carbon nucleus for the production of $\theta^0 \to 2\pi^0$ is ~ 0.1 mb. This cross section also applies approximately if the observed gamma rays originate from the process $\Lambda^0 \to n + \pi^0$. The kinematics of the latter process are more favorable than the kinematics of the θ^0 process for observing gamma rays at 90°. This condition compensates for the fact that only a single π^0 meson is produced in the Λ^0 decay instead of the two π^0 mesons produced in the θ^0 decay.

If an appreciable fraction of either Λ^0 hyperons or θ^0 mesons decay via π^0 mesons, and there is evidence that this may be the case,[9,10] the relatively low cross section for downstream gamma-ray production in p-carbon collisions observed here is striking compared to the cross section of ~ 1 mb observed by Fowler *et al.*[11] for the production of heavy unstable particles by 1.37-Bev pions on hydrogen.

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Angular Distribution in Electron-Photon Showers without the Landau Approximation

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The track-length angular distribution of electrons in an electron-photon cascade is calculated without the use of the Landau approximation which was invariably used in all previous work. The Tamm-Belenky model of the cascade is used. The results are presented in the form of a series, the first term of which is of the same form as the result yielded by the Landau approximation but with a modified value of the parameter E_s.

It is shown that the use of the Landau approximation introduces large errors not only for large values of the argument but for small values as well. The bearing of this result on previous theoretical work on the lateral distribution function is discussed.

1. INTRODUCTION

SEVERAL calculations of the track-length angular distribution of the electron-photon cascade have been made,[1–5] giving results in quite good agreement with each other especially at values of the angular variables $E\theta/E_s$ less than unity, where E_s, the "characteristic scattering energy," is 21 Mev and the angle θ between the directions of motion of the cascade electron and the primary particle is measured in radians. Several of these distributions are graphed in Figs. 1 and 2, where the essential agreement between them can be seen. However these calculations were all made under a common approximation, the Landau approximation, so there is as yet no check on the validity of this approximation nor upon the accuracy of these distributions.

Now the Landau approximation is the cascade equivalent of the well-known multiple-scattering approximation which, when applied to the elastic scattering of a particle without loss of energy, yields a Gaussian angular distribution. It has always been appreciated that because of the θ^{-4} dependence of the scattering cross section at large angles, the multiple-scattering approximation (in common with the Landau approximation) has no validity in the "tail" of the angular distribution. In the theory of multiple scattering this error in the tail can be corrected by the addition of a component corresponding to one or a few single scattering acts each through a large angle (the "single-scattering tail"). However the calculations of Snyder and Scott[6] and of Molière[7] of the angular distribution resulting from multiple scattering show that there is a considerable error in the Gaussian approximation to this distribution even at very small angles. The

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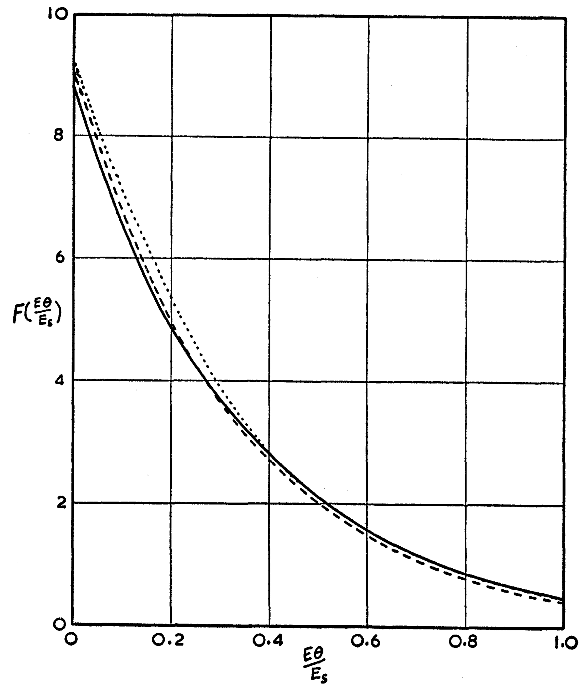

FIG. 1. Theoretical calculations of the track-length angular structure function for electrons of energy E in an electron-photon cascade. Full line: Moliere[1]; broken line: Belenky[2] with $q=2.289$; dotted line: Eyges and Fernbach.[4]

Gaussian distribution, as yielded by the multiple-scattering approximation, becomes accurate at small angles only when the scattering has continued over a path length of the order of several cascade units.

It would appear reasonable to think that this approximation would be of sufficient accuracy at small angles when applied to an electron-photon cascade, because such a cascade of a magnitude sufficient for its angular distribution to be studied must have developed over a path length of several cascade units. However, we can show by a very simple argument that the Landau approximation must inevitably be less accurate than is the multiple-scattering approximation when applied to scattering over a path length of the order of one cascade unit.

Although we are generally interested in cascades that have developed over a distance of several cascade units, the major contribution to the scattering of any particular electron in such a cascade comes from the last two or three cascade units immediately above it. This is because the probable energy of the "ancestor" of this particle at greater heights is much larger than that of the particle itself. Now the energies of the ancestors of the particle under consideration are not fixed quantities but can vary within the limits of a probability distribution. (If we knew this distribution we could use it to calculate the angular distribution of the cascade electrons, but this is by no means the simplest way to solve this problem.) Thus some of the electrons of energy E at a given depth had ancestors with energy very little greater than E at heights of several cascade units above, while others have just been produced from electrons (or photons) of energy very much greater than E—and, of course, we also have representatives of the infinity of gradations between these two extremes.

Electrons of the first type have, effectively, been scattered over a path length of several cascade units. Consequently the angular distribution of these particles, treated separately for the moment, will be accurately represented by the Landau approximation. However these same electrons are those (of the given energy E) which are scattered the most. The latter type of electrons, those with high-energy ancestors, will have been scattered much less, and will therefore make the larger contribution to the over-all distribution at small angles. But also, as they have been scattered over a distance small in comparison to a cascade unit, their angular distribution will be very inaccurately represented by the Landau approximation. Consequently the Landau approximation is even more inaccurate at small angles than one would expect from a simple analogy with multiple scattering.

We thus see that, although the various calculations of the track-length angular distribution mentioned above agree well with each other at small angles, it does not by any means follow that they bear any similarity to the true distribution. The purpose of this paper

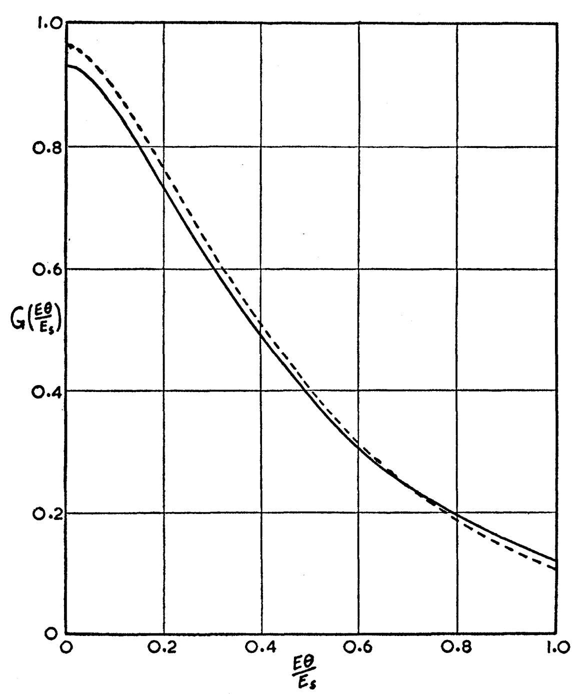

FIG. 2. Theoretical calculations of the projected track-length angular structure function for electrons of energy E in an electron-photon cascade. Full line: Belenky[2] with $q=2.145$; broken line: Kalos and Blatt.[5]

is to present a calculation of the track-length angular distribution of the electron-photon cascade in which the Landau approximation is not used, thus allowing for the first time a determination to be made of the errors involved through its use. This is made possible by the use of an approximation to the cascade cross sections themselves. The approximation used is exactly the same as that used by Belenky[2] in his calculation of the angular distribution under the Landau approximation. We calculate the track-length angular distribution using the Belenky model both with and without the Landau approximation. The "model" of the cascade we use was invented by Tamm and Belenky[8] for the purpose of yielding an analytical solution for the track-length average numbers when the ionization loss is included. We, like Belenky, are using the Tamm-Belenky model with the ionization term omitted. A glance at Figs. 1 and 2 shows that the angular distribution of the Tamm-Belenky model is very little different from that of other models when the Landau approximation is used. It is reasonable to assume that this model will be equally satisfactory when the Landau approximation is not used.

2. FORMAL SOLUTION FOR THE ANGULAR DISTRIBUTION

We define the "track-length angular distribution"

$$Q(E_0;E,\theta)dE\theta d\theta \qquad (1)$$

to be the average distance (in cascade units) travelled by electrons of energy in the range E to $E+dE$ while moving in a direction at an angle in the range θ to $\theta+d\theta$ to the direction of motion of the primary electron of energy E_0 which initiated the cascade. We also define the elastic scattering cross section

$$W(E,\theta)dt\theta d\theta \qquad (2)$$

to be the probability that an electron of energy E will, in travelling the distance dt, suffer an elastic scattering collision which deflects its direction of motion through an angle in the range θ to $\theta+d\theta$. Then

$$W_0=\int_0^\infty W(E,\theta)\theta d\theta \qquad (3)$$

is the total elastic scattering collision rate.

The equation satisfied by the angular distribution Q in the Tamm-Belenky model is

$$(q/E)\int_E^{E_0} Q(E_0;E',\theta)dE'-(q+W_0)Q(E_0;E,\theta)$$
$$+\int_0^\infty \theta' d\theta' \int_0^{2\pi}\frac{d\beta}{2\pi}W(E,\theta')Q(E_0;E,\theta'')$$
$$+\delta(E_0-E)\delta(\theta)/\theta=0, \qquad (4)$$

[8] I. Tamm and S. Belenky, J. Phys. (U.S.S.R.) **1**, 177 (1939).

where

$$\theta''=[\theta^2+\theta'^2-2\theta\theta'\cos\beta]^{\frac{1}{2}}, \quad q=2.289. \qquad (5)$$

The parameter q is defined as the coefficient occurring in the following approximate expression for the electron track-length average numbers which is accurate when E is much smaller than E_0 (see Rossi and Greisen[9]):

$$\int_0^\infty Q(E_0;E,\theta)\theta d\theta\equiv\pi(E_0;E)\equiv E_0/qE^2. \qquad (6)$$

One of the approximations made in the derivation of the Tamm-Belenky model is the substitution for the true bremsstrahlung cross section of an approximate cross section which yields a finite total collision rate for electrons. The value of this collision rate is q. Hence, when we are considering also the effects of elastic scattering events, the total collision rate for electrons becomes $q+W_0$.

Define the Hankel transforms

$$\bar{Q}(E_0;E,\phi)=\int_0^\infty J_0(\theta\phi)Q(E_0;E,\theta)\theta d\theta, \qquad (7)$$

$$\bar{W}(E,\phi)=\int_0^\infty J_0(\theta\phi)W(E,\theta)\theta d\theta-W_0. \qquad (8)$$

Then taking the Hankel transform of Eq. (4) yields

$$(q/E)\int_E^{E_0}\bar{Q}(E_0;E',\phi)dE'$$
$$+[\bar{W}(E,\phi)-q]\bar{Q}(E_0;E,\phi)+\delta(E_0-E)=0. \qquad (9)$$

This equation is easily solved by rewriting it in terms of the Hankel transform of the "integral angular distribution"

$$\bar{Q}_{\text{int}}(E_0;E,\phi)=\int_E^{E_0}\bar{Q}(E_0;E',\phi)dE', \qquad (10)$$

as it then becomes a simple first-order differential equation. The solution is

$$\bar{Q}_{\text{int}}(E_0;E,\phi)$$
$$=\frac{1}{q}\bar{V}(E_0,\phi)\exp\left\{\int_E^{E_0}\bar{V}(E',\phi)dE'/E'\right\}, \qquad (11)$$

where

$$\bar{V}(E,\phi)=\left[1-\frac{\bar{W}(E,\phi)}{q}\right]^{-1}. \qquad (12)$$

The angular distributions are most conveniently expressed in terms of the "angular structure function" $F(E_0;E,\theta)\theta d\theta$ and the "integral angular structure func-

[9] B. Rossi and K. Greisen, Revs. Modern Phys. **13**, 240 (1941).

tion" $F_{\mathrm{int}}(E_0;E,\theta)\theta d\theta$ which are defined by

$$\begin{aligned} F(E_0;E,\theta)&=Q(E_0;E,\theta)/\pi(E_0;E),\\ F_{\mathrm{int}}(E_0;E,\theta)&=Q_{\mathrm{int}}(E_0;E,\theta)/\pi_{\mathrm{int}}(E_0;E). \end{aligned} \tag{13}$$

These functions are normalized by

$$\int_0^\infty F(E_0;E,\theta)\theta d\theta=\int_0^\infty F_{\mathrm{int}}(E_0;E,\theta)\theta d\theta=1. \tag{14}$$

For the Tamm-Belenky model we have the very simple exact expressions for the average numbers

$$\begin{aligned} \pi_{\mathrm{int}}(E_0;E)&=E_0/qE,\\ \pi(E_0;E)&=E_0/qE^2+\delta(E_0-E)/q. \end{aligned} \tag{15}$$

Then from Eq. (11) we get

$$\bar{F}_{\mathrm{int}}(E_0;E,\phi) =\bar{V}(E_0,\phi)\exp\left\{-\int_E^{E_0}[1-\bar{V}(E',\phi)]dE'/E'\right\}. \tag{16}$$

By differentiating Eq. (11) with respect to E and then dividing by the average numbers $\pi(E_0;E)$, we can also obtain

$$\begin{aligned} \bar{F}(E_0;E,\phi)&=\bar{V}(E,\phi)\bar{F}_{\mathrm{int}}(E_0;E,\phi) \text{ for all } E<E_0,\\ \bar{F}(E_0;E_0,\phi)&=\bar{V}(E_0,\phi). \end{aligned} \tag{17}$$

It can be seen that the function $\bar{V}(E,\phi)$ is the Hankel transform of the track-length angular distribution resulting from pure elastic scattering together with an absorption coefficient q.

The structure functions F and F_{int} can be recovered from Eqs. (16) and (17) by application of the inverse transformation

$$F(E_0;E,\theta)=\int_0^\infty J_0(\theta\phi)\bar{F}(E_0;E,\phi)\phi d\phi. \tag{18}$$

We thus have an analytic expression for the track-length angular structure function of the electrons in the Tamm-Belenky model of the electron-photon cascade using a general expression for the elastic scattering cross section.

The "projected track-length angular structure function" $G(E_0;E,\theta)d\theta$ is defined by

$$G(E_0;E,\theta)=(1/\pi)\int_0^\infty F(E_0;E,(\theta^2+\eta^2)^{\frac{1}{2}})d\eta. \tag{19}$$

It represents the angular distribution that would be seen if all the tracks of the electrons were projected onto a plane containing the shower axis. Using the identity

$$\int_0^\infty J_0(\phi(\theta^2+\eta^2)^{\frac{1}{2}})d\eta=\frac{1}{\phi}\cos(\theta\phi), \tag{20}$$

we find that the projected structure function can be recovered from the Hankel transformed function $\bar{F}$ by the inversion formula

$$G(E_0;E,\theta)=(1/\pi)\int_0^\infty \cos(\theta\phi)\bar{F}(E_0;E,\phi)d\phi. \tag{21}$$

3. LANDAU APPROXIMATION AND ELIMINATION OF THE PRIMARY ENERGY

The solution obtained by Belenky for the angular distribution of the Tamm-Belenky model can now be obtained by the introduction of the Landau approximation. We simply substitute

$$\bar{W}(E,\phi)=-(E_s\phi/2E)^2 \tag{22}$$

into Eqs. (16) and (17), which then yield

$$\bar{F}_{\mathrm{int}}(E_0;E,\phi) =\left[1+\frac{1}{q}\left(\frac{E_s\phi}{2E_0}\right)^2\right]^{-\frac{1}{2}}\left[1+\frac{1}{q}\left(\frac{E_s\phi}{2E}\right)^2\right]^{-\frac{1}{2}}, \tag{23}$$

$$\bar{F}(E_0;E,\phi)=\left[1+\frac{1}{q}\left(\frac{E_s\phi}{2E_0}\right)^2\right]^{-\frac{1}{2}}\left[1+\frac{1}{q}\left(\frac{E_s\phi}{2E}\right)^2\right]^{-\frac{3}{2}}. \tag{24}$$

We see that the Hankel transform of the angular structure function depends upon the two variables ϕ/E and ϕ/E_0. It follows that the structure function itself is a function of the two variables $E\theta$ and $E_0\theta$. Now we can show that the dependence of this function upon the primary energy E_0 is of very little physical interest. This is because the track-length distribution is normally used as an approximation to the actual distribution at the depth of maximum development of the shower, and this is an approximation which is only valid when E_0 is much greater than E. We therefore have no *a priori* reason for believing that the dependence of the track-length distribution upon the primary energy bears any resemblance to the dependence of the distribution at maximum depth upon E_0. In fact it is easily shown that the second angular moment of our track-length distribution has a very different dependence on E_0 than has the second moment at maximum depth.

The various moments of the angular structure functions are easily obtained as the coefficients of the power series expansion of their Hankel transforms, thus:

$$\begin{aligned} \bar{F}(E_0;E,\phi)&=\int_0^\infty J_0(\theta\phi)F(E_0;E,\theta)\theta d\theta,\\ &=\sum_{n=0}^\infty \frac{\phi^{2n}}{4^n(n!)^2}\int_0^\infty \theta^{2n}F(E_0;E,\theta)\theta d\theta. \end{aligned} \tag{25}$$

Expanding Eqs. (23) in a series of powers of ϕ, we find that the second moment of the integral track-length structure function is

$$\langle\theta_{\mathrm{int}}^2\rangle_{\mathrm{Av}}=\frac{1}{2q}\left(\frac{E_s}{E}\right)^2[1+(E/E_0)^2]. \quad (26)$$

Thus the second integral angular moment decreases with increasing primary energy approaching, as E_0 tends to infinity, a value half of its value when E_0 equals E. On the other hand, the calculation by Chartres and Messel[10] of the second integral angular moment as a function of depth indicates that the value of the moment at the depth of maximum cascade development shows an *increase* with increasing primary energy.

This reasoning will still apply when we do not use the Landau approximation, for the second moment is not altered by the use of this approximation. We shall, therefore, for the remainder of this paper deal only with the angular structure function for infinite primary energy. We define

$$F(E,0)=\underset{E_0\to\infty}{\mathrm{limit}}F(E_0;E,0), \quad (27)$$

with a similar expression for the integral angular structure function. The inversion integral, Eq. (18), can now be evaluated analytically under the Landau approximation, yielding

$$F(E,\theta)=\int_0^\infty\frac{J_0(\theta\phi)\phi d\phi}{[1+(1/q)(E_s\phi/2E)^2]^{\frac{3}{2}}}=4q(E/E_s)^2\exp[-2q^{\frac{1}{2}}E\theta/E_s], \quad (28)$$

$$F_{\mathrm{int}}(E,\theta)=\int_0^\infty\frac{J_0(\theta\phi)\phi d\phi}{[1+(1/q)(E_s\phi/2E)^2]^{\frac{1}{2}}}=4q(E/E_s)^2\frac{\exp[-2q^{\frac{1}{2}}E\theta/E_s]}{2q^{\frac{1}{2}}E\theta/E_s}. \quad (29)$$

These expressions are greatly simplified if we express the structure function as a function of the single variable

$$u=2q^{\frac{1}{2}}E\theta/E_s \quad (30)$$

and renormalize it accordingly. We then have

$$F(u)=e^{-u};\quad F_{\mathrm{int}}(u)=e^{-u}/u. \quad (31)$$

4. CALCULATION OF THE ANGULAR STRUCTURE FUNCTION

To obtain a more accurate angular distribution than that given by the Landau approximation, we use for the elastic scattering cross section the simple analytic expression due to Goudsmit and Saunderson,[11] *viz.*,

$$W(E,\theta)=2W_0\theta_0^2/(\theta^2+\theta_0^2)^2,$$

TABLE I. Values of the parameters A and W_1 for several media.

Medium	Z	A (Mev)$^{-1}$	W_1
Carbon	6	123	1.60×10^5
Water	7.2	115	1.43×10^5
Air	7.4	114	1.41×10^5
Aluminum	13	93.8	9.85×10^4
Iron	26	71.4	5.98×10^4
Copper	29	68.0	5.44×10^4
Lead	82	34.8	1.41×10^4

where

$$\theta_0=\frac{1.13Z^{\frac{1}{3}}mc^2}{137E}[1.13+3.76(Z/137)^2]^{\frac{1}{2}}, \quad (32)$$

$$W_0\theta_0^2=\frac{137\pi(mc^2)^2}{E^2\ln(183Z^{-\frac{1}{3}})}[1+0.12(Z/82)^2]^{-1},$$

and m is the electronic mass. For the value of the angle θ_0, we have used the result of a more accurate estimation by Molière.[7]

This cross section yields a simple Hankel transform, namely

$$\bar{W}(E,\phi)=W_0[\theta_0\phi K_1(\theta_0\phi)-1], \quad (33)$$

where $K_1(x)$ is the modified Bessel function of the second kind.[12] Substituting this expression into Eqs. (17), (18), and (27) we find that the angular structure function becomes a function of $E\theta$ only, so renormalizing it by

$$\int_0^\infty F(E\theta)E\theta d(E\theta)=1, \quad (34)$$

we have

$$F(E\theta)=A^2\int_0^\infty\frac{J_0(AE\theta x)xdx}{1+W_1[1-xK_1(x)]}\times\exp\left\{-\int_0^x\left[1-\frac{1}{1+W_1[1-yK_1(y)]}\right]dy/y\right\}, \quad (35)$$

where

$$A^{-1}=E\theta_0 \quad (35a)$$

is a parameter, with the dimensions of energy, depending upon Z only; and

$$W_1=W_0/q, \quad (35b)$$

which is simply the ratio of the electron collision rates against elastic scattering and bremsstrahlung respectively, is also a function of the atomic number Z. Values of the parameters A and W_1 are entered in Table I for several media.

Owing to its dependence on the parameter W_1, the angular structure function is no longer a universal function valid for all media, as it was under the Landau approximation, but instead has to be recalculated for

[10] B. A. Chartres and H. Messel, Proc. Phys. Soc. (London) **A67**, 158 (1954).
[11] S. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, 24 (1940).
[12] G. N. Watson, *Bessel Functions* (Cambridge University Press, Cambridge, 1952), second edition, p. 78.

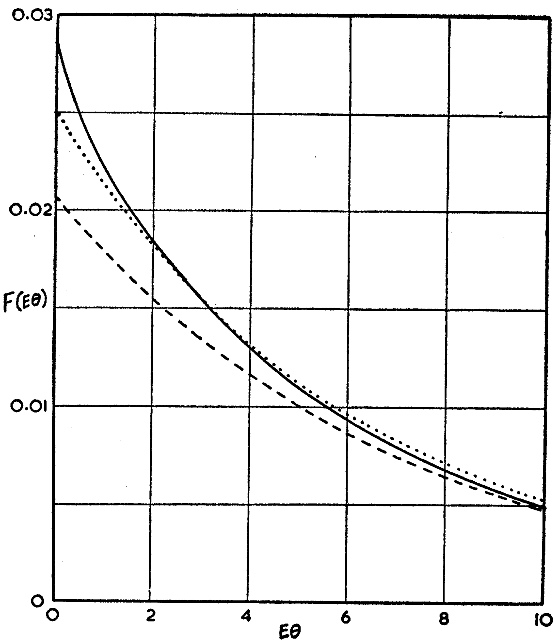

FIG. 3. Calculations of the track-length angular structure function $F(E\theta)E\theta d(E\theta)$ for electrons of energy E Mev in the Tamm-Belenky model of the electron-photon cascade in air. Full line: exact calculation; broken line: calculation based on the Landau approximation with $E_s=21$ Mev; dotted line: calculation based on the modified Landau approximation with $E_\theta=18.9$ Mev.

each medium. The full line in Fig. 3 is the result of a calculation of the angular structure function from Eq. (35) for air. The broken line in the same figure is the same function calculated under the Landau approximation. It can be seen that the Landau approximation is indeed very inaccurate at small angles. A comparison of Figs. 1 and 3 also shows that the error due to the use of the Landau approximation is very much greater than the differences between the structure functions derived from different models of the cascade. This justifies our use of the Tamm-Belenky model in order to eliminate the Landau approximation.

The projected angular structure function $G(E\theta)d(E\theta)$ is given by a similar expression to Eq. (35) with the kernel

$$A^2J_0(AE\theta x)x$$

replaced by

$$(1/\pi)A\cos(AE\theta x).$$

The full line in Fig. 4 is the projected angular structure function in air. The broken line is the same function calculated under the Landau approximation. It is apparent that the projected distribution is less sensitive than the unprojected one to the error involved in the Landau approximation.

5. SINGULARITY AT THE ORIGIN

In our numerical calculation of the angular structure function from Eq. (35), we have eliminated a singularity of the type $1/\theta$ from $F(E\theta)$. We now investigate this singularity.

The behavior of $F(E\theta)$ at small values of $E\theta$ depends upon the behavior of its Hankel transform at large values of the variable. Now $xK_1(x)$ is a monotonically decreasing function of x with the asymptotic behavior

$$xK_1(x)\sim(\tfrac{1}{2}\pi x)^{\frac{1}{2}}e^{-x}. \tag{36}$$

Hence for large values of x the exponential term in Eq. (35) behaves like

$$Cx^{-W_1/(1+W_1)}\simeq C/x, \tag{37}$$

where

$$C=\lim_{x\to\infty}\exp\left[-\int_0^x\left\{1-\frac{1}{1+W_1[1-yK_1(y)]}\right\}\frac{dy}{y}+\frac{W_1}{1+W_1}\ln x\right]\simeq 1\times10^{-3}\text{ in air.} \tag{37a}$$

Again, for sufficiently large values of x, the factor

$$1/\{1+W_1[1-xK_1(x)]\} \tag{38}$$

in Eq. (35) can be replaced by

$$1/(1+W_1)=7.1\times10^{-6}\text{ in air.} \tag{38a}$$

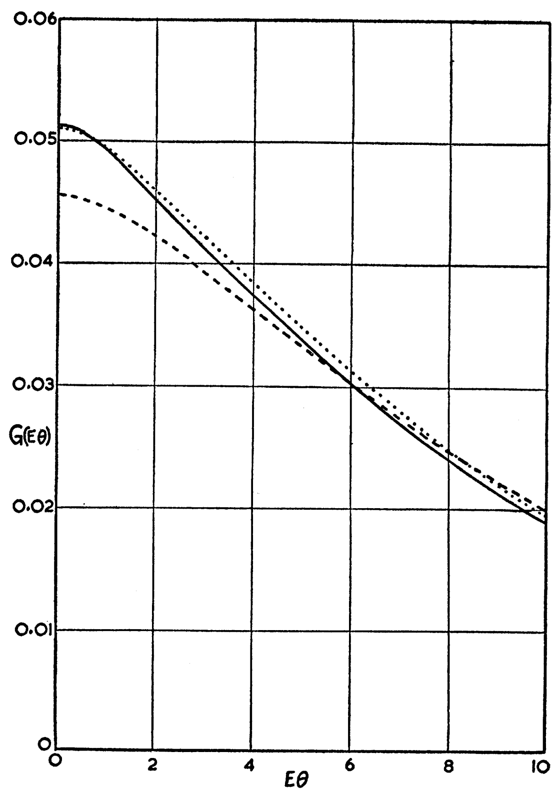

FIG. 4. Calculations of the projected track-length angular structure function $G(E\theta)d(E\theta)$ for electrons of energy E Mev in the Tamm-Belenky model of the electron-photon cascade in air. Full line: exact calculation; broken line: calculation based on the Landau approximation with $E_s=21$ Mev; dotted line: calculation based on the modified Landau approximation with $E_\theta=18.9$ Mev.

So the behavior of $F(E\theta)$ at very small $E\theta$ is, if we write $F_s(E\theta)$ for the singular part of $F(E\theta)$,

$$\begin{aligned} F_s(E\theta) &= A^2\times 7\times 10^{-9}\int_0^\infty J_0(AE\theta x)dx \\ &= 8\times 10^{-7}/E\theta. \end{aligned} \tag{39}$$

In our calculation of the angular structure function we eliminated this singular part, that is, we calculated

$$F_n = F - F_s. \tag{40}$$

The relative insignificance of the singular component F_s is made obvious when we note that it exceeds F_n in magnitude only for $E\theta$ less than 3×10^{-5} Mev; thus it would be completely invisible on the graph (Fig. 3) we have drawn. Furthermore, the proportion of the electrons which are represented by this singular component, which is obtained by integrating F_s from $E\theta=0$ to 3×10^{-5}, is

$$3\times 10^{-5}\times 8\times 10^{-7} = 2\times 10^{-11}. \tag{41}$$

Thus we see that a shower would have to contain 5×10^{10} electrons in order that just one electron could be observed which would verify the existence of this singularity in the angular distribution. We are therefore completely justified in neglecting the function $F_s(E\theta)$.

6. ANGULAR STRUCTURE FUNCTION AS A SERIES

The calculation we have performed of the angular structure function is one which must be repeated for each different medium as the parameter W_1, which varies from one medium to another, plays an important role in the expression. We can get around this difficulty by expanding the function in a manner analogous to that used by Molière[7] in his calculation of the angular distribution resulting from pure elastic scattering.

Expanding $xK_1(x)$ in a power series gives

$$\begin{aligned} 1-xK_1(x) &= \frac{x^2}{4}\sum_{n=0}^{\infty}\frac{(x^2/4)^n}{n!(n+1)!} \\ &\times\left[\frac{1}{n+1}+2\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}-\gamma\right)-\ln(x^2/4)\right], \end{aligned} \tag{42}$$

where $\gamma=0.5772157\cdots$. The approximation inherent in the Molière method essentially consists of taking only the first term in this series. Bethe[13] has justified this (in the case of multiple scattering) by showing that it is equivalent to using the multiple-scattering approximation for all scattering events that give deflections through angles small enough that the value of the angle θ_0 is significant. This argument is not sufficient here because, as we have seen, the Landau approximation is inherently less accurate than the multiple-scattering approximation. However, this approximation is justified in our case by the fact that in our numerical evaluation of the expression (35) we found that, for all values of $E\theta$ used excepting only $E\theta=0$, the values of x that contributed to the integral were so small that the first term of Eq. (42) was sufficiently accurate to yield the integral to five significant figures.

Substituting this approximate expression for $xK_1(x)$ and $yK_1(y)$ into Eq. (35) gives an expression for $F(E\theta)$ which can be considerably simplified by the use of the new angular variable,

$$u = 2q^{\frac{1}{2}}E\theta/E_\theta = 3.026(E\theta/E_\theta), \tag{43}$$

where E_θ, our new characteristic scattering energy which replaces E_s [compare Eq. (30)], is defined by

$$\frac{A^2E_\theta^2}{q} = W_1\left[1-2\gamma+\ln\left(\frac{A^2E_\theta^2}{q}\right)\right]. \tag{44}$$

We also define

$$B = A^2E_\theta^2/qW_1. \tag{45}$$

Then Eq. (35) becomes

$$\begin{aligned} F(u) &= \int_0^\infty \frac{J_0(ux)xdx}{1+x^2-(x^2/B)\ln(x^2)} \\ &\times\exp\left\{-\int_0^x\left[1-\frac{1}{1+y^2-(y^2/B)\ln(y^2)}\right]\frac{dy}{y}\right\}. \end{aligned} \tag{46}$$

We now expand this expression in a series of powers of $1/B$. Writing

$$F(u) = \sum_{n=0}^{\infty}\frac{F_n(u)}{B^n}, \tag{47}$$

we obtain

$$\begin{aligned} F_0(u) &= e^{-u}, \\ F_n(u) &= \int_0^\infty J_0(ux)f_n(x^2)xdx, \end{aligned} \tag{48}$$

where

$$f_1(x) = \frac{(3x/2)\ln x - \frac{1}{2}(1+x)\ln(1+x)}{(1+x)^{5/2}}, \tag{49}$$

$$\begin{aligned} f_2(x) = (1+x)^{-\frac{7}{2}}\bigg[&\frac{15}{8}\left(\frac{x\ln x}{1+x}\right)^2-\frac{3}{4}\left(\frac{x\ln x}{1+x}\right)\ln(1+x) \\ &+\tfrac{1}{8}[\ln(1+x)]^2+\frac{1}{2}\left(\frac{x\ln x}{1+x}\right) \\ &-\tfrac{1}{8}\ln(1+x)-\frac{1}{2}\int_0^x\frac{\ln y}{1+y}dy\bigg]. \end{aligned} \tag{50}$$

We now have a solution in terms of the two parameters E_θ and B which is of such a form that a change of these parameters is easily made without computing the in-

[13] H. A. Bethe, Phys. Rev. **89**, 1256 (1953).

TABLE II. Typical values of B and E_θ.

Medium	Z	$E\theta$ (Mev)	B
Carbon	6	18.8	14.58
Water	7.2	18.9	14.46
Air	7.4	18.9	14.44
Aluminum	13	19.0	14.06
Copper	29	19.1	13.42
Lead	82	17.9	11.96

tegrals over again. Typical values of B and E_θ are given in Table II, while values of $F_0(u)$, $F_1(u)$, and $F_2(u)$ are given in Table III—The values of $F_2(u)$ are not very accurate and should be treated as order-of-magnitude estimates only.

A very interesting point which arises from this expansion of the structure function is that the first term in the series is of the same form as the solution given by the Landau approximation. Hence, for applications in which a high degree of accuracy is not required, we can still use the Landau approximation as long as we replace the characteristic scattering energy E_s by the value of E_θ appropriate to the medium. This procedure is then equivalent to using the exact scattering cross section but taking only the first term of the series in Eq. (47). The dotted line in Fig. 3 is the angular structure function found for air when this "modified Landau approximation" is used. It can be seen that the modification of the value of E_s greatly improves the accuracy of the distribution at small angles, although it still does not reproduce the rather sharp peak of the more exact function.

When a higher accuracy is required, more terms of the series equation (47) must be used. The three curves in Fig. 5 give the percentage error involved in the use of 1, 2, or 3 terms respectively of this series. These errors were found by comparing the results of this calculation with that of the exact expression, Eq. (35).

The projected structure function $G(u)$ can be expanded in exactly the same way, yielding

$$G_0(u)=(u/\pi)K_1(u),$$
$$G_n(u)=\frac{1}{\pi}\int_0^\infty \cos(ux)f_n(x^2)dx. \tag{51}$$

Values of $G_0(u)$ and $G_1(u)$ are given in Table IV. Once again the first term, $G_0(u)$, is identical in form to the

TABLE III. Values of $F_0(u)$, $F_1(u)$, and $F_2(u)$.

u	$F_0(u)$	$F_1(u)$	$F_2(u)$
0	1.0000	1.386	7
$\frac{1}{8}$	0.8825	0.302	2.5
$\frac{1}{4}$	0.7788	0.154	1.2
$\frac{1}{2}$	0.6065	0.005	0.2
1	0.3679	−0.081	−0.08
2	0.1353	−0.046	
4	0.0183	0.004	
8	0.0003	0.001	

solution given by the Landau approximation but with the modified value of E_s. The dotted line in Fig. 4 is the result given by the use of the modified Landau approximation. Owing to the relative insensitivity of the projected function to errors in the scattering cross section, only the two terms G_0 and G_1 are necessary to give a distribution indistinguishable from the one given by the more exact calculation.

7. DISCUSSION

We have presented here the results of a calculation of the track-length angular structure function and the projected track-length angular structure function of electrons in an electron-photon cascade. These functions

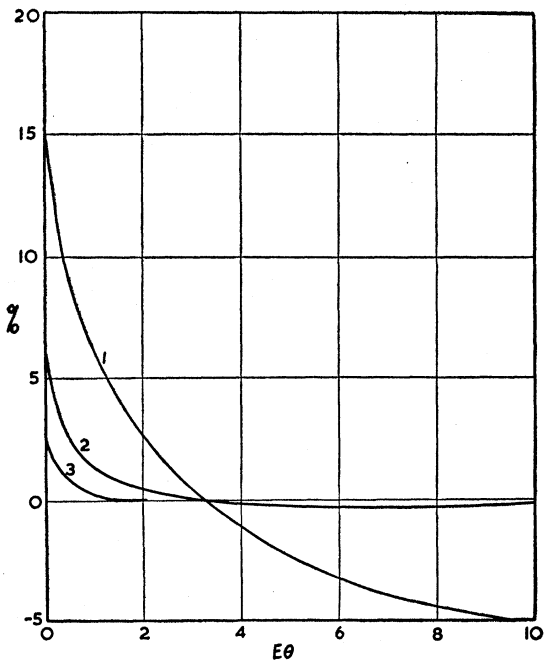

FIG. 5. The percentage error involved in the calculation of the track-length angular structure function of electrons in the Tamm-Belenky model of the electron-photon cascade using 1, 2, or 3 terms, respectively, of the series equation (47).

are a very good approximation to the analogous distributions at the depth of maximum development of a cascade—the approximation improving with increasing values of E_0/E.

The results of this calculation are expressed in the form of values of five universal functions F_0, F_1, F_2, G_0, and G_1. Using the values of the two parameters E_θ and B appropriate to the medium under consideration, the angular structure functions can be obtained very easily from these functions by the use of Eqs. (43) and (47). The error involved in this estimation is less than 1% for all angles except, in the case of the unprojected function, values of $E\theta$ less than 1 Mev where the error never exceeds 3 percent.

The only remaining source of error in this calculation is that which arises from the use of the Tamm-Belenky model to represent the electron-photon cascade. We can estimate this error by comparing the angular distributions yielded by this and other models under the Landau approximation. This is done in Figs. 1 and 2. When we note that the calculations of Molière and of Kalos and Blatt also used an approximate model of the cascade and that the angular distribution of Eyges and Fernbach was obtained from the angular moments by a graphical method, and when we compare the very small differences between the results given by these different methods with the larger error due to the use of the Landau approximation, we see that our calculation is, indeed, the most accurate estimation of the track-length angular distribution yet made.

However, we must point out three limitations on the range of validity of our calculation.

(1) The first is that due to the use of the "small angle approximation." The derivation of Eq. (4) is based upon the assumption that all angles θ are so small that the angle can be equated to its sine or tangent. As the mean angle is of the order of E_s/E [see Eq. (26)], this limits the range of validity to energies much greater than 21 Mev.

TABLE IV. Values of $G_0(u)$ and $G_1(u)$.

u	$G_0(u)$	$G_1(u)$
0	0.3183	0.0000
$\frac{1}{8}$	0.3116	−0.0092
$\frac{1}{4}$	0.2982	−0.0202
$\frac{1}{2}$	0.2636	−0.0359
1	0.1916	−0.0415
$\frac{3}{2}$	0.1324	−0.0295
2	0.0890	−0.0143

(2) The second limitation is that due to the neglect of ionization loss. This limits the range of validity to energies much greater than the critical energy whose value decreases with increasing atomic number, ranging from 84 Mev in air to 7.6 Mev in lead. Thus whether restriction (1) or (2) is the more important depends upon the medium.

(3) The third restriction is due to the neglect of the finite size of the scattering nuclei in the derivation of the elastic scattering cross section, Eq. (32). This results in the cross section, and hence the final angular distribution, being inaccurate for all angles θ of the order of magnitude of or greater than θ_1, where

$$E\theta_1 \simeq 100 Z^{-\frac{1}{3}} \text{ Mev.} \tag{52}$$

This limitation is most restrictive in the heaviest media; in lead we have $E\theta_1 \simeq E_s \simeq 20$ Mev.

The results of our calculation settle for once and for all the question of the validity of the Landau approximation at both small and large angles. It has been realized in the past that the Landau approximation could not possibly yield correct results for large values of θ, but, by using physical arguments, it was strongly argued that the approximation could have little or no effect on calculations for small values of θ. The results presented in Fig. 3 show how wrong such physical arguments were and that the use of the Landau approximation yields incorrect results for both large and small θ. Incidentally, it also shows the danger of using purely physical arguments to justify mathematical approximations.

The present result also shows that all previous calculations on the radial distribution function which used the Landau approximation contain large inaccuracies at small and large distances from the shower axis. Agreement between theory and experiment may be due to the inability of experiments to yield sufficiently accurate results.

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