TABLE III. Comparison of experimental and theoretical values of average secondary asymmetry for zenith angle 45° at Echo Lake.

Channel	Theoretical asymmetry	Experimental asymmetry
I	0.162	0.141 ± 0.023
II	0.208	0.153 ± 0.088

C. Comparison with Experiment

Numerical computations were carried out for a zenith angle of 45° at Echo Lake (geomagnetic latitude $\lambda = 48.4°$N, geomagnetic longitude $\omega = 112.7°$W, atmospheric depth $H = 705$ g cm^{-2}). The temperature variation with atmospheric depth was taken from the curve given by Olbert[2] for 40° geographic latitude. The horizontal magnetic field of the earth was taken to be 0.22 gauss [3] the corresponding value of γ being 3.9×10^{-2}.

The critical rigidity at Echo Lake is 2.7 Bv at the vertical, and 2.3 and 4.2 Bv at 45° west and east respectively. These figures were obtained from curves given by Vallarta,[4] corrected according to the procedure given in reference 1. The corresponding values of a are 522 g cm^{-2} for the vertical direction, and 519 and 538 g cm^{-2} for west and east.

The theoretical value of the average secondary asymmetry has been compared with the experimental value in Table III. It is seen that the theory is wholly adequate to explain the observed asymmetry.

[2] S. Olbert, Phys. Rev. **92**, 454 (1953).

[3] F. E. Fowle, *Smithsonian Physical Tables* (Smithsonian Institution, Washington, 1934), eighth revised edition, p. 593.

[4] M. S. Vallarta, Phys. Rev. **74**, 1837 (1948).

PHYSICAL REVIEW VOLUME 104, NUMBER 2 OCTOBER 15, 1956

Necessity of Singularities in the Solution of the Field Equations of General Relativity*

ARTHUR KOMAR
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received May 25, 1956; revised version received July 30, 1956)

The Friedmann solution of the field equations of general relativity predict the expansion of the universe from a singular instant in time. This paper considers the behavior of universes which are less symmetric than the Friedmann model, and which have more general fields that carry stress and energy and produce gravitation. We treat the nonsymmetric problem as one treats the symmetric problem, via a "co-moving coordinate system" such that in it free infinitesimal test particles once at rest remain at rest. In such a coordinate system and with standard assumptions about the stress energy tensor we establish that the solutions of the field equations of the general theory of relativity necessarily have singularities at finite time. The considerations are independent of the symmetry, topology or boundary conditions assumed for the space-like three-dimensional hypersurfaces.

INTRODUCTION

A PRINCIPAL success of the general theory of relativity in the realm of cosmology is given by the Friedmann solution[1] of the field equations. This solution, which employs the assumptions that the universe is spacially isotropic and that the state of matter may be represented by incoherent dust, yields the result that the universe is not stationary, but is rather in a state either of expansion from a singular point in time (which would correspond to creation), or of contraction toward a singular point in time (which would correspond to annihilation). The question naturally arises whether such singular points are a consequence of the particular symmetry presupposed in Friedmann's model, or whether perhaps for more general distributions of matter one need not expect instants of creation or annihilation of the universe. The purpose of this paper is to show that singularities in the solution of the field equations of general relativity are to be expected under very general hypotheses (enumerated specifically below), and in particular that the singular instant of creation (or annihilation) necessarily would occur at a finite time in the past (or future, respectively).

ENUMERATION OF HYPOTHESES

We make the following assumptions:

(A) The universe is a topological product of a three-dimensional hypersurface and a line. The line represents the direction of time, while the hypersurface represents space.

(B) We may select a set of these space-like hypersurfaces which are geodesically parallel for all time. (Physically, this corresponds to the assumption that the average motion of matter throughout the universe is sufficiently uniform that a coordinate system can be chosen so that at each point free infinitesimal test particles are fixed relative to this coordinate frame. For the case when the distribution of matter in the

* Based on one section of a thesis submitted to Princeton University in partial fulfillment of the requirements for the Ph.D. degree, April, 1956.

[1] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1953).

universe may be represented as incoherent dust, such a frame is frequently referred to as a "co-moving coordinate system." We should note, however, that this situation of being able to select a set of space-like geodesically parallel hypersurfaces for all times is not as general as one might at first glance expect. For even such a highly symmetric solution of the Einstein field equations as that of Wheeler's geon[2] does not have this property.)

(C) The Einstein field equations

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=KT_{ij} \qquad (1)$$

hold (where K is a positive constant, equal to $8\pi G/c^4$). We do not consider the possibility of a cosmological term.

(D) The energy momentum tensor T_{ij} is to have the properties that T_{44} and the trace T is non-negative, and that $T_{44}=0$ implies $T_{ij}=0$. These properties are in fact true of all classical energy momentum tensors if we reject the possibility of negative pressure terms. (A constant negative pressure term in the energy-momentum tensor is evidently an alternative way of interpreting or introducing a positive cosmological term. However, Einstein, the introducer of this cosmological term, has since given strong reasons *against* it.)

(E) The metric tensor on the space-like hypersurfaces is assumed to be positive-definite.

(F) The metric tensor g_{ij} of the four-space is assumed to have the Minkowski signature $(1, 1, 1, -1)$.

FORM OF THE METRIC, RICCI, AND RIEMANN TENSORS

If we select the coordinate system such that the geodesically parallel hypersurfaces hypothesized in (B) have the equations

$$x^4=\text{const}, \qquad (2)$$

where the constant is taken equal to the geodesic distance of the surface from a fixed base surface, the metric assumes the form[3]

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta-(dx^4)^2. \qquad (3)$$

(A summation convention from 1 to 3 is understood on Greek indices.) As is well known, such a coordinate system can always be chosen, but it will in general become singular after a finite time interval. To avoid this eventuality, we postulate hypothesis (B). The $g_{\alpha\beta}$ is evidently the metric tensor on the hypersurfaces given by Eq. (2) and via hypothesis (E) it is positive-definite. If in addition we define on each of these hypersurfaces the symmetric tensor

$$\chi_{\alpha\beta}\equiv\partial g_{\alpha\beta}/\partial x^4, \qquad (4)$$

we find that we can write the Ricci and Riemann tensors thus:

$$\begin{aligned} R_{\alpha\beta}&=P_{\alpha\beta}+\text{terms in }\chi_{\alpha\beta},\\ R_{\alpha 4}&=\text{terms in }\chi_{\alpha\beta},\\ R_{44}&=-\tfrac{1}{2}(\partial\chi_\alpha{}^\alpha/\partial x^4)-\tfrac{1}{4}\chi_{\alpha\beta}\chi^{\alpha\beta},\end{aligned} \qquad (5)$$

$$\begin{aligned} R_{\alpha\beta\gamma\delta}&=P_{\alpha\beta\gamma\delta}+\text{terms in }\chi_{\alpha\beta},\\ R_{\alpha 4\beta\gamma}&=\text{terms in }\chi_{\alpha\beta},\\ R_{\alpha 4\beta 4}&=\text{terms in }\chi_{\alpha\beta}.\end{aligned} \qquad (6)$$

$P_{\alpha\beta\gamma\delta}$ and $P_{\alpha\beta}$ are respectively the Riemann and Ricci tensors constructed from the metric $g_{\alpha\beta}$ of the hypersurfaces of Eq. (2). All raising and lowering of Greek indices is understood to be done with respect to this metric.

PROOF OF THE EXISTENCE OF SINGULARITIES

We may rewrite the Einstein field equations (1) in the form

$$R_{ij}=K(T_{ij}-\tfrac{1}{2}g_{ij}T). \qquad (7)$$

We thus obtain from assumption (D) and Eq. (5)

$$R_{44}=-\tfrac{1}{2}\frac{\partial\chi_\alpha{}^\alpha}{\partial x^4}-\tfrac{1}{4}\chi_{\alpha\beta}\chi^{\alpha\beta}=K(T_{44}+\tfrac{1}{2}T)\geqslant 0. \qquad (8)$$

If we transform momentarily to a normal coordinate system[3] at a point on the space-like hypersurface, we see by an application of the Schwarz inequality

$$|\chi_\alpha{}^\alpha|=\chi_{11}+\chi_{22}+\chi_{33}\leq\sqrt{3}[(\chi_{11}{}^2+\chi_{22}{}^2+\chi_{33}{}^2)]^{\frac{1}{2}}, \qquad (9)$$

and therefore

$$(\chi_\alpha{}^\alpha)^2\leq 3(\chi_{11}{}^2+\chi_{22}{}^2+\chi_{33}{}^2)\leq 3\chi_{\alpha\beta}\chi^{\alpha\beta}. \qquad (10)$$

Since this is a scalar relationship it holds in general, independent of the choice of coordinate system. Combining Eqs. (10) and (8) we obtain

$$\frac{-\partial(\chi_\alpha{}^\alpha)}{\partial x^4}\geq\frac{(\chi_\alpha{}^\alpha)^2}{6}\geq 0. \qquad (11)$$

Let us consider now an instant when $\chi_\alpha{}^\alpha<0$. For simplicity of notation we define

$$f\equiv-\chi_\alpha{}^\alpha;\quad x^4\equiv t, \qquad (12)$$

and note that if $f_0\equiv f(t_0)>0$ at some time t_0, then for all subsequent times $f\geq f_0>0$, due to the monotonic character of the solutions of Eq. (11). We can now write Eq. (11)

$$\frac{d(1/f)}{dt}\leq-\frac{1}{6}. \qquad (13)$$

If we integrate this we find

$$1/f-1/f_0\leq\tfrac{1}{6}(t-t_0). \qquad (14)$$

[2] J. A. Wheeler, Phys. Rev. **97**, 511–536 (1955).
[3] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1949).

We now observe that as t increases, the right-hand side of (14) becomes increasingly negative. However, since we are assuming for the moment $f_0 > 0$ (and therefore $f > 0$), the left-hand side can never become more negative than $-1/f_0$, and in particular this occurs when $f = \infty$. If we call the corresponding time at which this occurs t_∞, we see from Eq. (14) that

$$t_0 < t_\infty \leq \frac{6}{f_0} + t_0 < \infty. \tag{15}$$

For the case when $f_0 < 0$, we observe that for all earlier times $f \leq f_0 < 0$. Therefore if we define a new function $g \equiv -f = \chi_\alpha{}^\alpha$, and a new time variable $\tau = -t$, we see that we revert to the previous case. (I am indebted to Professor V. Bargmann for indicating the above considerations.)

We may therefore conclude that unless $\chi_\alpha{}^\alpha$ is identically zero it will already diverge at a finite time (either in the past or future according to whether $\chi_\alpha{}^\alpha > 0$ or $\chi_\alpha{}^\alpha < 0$). We should note that one cannot easily determine whether the singularity is in the coordinate system or whether the space itself is singular. Taub[4] has pointed out that there is as yet no well-defined way of determining what constitutes an essential singularity within the general theory of relativity.

If, however, we take $\chi_\alpha{}^\alpha \equiv 0$, we see from Eq. (8) and the fact that $g_{\alpha\beta}$ is positive-definite that

$$\chi_{\alpha\beta} = 0, \tag{16a}$$

$$T_{44} = 0; \quad T = 0. \tag{16b}$$

From assumption (D) we then have

$$T_{ij} = 0, \tag{17}$$

and thus, from Eq. (7),

$$R_{ij} = 0. \tag{18}$$

Combining Eqs. (5), (16a), and (18) we find

$$P_{\alpha\beta} = 0. \tag{19}$$

For a three-dimensional manifold the Ricci tensor is equivalent to the Riemann tensor.[3] Consequently we have

$$P_{\alpha\beta\gamma\delta} = 0. \tag{20}$$

Thus from Eq. (6) we find

$$R_{ijkl} = 0, \tag{21}$$

and therefore the space is flat. [There are many other sets of postulates under which one can show that the Einstein field equations (1) imply that the space is flat.[5] Furthermore, no nontrivial, nonsingular solution of Eq. (1) has yet been found. However, the possibility that such a solution may exist still remains open.]

CONCLUSION

We therefore find that if we want a nontrivial (i.e., nonflat) solution of the field equations of general relativity we must be prepared either: (A) to allow for singularities (as for example in the Schwarzschild[6] or the Friedmann[1] solutions); (B) to permit the possibility of a cosmological term or a negative pressure term (as for example in the Einstein cylinder universe[1]); or (C) to consider spaces which do not have the simplifying property of containing a set of geodesically parallel space-like hypersurfaces for all times. It is of particular interest that, in the case which we considered, of a comoving coordinate system, a singularity necessarily occurs at a finite time independent of any choice of symmetry, topology or boundary conditions for the space-like hypersurfaces. For, if we reject the possibility of representing finite distributions of matter by means of singularities, and if the only singularity which we are prepared to admit is one which corresponds to a creation of the universe, we have as a necessary consequence of the considerations (and *assumptions*) of this paper that the creation occurred at a finite time in the past.

ACKNOWLEDGMENTS

I would like to express my appreciation to Professor J. A. Wheeler and to Professor V. Bargmann for their many helpful discussions and suggestions which enabled me to prepare this paper.

[4] A. H. Taub, Ann. Math. **53**, 472–490 (1951).

[5] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnetisme* (Masson et Cie, Paris, 1955).

[6] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Cambridge, 1951).