Amplitude Dependence of the Velocity of Second Sound*†

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The amplitude dependence of the velocity of second sound in liquid helium was measured as a function of temperature between 0.936°K and 2.045°K. The results are in agreement with the theoretical predications of Temperley and Khalatnikov. The possibility that amplitude effects are the cause of the discrepancies in the various published values of the velocity of second sound is discussed. An explanation is offered for the anomalous dependence, observed by Osborne, of the velocity of large amplitude signals on the distance between the heater and the bolometer.

INTRODUCTION

Of all the unusual properties displayed by liquid helium below the λ point (2.18°K), perhaps the most striking is the ability of helium II to conduct heat in the form of a wave motion rather than the usual diffusion and convection current processes.

Second sound was first detected in 1944.[1] Since then, measurements of the amplitude-independent velocity of second sound have been carried out over a temperature range from 2.18°K down to about 0.01°K.[2–9] However, the only experimental investigation of the amplitude dependence of the velocity of second sound was a qualitative one made by Osborne in 1951.[10]

The amplitude-independent velocity measurements give a method of checking the two-fluid hydrodynamical equations for helium II up to terms linear in v_n and v_s, where v_n and v_s are the normal fluid and superfluid velocities, respectively. Thus, there is now general agreement on the first order hydrodynamical equations for helium II. However, as has been pointed out,[11,12] there is no such agreement on the terms quadratic in v_n and v_s. When these terms are considered in the derivation of the velocity of second sound, an amplitude-dependent velocity results. The amplitude dependence of the velocity of second sound has been derived by Temperley[13] and Khalatnikov.[14] It is the purpose of this experiment to check their theoretical predictions experimentally.

Temperley's and Khalatnikov's results may be presented in the form of a dimensionless coefficient $\tau_2 \equiv (c_2 - c_{20})/v_n$, where c_{20} is the zero amplitude value for the velocity of second sound, c_2 is the velocity at a point on the profile of a second sound signal, and v_n is the normal fluid velocity at that point.

Temperley obtained the expression $\tau_2 = 2-(\rho/\rho_s)$, where ρ = total fluid density and ρ_s = superfluid density. Temperley deliberately omitted the term due to the change in c_{20} with temperature $[(\partial c_{20}/\partial T)\Delta T$, where ΔT temperature above the ambient of a point on the profile of a second sound signal$]$ because he felt the term $2-(\rho/\rho_s)$ was much more important. However, for the sake of completeness, the term due to $(\partial c_{20}/\partial T)\Delta T$ is added to Temperley's result. Thus, indicating Temperley's τ_2 by τ_{2T}, we have

$$\tau_{2T} = 2 - \frac{\rho}{\rho_s} + \frac{ST}{C}\frac{\partial}{\partial T}\ln c_{20},$$

where

$$\left[\frac{ST}{C}\frac{\partial}{\partial T}\ln c_{20}\right] v_n = \frac{\partial c_{20}}{\partial T}\Delta T,$$

S = entropy and C = specific heat.

For τ_2, Khalatnikov gives

$$\tau_{2K} = \frac{ST}{C}\frac{\partial}{\partial T}\ln\left(c_{20}^3\frac{C}{T}\right),$$

where τ_{2K} refers to Khalatnikov's expression for τ_2.

Temperley assumed for the entropy, $S = S_\lambda(\rho_n/\rho)$, an expression experimentally valid above 1.4°K and a good approximation above 1.0°K. When this expression for S is substituted into Khalatnikov's equation for τ_2, τ_{2K} reduces to τ_{2T}. Below 1.4°K τ_{2K} and τ_{2T} slowly diverge. For example,

$$\text{at } 1.4°\text{K},\ 2-\frac{\rho}{\rho_s} = \frac{ST}{C}\frac{\partial}{\partial T}\ln\left(c_{20}^2\frac{C}{T}\right) = 0.90;$$

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[1] V. Peshkov, J. Phys. U.S.S.R. **8**, 381 (1944).
[2] Lane, Fairbank, and Fairbank, Phys. Rev. **71**, 600 (1947).
[3] D. V. Osborne, Nature **162**, 213 (1948).
[4] R. D. Maurer and M. A. Herlin, Phys. Rev. **76**, 948 (1949).
[5] J. R. Pellam, Phys. Rev. **75**, 1183 (1949).
[6] J. R. Pellam and S. B. Scott, Phys. Rev. **76**, 869 (1949).
[7] K. R. Atkins and D. V. Osborne, Phil. Mag. **41**, 1078 (1950).
[8] V. Peshkov, J. Exptl. Theoret. Phys. (U.S.S.R.) **18**, 867, 951 (1948); **19**, 270 (1949); **23**, 687 (1952).
[9] deKlerk, Hudson, and Pellam, Phys. Rev. **93**, 28 (1954).
[10] D. V. Osborne, Proc. Phys. Soc. (London) **A64**, 114 (1951).
[11] R. B. Dingle, Phil. Mag. Suppl. **1**, 111 (1952).
[12] J. G. Daunt and R. S. Smith, Revs. Modern Phys. **26**, 172 (1954).
[13] H. N. V. Temperley, Proc. Phys. Soc. (London) **A64**, 105 (1951).
[14] I. M. Khalatnikov, Doklady Acad. Nauk, S.S.S.R. **79**, No. 2, 237 (1951). See also J. Exptl. Theoret. Phys. (U.S.S.R.) **23**, 253 (1952).

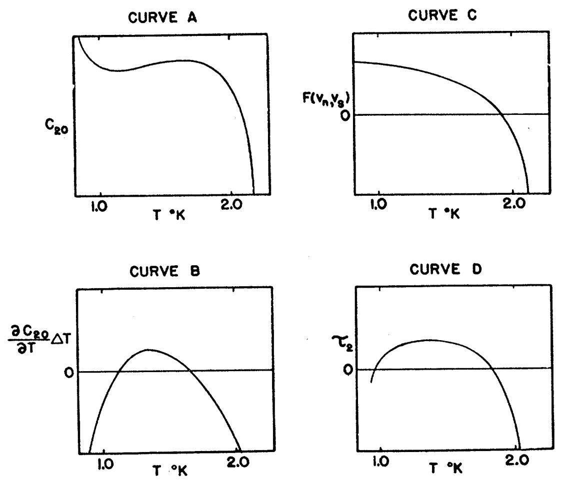

FIG. 1. Graphs of the terms contributing to the amplitude dependence of the velocity of second sound. Curve A: The zero amplitude velocity of second sound *vs* temperature. Curve B: The change in velocity of a signal of temperature amplitude ΔT, due to the variation of the velocity of second sound with temperature *vs* temperature. Curve C: The net normal fluid and superfluid stream velocity per unit normal fluid velocity in a second sound signal *vs* temperature. Curve D: τ_2—sum of Curve B and Curve C with Curve B divided by normal fluid velocity.

at 1.0°K, $2-\frac{\rho}{\rho_s}=0.993$,

$$\text{while } \frac{ST}{C}\frac{\partial}{\partial T}\ln\left(c_{20}^2\frac{C}{T}\right)=0.96;$$

$$\text{at } 0°\text{K, } 2-\frac{\rho}{\rho_s}=1.0, \text{ while } \frac{ST}{C}\frac{\partial}{\partial T}\ln\left(c_{20}^2\frac{C}{T}\right)=\frac{2}{3}.$$

τ_2 is physically composed of two terms. The significance of these two terms may be seen with the aid of Fig. 1. Consider a large rectangular second sound pulse (the carrier pulse) with a short one (the mark pulse) riding on top (e.g., Fig. 2). The two amplitude-dependent terms which will affect the velocity of the mark pulse arise as follows:

(1) The mark pulse will be traveling in liquid helium which will be slightly warmer than the ambient bath temperature owing to the temperature of the carrier pulse. Since the velocity of second sound is a function of temperature (Curve A, Fig. 1), the mark pulse will move either faster or slower than the carrier pulse depending on whether $(\partial c_{20}/\partial T)\Delta T$ is positive or negative (Curve B, Fig. 1).

(2) The mark pulse will be traveling in a current or stream of normal fluid and superfluid due to the relative motion of these two fluids in the carrier pulse. The mark pulse will be carried along by the resultant of these two opposing velocities.

From the second sound relationship of zero momentum transfer $\rho_n v_n + \rho_s v_s = 0$, we see that when $\rho_n < \rho_s$ (i.e., below 1.93°K), $|v_n| > |v_s|$. Therefore, below 1.93°K the mark pulse will be carried along by a net stream velocity $(v_n + v_s)$ which is in the direction of propagation of the second sound signal. Above 1.93°K, where $|v_s| > |v_n|$, the net stream velocity will be opposite to the direction of propagation and the mark pulse will be retarded.

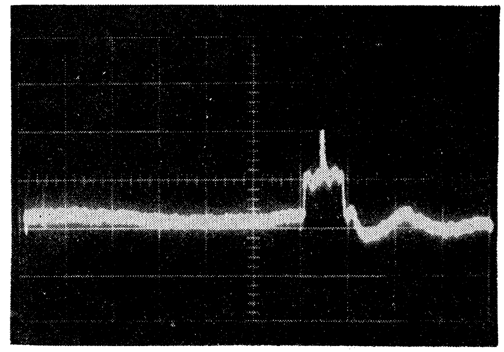

FIG. 2. Photograph of oscilloscope screen showing received second sound signal with the mark pulse riding on top of the carrier pulse. The oscilloscope sweep was started simultaneously with the transmission of the second sound signal. In an actual measurement of τ_2, the mark pulse was made very much smaller than shown in this photograph.

The term caused by this effect is sketched in Curve C, Fig. 1. The theoretical velocity change (in the temperature region above 1.4°K) due to the net stream velocity, may be obtained from τ_{2T} by multiplying $2-(\rho/\rho_s)$ by v_n. Thus $\Delta c_2 = [2-(\rho/\rho_s)v_n] = v_n + v_s$. Therefore, we obtain the result that the change in the velocity due to this effect is equal to the net stream velocity of the normal and superfluid components.

τ_2 given in Curve D, Fig. 1 is then simply the sum of Curves B and C with Curve B divided by v_n.

EXPERIMENTAL DETAILS

The amplitude dependence of the velocity of second sound was measured by means of a pulse technique. In the pulse method used, the time of flight of a small second sound pulse was measured over a known distance. Then this small pulse (the mark pulse) was superimposed near the middle of a pulse of relatively long duration (the carrier pulse), Figs. 2 and 3. The variation of the velocity of second sound with amplitude was measured by noting the change in the time of flight of the mark pulse across the known distance when the carrier pulse of measured amplitude was on and then off.

In order to insure that no error arose from temperature fluctuations of the bath, an electronic temperature control[15] was used above 1.5°K. When the temperature control was not used, a set of measurements was completed in about ten seconds—a time short compared to any significant temperature variations of the bath.

Figure 4 shows a simplified block diagram of the second sound apparatus. The blocking oscillator in the upper left-hand corner of the diagram is the master trigger for the entire apparatus. The blocking oscillator

[15] W. S. Boyle and J. B. Brown, Rev. Sci. Instr. **25**, 359 (1954).

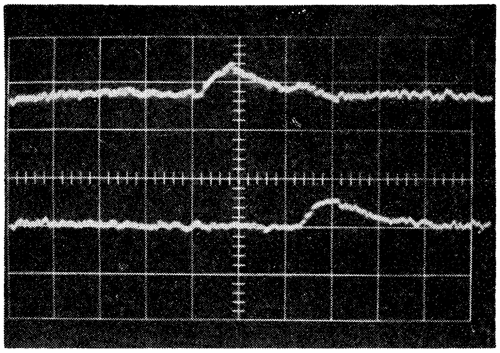

FIG. 3. Photograph of oscilloscope screen showing the change in the time of flight of the mark pulse when transmitted on top of the carrier pulse (upper trace). Full length of sweep represents 100 μsec. The leading and trailing edges of the carrier pulse are not visible because the oscilloscope sweep was started after the arrival of the leading edge and completed before the arrival of the trailing edge of the carrier pulse at the bolometer.

initiates a second sound pulse approximately once every second when free-running. The blocking oscillator could also be manually keyed for single-pulse operation. This low repetition rate caused very little heat, due to transmission of second sound, to be introduced into the cryogenic apparatus.

The oscilloscope sweep delay and the mark pulse were started simultaneously. Thus, the time of flight of the mark pulse was read directly from the accurately calibrated Tektronix 535 oscilloscope without making any correction for the delay between the start of the carrier pulse and the start of the mark pulse. The oscilloscope sweep delay was accurately calibrated and continuously variable. When recording the change in time of flight of the mark pulse, the sweep delay was set so that the oscilloscope did not sweep until the mark pulse had almost arrived at the bolometer. Then, the oscilloscope trace was swept quickly across the oscilloscope screen. This fast sweep magnified the shift in position of the mark pulse on the screen. For example, at a representative setting, the sweep would be delayed for about 5000 microseconds. Then, the oscilloscope trace would be swept across the 10-cm screen in 100 or 200 microseconds.

The transmitter output was fed into a heater made of aluminum film evaporated on plate glass. Lines were scratched through the aluminum film to increase the electrical resistance.

Because of the desirability of rapid thermal rise time and constant resistance in the liquid helium temperature range, the aluminum-film heater was used in preference to either the carbon-film heater[5] or the heater wound with fine resistance wire.[1]

The carbon film heater while having a relatively rapid thermal rise-time, has the distinct disadvantage of having a strongly temperature-dependent resistance.[16]

[16] H. A. Fairbank and C. T. Lane, Rev. Sci. Instr. 18, 525 (1947).

Because of the Kapitza film,[17] the temperature at the surface of the second sound transmitter is of the order of magnitude 1°K above the ambient temperature while a signal is being transmitted. Thus, in order to know the power input into a carbon-film heater, it would be necessary to measure both the voltage and the current during the output pulse.

The wire sound heater has a resistance virtually independent of temperature if wound of wire such as constantan or manganin. This heater, however, was found to have a much slower thermal rise time than the aluminum-film heater. The slower rise time made it more difficult to pick a unique time for the arrival of the mark pulse.

The heater and the bolometer resistor were supported at opposite ends of a cylindrical Lucite tube. Besides providing an accurate heater-bolometer spacing, the tube directed the flow of second sound so that the wave motion up the tube approximated a plane wave. The sensitive part of the bolometer resistor was a small strip centered on the axis of the tube. Thus, only the central portion of the second sound signal was detected. Any edge effects due to attenuation or distortion of the second sound signal by the plastic tube were thereby avoided.

Two tubes cut from Lucite were used for this experiment. Corrections due to the contraction of the Lucite when cooled to liquid helium temperatures[18,19] were made for the change in length and cross-sectional area of the tubes. One tube was 10.00 cm long and the other was 5.00 cm long when cooled to liquid helium temperatures. The cross-sectional area of both tubes was 9.27 cm^2 at the temperature of liquid helium.

The temperature of the liquid helium was determined from the pressure of the saturated vapor in equilibrium with the liquid. The conversion between vapor pressure and temperature was made using "the agreed scale of 1948."[20]

RESULTS

The experimental points for τ_2 are shown in Fig. 5. The solid line represents

$$\tau_{2K}=\frac{ST}{C}\frac{\partial}{\partial T}\left(c_{20}{}^{3}\frac{C}{T}\right),$$

as given by Khalatnikov. As discussed above, this curve is essentially the same in the region above 1°K as that predicted by Temperley.

In plotting Khalatnikov's expression for τ_2, it was found that the curve was very sensitive to the values used for the velocity of second sound. The various

[17] P. L. Kapitza, J. Phys. U.S.S.R. 4, 181 (1941).

[18] H. L. Laquer and E. L. Head, Proceedings of the Schenectady Cryogenics Conference, October 6–7, 1952 (unpublished), p. 176.

[19] Giauque, Geballe, Lyon, and Fritz, Rev. Sci. Instr. 23, 169 (1952).

[20] H. Van Dijk and D. Shoenberg, Nature 164, 151 (1949).

published values for the velocity of second sound[1,2–9,21] differ by about 1%. Not only do the magnitudes differ, but the derivative of the velocities with respect to temperature are in much wider disagreement. Some sources give a positive value for dc_{20}/dT, while other sources give a negative value for the derivative at the same temperature. Thus, the theoretical curve may fall within wide limits. For example, different values for the velocity of second sound give a lower crossover temperature for τ_2 which ranged from 0.86°K to almost 1°K. Various published values for the velocity of second sound[2,4,5,8,9,21] were averaged together for the calculation of the theoretical curve below about 1.8°K.

The various measurements of the velocity of second sound are in good agreement in the temperature region where τ_2 is small (near 1.87°K), and they diverge in the regions where τ_2 is large. The standing-wave technique, which claims the highest accuracy for the velocity measurements, is probably the most seriously affected by any amplitude-velocity dependence. Because of resonance, the heat current density near the center of a standing second sound wave may be of the order of a hundred times greater than that measured at the heater input.[22] Thus, it is probable that amplitude effects are the cause of the discrepancies in the velocity measurements in the regions where τ_2 is large.

For the temperature region above 1.4°K, it was found possible to calculate values for the theoretical curve without using any data for the velocity of second sound. By assuming that the density of the normal fluid and the total entropy are proportional and that $\rho_n=\rho(T/T_\lambda)^r$,

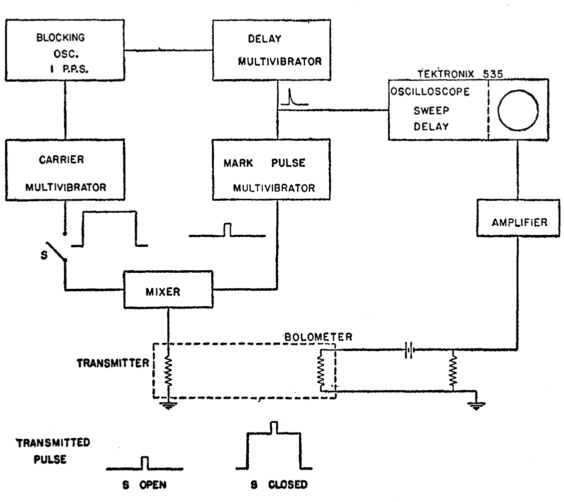

FIG. 4. Simplified block diagram of apparatus for measuring the amplitude dependence of the velocity of second sound. Blocking oscillator initiates the carrier pulse and, after a delay, the mark pulse. The oscilloscope sweep delay and the mark pulse are started simultaneously so that the time of flight of the mark pulse is given directly by the oscilloscope. By means of switch S, the mark pulse may be transmitted either alone or on top of the carrier pulse.

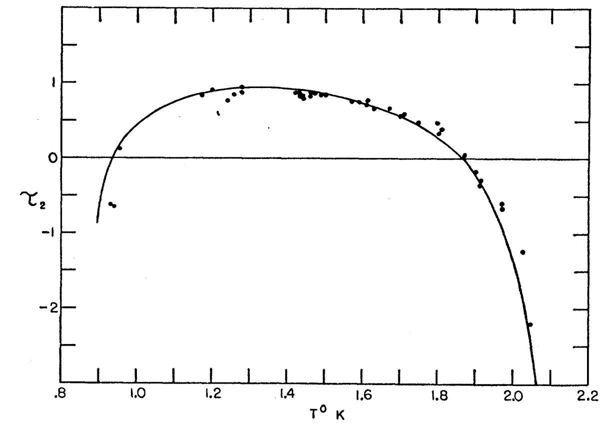

FIG. 5. Experimental points for $\tau_2\equiv(c_2-c_{20})/v_n$, where c_{20} is the zero amplitude value for the velocity of second sound, c_2 is the velocity at a point on the profile of a second sound signal, v_n is the normal fluid velocity at that point. The solid line represents $\tau_2=(ST/C)(\partial/\partial T)[c_{20}{}^3(C/T)]$ as given by Khalatnikov,[14] where S is the entropy, T is the absolute temperature, and C is the specific heat. Temperley's result[13] is essentially identical with this curve above 1°K.

where ρ=total density of liquid helium, ρ_n=normal fluid density and $r=5.4$, it is possible to put the expression for τ_2 in a form which is a function only of the temperature $\tau_2=2.59-(3/2)[1-(T/T_\lambda)^{5.4}]^{-1}$. The two foregoing assumptions have been experimentally verified[23,24] between 1.4°K and the λ point. There is, however, an uncertainty in the value of the exponent r of about ±0.1. However, calculations show that an error in the value of r of 0.1 will cause the magnitude of τ_2 at 2°K to be in error by 0.1, diminishing to an error of about 0.01 at 1.4°K. Another possible source of error in plotting the theoretical expression for τ_2 comes from the values used for the specific heat.[24] These values have a probable error of about ±5%.

The two methods for evaluating the theoretical expression for τ_2 gave practically identical results between 1.4°K and 1.8°K. Above 1.8°K the latter method was considered the most reliable.

The theoretical expressions for τ_2 were derived under the assumption that $W=v_n-v_s$ is small compared to c_{20}. Therefore, it was necessary to observe this criterion in the course of the experiment in order that the measurements be a fair check of the theories. Using the second sound relationship $v_n\rho_n+v_s\rho_s=0$, it may be shown that $W=(\dot{H}/\rho ST)(1+\rho_n/\rho_s)$. The values of $\dot{H}$ used in the measurements were kept small enough so that W was of the order of 1% of c_{20}.

The amplitude of the mark pulse should ideally approach zero in order that the time of flight of the mark pulse should depend solely on the temperature and the amplitude of the carrier pulse. Two methods were used in the experiment to attain the approximation of a mark pulse of zero amplitude:

[21] V. Peshkov, J. Phys. U.S.S.R. **10**, 389 (1946).
[22] J. R. Pellam and W. B. Hanson, Phys. Rev. **85**, 216 (1952).

[23] deKlerk, Hudson, and Pellam, Phys. Rev. **89**, 662 (1953).
[24] Kramers, Wasscher, and Gorter, Physica **18**, 329 (1952).

TABLE I. Values for $\tau_2 \equiv (c_2 - c_{20})/v_n$ and $\Gamma_2 \equiv (c_2 - c_{20})/\dot{H}$ from the smoothed experimental curves. c_{20} is the zero amplitude value for the velocity of second sound, c_2 is the velocity at a point on the profile of a second sound signal, v_n is the normal fluid velocity at that point, and $\dot{H}$ is the heat current density at that point. The estimated error for these values of τ_2 and Γ_2 is $\pm 10\%$.

T°K	τ_2	$\Gamma_2\left(\frac{\text{cm}^3}{\text{watt sec}}\right)$
0.946±0.01	0.00	0
0.95	0.075	45
1.00	0.45	180
1.05	0.63	180
1.10	0.73	150
1.15	0.80	125
1.20	0.83	91
1.25	0.84	69
1.30	0.85	53
1.35	0.85	41
1.40	0.85	32
1.45	0.84	25
1.50	0.82	19
1.55	0.77	14.4
1.60	0.72	10.9
1.65	0.67	8.2
1.70	0.59	6.0
1.75	0.48	4.1
1.80	0.35	2.5
1.85	0.123	0.74
1.873±0.005	0.00	0.00
1.90	−0.16	−0.80
1.95	−0.51	−2.2
2.00	−0.95	−3.5
2.05	−2.2	−7.0

(1) Mark pulses of various amplitudes were transmitted and the times of flight were extrapolated to zero amplitude.

(2) For the carrier pulse amplitudes used, the velocity of second sound was found to be a linear function of the amplitude. Therefore, the mark pulse amplitude when transmitted alone was set equal to its amplitude when transmitted on top of the carrier pulse. Thus, the effect of the amplitude of the mark pulse on the change in its time of flight was essentially canceled. Of course, this method yielded absolute values for the times of flight of the mark pulse which were slightly in error. However, the mark pulse amplitudes used were so small as to give an error of less than 0.1% in the time of flight.

Table I gives values for τ_2 taken from a smoothed experimental curve. Table I also lists smoothed experimental values for Γ_2, where $\Gamma_2 \equiv (c_2 - c_{20})/\dot{H} = \tau_2 v_n/\dot{H}$. The estimated error for the values of τ_2 and Γ_2 is approximately $\pm 10\%$.

Osborne has made qualitative measurements[10] of τ_2 which are in agreement with the general shape of the curve obtained in this experiment. Osborne's measurements were made by observing the shape of received second sound pulses. A received pulse with a steep leading edge and an elongated trailing edge indicated that τ_2 is positive. At the temperatures, where τ_2 is negative, a received pulse has an elongated leading edge and a steepened trailing edge.

At temperatures around 2°K, Osborne observed an anomalous effect. For large-amplitude pulses and a short distance between heater and bolometer, the received pulse had a steepened leading edge and an elongated trailing edge, indicating a positive value for τ_2. However, when the distance between the heater and bolometer was increased, the shape of the received pulse changed so as to indicate a negative value for τ_2. As a specific example, Osborne gives $\dot{H} = 3.5$ watts/cm², $T = 2.12$°K, $\tau_2 > 0$ when d (distance between heater and bolometer) ≤ 5 cm, $\tau_2 < 0$ when $d > 5$ cm.

This anomaly is probably explained on the basis of sufficient local heating at the surface of the wire-wound second sound heater to raise the temperature of the liquid at the surface of the wire above the λ point. The effect of heating the liquid above the λ point would be to produce a second sound pulse with a very sharp rise and a long trailing edge since heat in the nonsuperfluid region would be conducted to the helium II by normal thermal conduction processes. If the bolometer is sufficiently close to the heater, the received pulse will have the shape of the transmitted pulse since the second order effects would not have enough time to change the shape of the pulse.

An effect similar to that described above was observed by us when the second sound heater consisted of a wire wound element. No such anomaly was observed when excessive local heating was avoided by use of the aluminum film heater.

CONCLUSION

Within the experimental errors inherent in this experiment and the uncertainties in the evaluation of the theoretical curve, the experimental results are in agreement with the theories of Temperley and Khalatnikov. Since the theoretical expressions for τ_2 were derived by solving the two-fluid hydrodynamical equations of motion for helium II to terms of second order, it may be concluded that within experimental error, the second order terms in the hydrodynamical equations used by Khalatnikov and Temperley have been verified between approximately 1°K and 2°K.

The second sound velocity was found to be a linear function of the heat current density, $\dot{H}$, up to the highest values used ($\dot{H} = 1.6$ watts/cm²). No evidence of any higher order terms was detected.

ACKNOWLEDGMENTS

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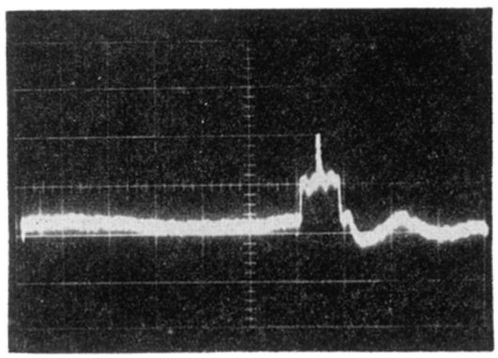
We particularly wish to acknowledge the generous assistance of G. K. Walters, E. Lynch, and W. McCormick in taking the experimental data.



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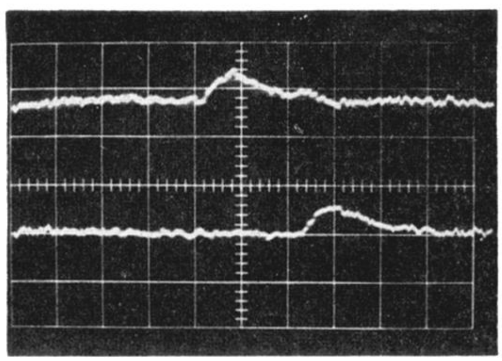

FIG. 3. Photograph of oscilloscope screen showing the change in the time of flight of the mark pulse when transmitted on top of the carrier pulse (upper trace). Full length of sweep represents 100 μsec. The leading and trailing edges of the carrier pulse are not visible because the oscilloscope sweep was started after the arrival of the leading edge and completed before the arrival of the trailing edge of the carrier pulse at the bolometer.