucleon" and dynamic "nuclear" wall is defined as follows. A particle is scattered by a potential well whose wall is variable in position and bound by a quantum-mechanical harmonic-oscillator potential. The degrees of freedom are the position coordinate, r, of the particle and the "deformation" coordinate, α, of the wall, defined by $R=R_0(1+\alpha)$, where R is the position coordinate of the wall and R_0 its equilibrium value in the absence of interaction. For simplicity the nucleonic potential is taken to be $-V_0$ for $r<R$ and zero for $r>R$. An infinite wall is supposed to be placed at $r=0$, so that the wave function vanishes there and need be defined only for $r\geq 0$. The Hamiltonian is written

$$\begin{aligned} H&=T_p+V(r,\alpha)+H_T(\alpha),\\ T_P&=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2},\\ H_T(\alpha)&=-\frac{\hbar^2}{2\mathcal{I}}\frac{\partial^2}{\partial\alpha^2}+\frac{1}{2}C\alpha^2,\\ V(r,\alpha)&=-V_0\int_0^{R_0(1+\alpha)}\delta(r-\rho)d\rho, \end{aligned} \tag{13}$$

where m and $\mathcal{I}$ are, respectively, the mass of the particle and the product of the mass of the wall by R_0^2, and C is the stiffness constant for the wall vibration. The normalized harmonic-oscillator eigenfunctions of H_T are denoted by $h_t(\alpha)$ and the corresponding eigenvalues by $\epsilon_t=(t+\frac{1}{2})\hbar\omega$ $(t=0,1,2,\cdots)$, where $\omega=(C/\mathcal{I})^{\frac{1}{2}}$. Thus the $h_t(\alpha)$ are to be identified with the $\Psi_\Gamma(\xi)$ of Sec. 2. The adiabatic wave function $\chi_{kt}^{(+)}(r,\alpha)$ for the process initiated by the particle with energy $\mathcal{E}$ incident on the wall in state t may be written explicitly as

$$\chi_{kt}^{(+)}(r,\alpha)=w_k^{(+)}(r,\alpha)h_t(\alpha),$$

with $w_k^{(+)}$, the partial adiabatic wave function, given by

$$\begin{aligned} w_k^{(+)}(r,\alpha)&=\sqrt{2}\frac{k}{K}e^{i\zeta}\left(\cos^2KR+\frac{k^2}{K^2}\sin^2KR\right)^{-\frac{1}{2}}\sin Kr \qquad (r\leq R),\\ &=\sqrt{2}e^{i\zeta}\sin(kr+\zeta), \qquad (r\geq R), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \zeta&=\zeta(\alpha)=-kR+\tan^{-1}\left(\frac{k}{K}\tan KR\right),\\ R&=R(\alpha)=R_0(1+\alpha), \end{aligned}$$

and K is the inside wave number: $K^2=k^2+(V_0/D_0)$ with $D_0=\hbar^2/2mR_0^2$, $w_k^{(+)}$ being normalized to one particle per unit distance in r. It is the phase shift ζ rather than the energy $\mathcal{E}$ which depends on α for adiabatic wave functions in the particle continuum. For $r>R$, $\chi_{kt}^{(+)}$ may also be written

$$\chi_{kt}^{(+)}(r,\alpha)=\phi_{kt}(r,\alpha)+\sqrt{2}e^{i\zeta(\alpha)}\sin\zeta(\alpha)e^{ikr}h_t(\alpha),$$

where $\phi_{kt}(r,\alpha)=\sqrt{2}\sin kr\cdot h_t(\alpha)$, the incident wave. The limit in which we are interested may be expressed in terms of parameters of the wall Hamiltonian: $\lambda\to 0$,

$\mathcal{I}=\lambda^{-1}\mathcal{I}_0$, $C=\lambda C_0$, $\hbar\omega=\lambda\hbar\omega_0$, the $\mathcal{I}_0$, C_0, $\hbar\omega_0$, being fixed and finite. The zero-point oscillation amplitude of the wall, $(\hbar\omega/2C)^{\frac{1}{2}}$, remains constant during the limiting process.

Corresponding to Eq. (11), the adiabatic amplitude for the process $t\rightarrow t'$ is

$$f_{t't}=2\int_{-\infty}^{\infty}d\alpha h_{t'}{}^{*}(\alpha)e^{i\zeta(\alpha)}\sin\zeta(\alpha)h_t(\alpha). \qquad (15)$$

In terms of $f_{t't}$, the probability for the process (number of transitions $t\rightarrow t'$ per unit time÷number of particles incident per unit time) is $\pi_{t't}=|f_{t't}|^2$. [If $e^{i\zeta}\sin\zeta$ is expanded to first degree in α, the result for $\pi_{t't}$ is identical with that based on a first-order expansion of the original interaction (13),

$$-V_0\int_0^{R_0}\delta(r-\rho)d\rho-V_0R_0\alpha\delta(r-R_0), \qquad (16)$$

when the second term in (16) is treated to first order in a distorted-wave Born approximation and k' is set equal to k.]

Numerical calculations of the probabilities $\pi_{t't}$ by another method (to be given in B) show that resonance structure associated with formation of quasi-bound intermediate states dominates the energy dependence of these quantities, except when the coupling is very weak and the resonances consequently very narrow. Furthermore, there appears no reason to believe that the results calculated from the adiabatic approximation (15) are more nearly correct than those obtained from a simple distorted-wave Born approximation. As already stated, therefore, the above formulation for this example is intended only to illustrate the concepts of the adiabatic method.

4. FORMULATION FOR NUCLEAR ROTATIONAL EXCITATION BY NEUTRON SCATTERING

It is reasonable to apply the adiabatic treatment to the inelastic scattering of neutrons with excitation of rotational levels in strongly deformed target nuclei. This process is visualized as occurring by way of the interaction of the neutrons with the deformed potential well presented by such nuclei according to the unified model.[3] The adiabatic formulation and a brief discussion of this problem will be given here.

The target nucleus is regarded as describable by the usual strong-coupling wave function of the unified model[3] and is assumed to be axially symmetric. It is assumed that the coupling of the incident particle with the intrinsic modes of excitation of the target at moderate energies may be neglected: only the collective rotational excitations are considered. Then, apart from symmetrization, the target wave function is just $D^I{}_{MK}(\theta_i)$, θ_i being the Euler angles of the principal axes of the nucleus relative to space-fixed axes. The $D^I{}_{MK}(\theta_i)$ are therefore to be identified with the $\Psi_\Gamma(\xi)$ of Sec. 2. The adiabatic elastic scattering amplitude, $f(\theta,\phi,\theta_i)$, that appears in Eq. (11) is the amplitude for scattering by the deformed potential well (perhaps complex) with orientation θ_i relative to the direction of the incident beam.[7]

An incident wave e^{ikz} can be written in terms of coordinates (r,θ',ϕ') measured relative to the principal axes as

$$e^{ikz}=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}a_le^{ik\theta_3}\tilde{D}^l{}_{0k}(\theta_1)j_l(kr)Y_{lk}(\theta',\phi'),$$

where

$$a_l\equiv 2\pi^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}i^l,\quad \tilde{D}^l{}_{mm'}(\theta_1)\equiv e^{-i(m\theta_2+m'\theta_3)}D^l{}_{mm'}(\theta_i),$$

and use is made of the relation

$$Y_{lm}(\theta,\phi)=\sum_{m'=-l}^{l}D^l{}_{mm'}(\theta_i)Y_{lm'}(\theta',\phi'). \qquad (17)$$

Correspondingly, outside the maximum radius of interaction the adiabatic wave function for the scattered particle may be written

$$w_k{}^{(+)}(\mathbf{r},\theta_i)=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}e^{ik\theta_3}[a_l\tilde{D}^l{}_{0k}(\theta_1)j_l(kr)-id_{lk}(\theta_1)h_l{}^{(1)}(kr)]Y_{lk}(\theta',\phi'), \qquad (18)$$

where $h_l{}^{(1)}$ is an *outgoing* spherical Hankel function. The orientation-dependent coefficients $d_{lk}(\theta_1)$ are obtained from the solution of the elastic scattering problem most naturally in the form

$$d_{lk}(\theta_1)=\sum_{l'=|k|}^{\infty}A_{lk}{}^{l'}\tilde{D}^{l'}{}_{0k}(\theta_1). \qquad (19)$$

The $A_{lk}{}^{l'}$ depend only upon the incident energy and the form and spatial shape of the potential well. They vanish when $l-l'$ is odd. These important coefficients may be calculated by integrating through the region of interaction the set of coupled differential equations for the radial components in the expansion of the wave function in terms of spherical harmonics relative to the nuclear axes. From (19) one can rewrite (18):

$$w_k{}^{(+)}(\mathbf{r},\theta_i)=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\sum_{l'=|k|}^{\infty}e^{ik\theta_3}[a_l\delta_{ll'}j_l(kr)-iA_{lk}{}^{l'}h_l{}^{(1)}(kr)]\tilde{D}^{l'}{}_{0k}(\theta_1)Y_{lk}(\theta',\phi').$$

Transforming back to space-fixed axes by the relation inverse to (17), one then obtains for the scattering amplitude

$$f(\theta,\theta_1,\phi-\theta_2)=-k^{-1}\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\sum_{m=-l}^{l}\sum_{l'=|k|}^{\infty}\times i^{-l}A_{lk}{}^{l'}\tilde{D}^l{}_{mk}{}^{*}(\theta_1)\tilde{D}^{l'}{}_{0k}(\theta_1)Y_{lm}(\theta,\phi)e^{-im\theta_2}.$$

[7] Because of the assumed axial symmetry $f(\theta,\phi,\theta_i)=f(\theta,\theta_1,\phi-\theta_2)$, where $(\theta_1,\theta_2,\theta_3)=(\theta,\phi,\psi)$ in the notation, for example, of H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, Cambridge, 1950). (θ_1, in particular, is the angle between the symmetry axis and the direction of the incident beam.)

The amplitude (11) for the transition $I, M \to I', M'$ within a rotational band characterized by a projection K of angular momentum along the nuclear symmetry axis becomes

$$f^K{}_{I'M';IM}(\theta,\phi) = -k^{-1}(2I'+1)^{\frac{1}{2}}(2I+1)^{\frac{1}{2}} \times \sum_{l'=|M'-M|}^{\infty} \sum_{k=-l'}^{l'} \sum_{L=|I'-l'|}^{I'+l'} \sum_{l=|k|}^{\infty} \times i^{-l'}(2L+1)^{-1}(I'l'M', M-M'|LM) \times (I'l'Kk|L, K+k)(IlM0|LM) \times (IlKk|L, K+k)A_{l'k}{}^{l} \cdot Y_{l', M-M'}(\theta,\phi), \quad (20)$$

where $(ab\alpha\beta|c\gamma)$ denotes a Clebsch-Gordan coefficient $[(ab\alpha\beta|abc\gamma)$ in the notation of Condon and Shortley$]$. In obtaining (20), the integration formula[8]

$$\int D^{L}{}_{MK}{}^{*}(\theta_i)D^{l_1}{}_{m_1k_1}(\theta_i)D^{l_2}{}_{m_2k_2}(\theta_i)\sin\theta_1 d\theta_2 d\theta_3 = \frac{2(2\pi)^2}{2L+1}(l_1l_2m_1m_2|LM)(l_1l_2k_1k_2|LK)$$

was used. Finally, the differential cross section for the process $I \to I'$ with an unpolarized beam is

$$d\sigma^K{}_{I'I}(\theta) = (2I+1)^{-1} \sum_{M=-I}^{I} \sum_{M'=I'}^{I'} \times |f^K{}_{I'M';IM}(\theta,\phi)|^2 d\Omega. \quad (21)$$

Equations (20) and (21) express the desired cross sections directly and algebraically in terms of the elastic scattering coefficients $A_{l'k}{}^{l}$.

One will expect that the lowest few rotationally excited states all contribute significantly to the wave function for the scattering, due to the relatively strong coupling between the rotational motion and the incident wave for the large deformations envisaged. However, one will also expect that the contributions tail off rapidly with increasing spin (and hence excitation) when the corresponding lowest requisite neutron orbital angular momentum is such that the classically computed distance of nearest approach at the energy in question is approximately equal to the (maximum) nuclear radius. Further, the lower rotational levels are relatively closely spaced. These circumstances, in accord with the discussion of Sec. 2, indicate the adequacy and suggest the appropriateness of the adiabatic approximation in this problem. Its advantage consists in the greater simplicity of the sets of coupled differential and algebraic equations which arise in the problem of elastic scattering by a deformed well relative to those which arise in the problem obtained without use of the adiabatic approximation. In this latter (nonadiabatic) formulation one must consider the nucleonic and rotational motions simultaneously. A disadvantage of the adiabatic approximation is that it cannot be carried through in a representation in which the total angular momentum and its z projection are diagonal; the number of channels requiring separate treatment is correspondingly larger.

In connection with the adiabatic approximation, one may consider the semiclassical limit, in which the parameters ξ of Sec. 2, following classical equations of motion, modulate the scattering amplitude $f(\theta,\phi,\xi(t))$. Such a description may be appropriate when, in addition to the conditions for validity of the adiabatic approximation, is satisfied also the condition that the motion of the target be only negligibly affected by the transition induced by the scattering, a condition which requires, in particular, that $|\epsilon_{\Gamma'}-\epsilon_{\Gamma}| \ll \epsilon_{\Gamma}$ (if $\epsilon_{\Gamma_0}=0$ for the lowest level Γ_0). This modulation is the same in principle as that involved in the description of the Raman effect at the lowest level of approximation.[3] There the amplitude, f, of the scattered light depends upon the molecular orientation, ξ; this orientation is calculated from the classical equation of motion for the rotation. In the present problem of neutron-induced nuclear rotational Raman effect, as it may be called, the expression given for the partial cross sections in terms of Clebsch-Gordan coefficients must likewise reduce in the semiclassical limit to a simple form clearly related to the idea of modulation. To effect this reduction directly one must derive a suitable limiting formula for the Clebsch-Gordan coefficients, an interesting project in itself. However, one can by-pass this analysis and write down the semiclassical result at once. For this purpose one has only to Fourier analyze the scattering amplitude with respect to the time-dependence of the angles that specify the nuclear orientation.

A numerical calculation of neutron-induced nuclear rotational excitation for targets of spin zero is now being carried out in the nonadiabatic formulation with cutoff at a finite maximum spin ("Tamm-Dancoff approximation") by Dr. L. Wilets and the author.

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[8] E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Friedrich Vieweg and Son, Braunschweig, 1931).