greater than $\frac{1}{2}$. If the ground state spin is assumed to be $\frac{1}{2}$, then these two excited levels can be used to calculate a decoupling parameter $a=-0.071$, and the next rotational level is predicted at 134.5 kev. The vibration-rotation interaction correction, which is proportional to $I^2(I+1)^2$, for this 7/2 level amounts to about -2 kev, bringing the energy of the expected level into good agreement with the one observed at 132 kev. This apparent sequence of rotational levels seems to be fortuitous, however, since it is not supported by the observed ground state spin, Coulomb excitation studies,[3] or character of the beta transitions.

TABLE III. The resolution of the beta spectrum of Tb^{161}.

Maximum energy, kev	Percent abundance	Log ft	ΔI, parity
531±10	68%	6.7	0 or 1, yes
447±10	22%	6.9	0 or 1, yes
405±10	10%	7.2	0 or 1, yes

PHYSICAL REVIEW VOLUME 104, NUMBER 2 OCTOBER 15, 1956

Fluctuations of Nuclear Reaction Widths*

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(Received July 5, 1956)

The fluctuations of the neutron reduced widths from the resonance region of intermediate and heavy nuclei have been analyzed by a statistical procedure which is based on the method of maximum likelihood. It is found that a chi-squared distribution with one degree of freedom is quite consistent with the data while a chi-squared distribution with two degrees of freedom (an exponential distribution) is not. The former distribution corresponds to a Gaussian distribution for the reduced-width amplitude, and a plausibility argument is given for it which is based on the consideration of the matrix elements for neutron emission from the compound nucleus and of the central limit theorem of statistics. This argument also suggests that within the framework of the compound-nucleus theory all reduced-width amplitudes have Gaussian distributions, and that many of the distributions for the various channels may be independent. One consequence of the latter suggestion is that the total radiation width for a given spin state which is formed in neutron capture will be essentially constant, in agreement with some observations, because it is the sum of many partial radiation widths. The fluctuations of the provisional fission widths of U^{235} are best described by a chi-squared distribution with about $2\frac{1}{2}$ degrees of freedom, indicating that there are effectively only a few independently contributing fission channels.

I. GENERAL REMARKS

SEVERAL hundred resonances have been observed in the Brookhaven fast chopper work on total neutron cross sections of intermediate and heavy nuclei in the neutron energy range up to several hundred electron volts.[1] For many of these resonances it has been possible to deduce the neutron width Γ_n and the velocity-independent reduced width $\Gamma_n{}^0=\Gamma_n/E_0{}^{\frac{1}{2}}$, where E_0 is the resonance energy.[2,3] In a typical sample of from ten to fifteen resonances the reduced widths are observed to fluctuate violently, the ratio of the largest to the smallest being as high as several hundred. Indeed, Hughes and Harvey[4] have recently shown that the aggregate of the reduced-width data for fourteen nuclides is reasonably consistent with exponential-like distributions, one of the form $x^{-\frac{1}{2}}\exp(-\frac{1}{2}x)$ and another of the form $\exp(-x)$, where $x=\Gamma_n{}^0/\langle\Gamma_n{}^0\rangle_{\text{Av}}$. In view of the importance to nuclear reaction theory and to nuclear engineering of knowing which of the two distributions is more likely to be the correct one, we have made a more quantitative statistical analysis of the data. This analysis shows that the former distribution is quite consistent with the data, whereas to the latter one it assigns a very small probability of being correct. The most significant consideration of our analysis which enables this distinction to be made is the accounting for the possibility that levels with small widths, of which there are predicted to be a relatively large number in the former distribution, will not be observed. A second consideration which also enhances this distinction is the accounting for the errors introduced when finite-sample averages are used as estimates for the infinite-sample (population) averages $\langle\Gamma_n{}^0\rangle_{\text{Av}}$.[5]

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] Seidl, Hughes, Palevsky, Levin, Kato, and Sjöstrand, Phys. Rev. **95**, 476 (1954).

[2] Harvey, Hughes, Carter, and Pilcher, Phys. Rev. **99**, 10 (1955).

[3] D. J. Hughes and J. A. Harvey, *Neutron Cross Sections*, Brookhaven National Laboratory Report BNL-325 (Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1955).

[4] D. J. Hughes and J. A. Harvey, Phys. Rev. **99**, 1032 (1955).

[5] The existence of a reasonably well-defined average neutron reduced width is assumed here. The existence of such an average is suggested by the work of Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954) and of Lane, Thomas, and Wigner, Phys. Rev. **98**, 693 (1955).

The $x^{-\frac{1}{2}}\exp(-\frac{1}{2}x)$ distribution is also more reasonable on theoretical grounds. It corresponds to a Gaussian distribution for the reduced-width amplitude (that is, for $x^{\frac{1}{2}}$) for neutron emission from the compound nucleus while the $\exp(-x)$ reduced-width distribution corresponds to an amplitude distribution which assigns a zero probability for a zero amplitude. The reduced-width amplitude is proportional to an integral (matrix element) of the product of a wave function for the state of the compound nucleus and a wave function for the neutron channel, the integral being over the nuclear configuration space of many dimensions (essentially $3A$, where A is the number of nucleons).[6] In the spirit of the compound-nucleus theory, the former wave function is presumed to be very complex as a result of the strong nuclear interactions, and the wave functions for the various states are presumed to be essentially unrelated to each other. One may regard the matrix element as composed of contributions from many "cells" of the configuration space, the sign of the contribution from a particular cell being positive with the same probability that it is negative, and the sign and magnitude of a particular contribution being random from level to level and independent of the signs and magnitudes of the contributions from the other cells. The over-all size of each cell may be supposed to be such that each linear dimension is about 1×10^{-13} cm, the characteristic wavelength of a nucleon in the nucleus, so that in a heavy nucleus there will be a very large number of independently-contributing cells. In consideration of the central-limit theorem of statistics,[7] it may be expected that the probability distribution for the matrix elements (that is, for the sum over cells) will be approximately normal (that is, Gaussian) with zero mean and asymptotically normal in the limit as the number of effective independently-contributing cells becomes infinitely large (as in a hypothetical, infinitely heavy nucleus).[8]

The above arguments are not intended to constitute a derivation for the normal distribution but they do make it a plausible one. Departures from normality are to be expected. For example, according to the Wigner limit the reduced widths Γ_n^0 cannot exceed several thousand electron volts. However, this limit is millions of times larger than typical average reduced widths of heavy nuclei, so that the truncation occurs far out in the tail of the distribution and will not significantly affect the present considerations.[9]

It seems reasonable to hypothesize that all reduced-width amplitudes for levels of the compound nucleus (that is, levels of fairly high excitation) will be distributed approximately normally with zero means when sampled from level to level[10]; the variances of these distributions are just the average reduced widths. The amplitudes associated with the *partial* radiative capture widths for the compound states will also be assumed to have such a distribution because these amplitudes are proportional to matrix elements involving the wave functions of the compound states (as well as to wave functions for the final states). On the other hand, the distribution for the *total* radiation width for a particular spin state which is involved in neutron capture is expected to be very narrow, its variance being inversely proportional to the number of contributing partial widths. However, this demonstration (Sec. III) involves the additional assumption that the distributions for the various partial widths are independent,[11] which assumption is more open to question than is the normality assumption. Although the independence assumption may be applicable to the bulk of the transitions to states of fairly high excitation, which states are dense and presumably complex like the states of the compound nucleus, it may not be applicable to the direct transitions to low-lying states which may not be sufficiently complex. That is, several low-lying states might be identical in all respects except for a single property which is not involved in the radiative matrix elements except possibly through a common factor; the distributions of the matrix elements to these several states would then be completely correlated, although they could still be normal. However, for the treatment of III such transitions are combined as a single "independently-contributing" transition. The above-mentioned variance of the distribution for the total radiation width is then inversely proportional to the number of "independently-contributing" partial widths rather than to the actual number of partial widths. It is hard to say just what difference is to be expected of these two numbers; it would depend on the relative importance of the contributions from transitions to the low-lying

[6] More precisely, the matrix element is a "surface" integral in the sense that the coordinate for the relative separation of the neutron and the residual nucleus is set equal to the channel radius. See Eq. (17) of E. P. Wigner and L. Eisenbud, Phys. Rev. 72, 29 (1947) for the precise definition.

[7] See, for example, Harald Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1945), Sec. 17.4.

[8] In this connection it is important to realize that the matrix elements can always be made real. See the discussion following the equation referred to in reference 6.

[9] The relation between the reduced width γ^2 of the Wigner-Eisenbud theory (reference 6) and the reduced width Γ_n^0 which is commonly used in the discussions of low-energy neutron resonance is $\Gamma_n^0=\Gamma_n/E_0(\text{ev})^{\frac{1}{2}}=0.4390\gamma^2a\times10^{-3}$, where γ^2 has the same energy unit as Γ_n^0, and a is the channel radius in units of 10^{-13} cm; although γ^2 refers to the center-of-mass system, the energies of Γ_n^0 and E_0 refer to the laboratory system. (As it is not necessary to specify a nuclear radius in the consideration of s-wave neutron widths, one may prefer to regard the reduced width as the product γ^2a, which has the dimensions of energy times distance.) The upper limit of γ^2 is approximately $\hbar^2/Ma^2$, where M is the neutron mass, or $40/a^2$ Mev when a is specified in the above unit. The upper limit of Γ_n^0 is therefore $18/a$ kev.

[10] The proton reduced widths of the 2^+ levels of Cu^{59} which are excited when Ni^{58} is bombarded by protons with energies from $2\frac{1}{2}$ to 5 Mev are observed to be distributed in an exponential-like manner [J. P. Schiffer (private communication)]. For an abstract on this experiment, see Moore, Schiffer, and Class, Bull. Am. Phys. Soc. Ser. II, 1, 39 (1956).

[11] Distributions which are independent are of course also uncorrelated, but the converse is not generally true. However, for the discussions which follow it is well to keep in mind that normal distributions which are uncorrelated are also independent; (see Sec. 24.1 of reference 7).

states compared to the contributions from the states of fairly high excitation.

The distribution of fission widths is considered in Sec. IV. Offhand, one might expect this distribution to be very narrow because, like radiative capture, there are many final states for the fission process, one for each possible fragment pair and additional ones for each possible pair of excitation states that are energetically accessible. The amplitudes associated with the partial widths for these states are expected to have normal distributions. However, if in the fission act the nucleus passes through only one or a few well-ordered nuclear states (fission channels) which describe the saddle-point configuration,[12] the various partial widths would be highly correlated (see the appendix). Indeed, if there were only one such state, the partial width distributions would be completely correlated, and the total fission width would be expected to have the one-channel distribution $x^{-\frac{1}{2}}\exp(-\frac{1}{2}x)$.

The distributions of nuclear reaction widths enter into the considerations of the averages and the fluctuations of the cross sections for nuclear reactions which proceed through compound-nucleus formation. Although in some of the previous published work the randomness of the signs of the reduced-width amplitudes was considered, it was generally not suspected that the fluctuations of the magnitudes were large enough to warrant detailed considerations.[13] The average cross section for compound elastic scattering of neutrons, for example, is now found to depend critically on the extent of the fluctuations of the neutron reduced widths. Thus, the cross section predicted using the $x^{-\frac{1}{2}}\exp(-\frac{1}{2}x)$ distribution is twice as large as that predicted using the less-violent $\exp(-x)$ distribution, and in the excitation region where the levels overlap it is many times larger than the prediction for constant widths.[14] As another example, the average capture-to-fission ratio of U^{235} is found to exceed the ratio of the average capture width to the average fission width by an amount which depends on the extent of the fission width fluctuations.[15] These matters will be discussed later in more detail.

II. NEUTRON WIDTHS

The distributions $x^{-\frac{1}{2}}\exp(-\frac{1}{2}x)$ and $\exp(-x)$ belong to the class of chi-squared distributions

$$P(x;\rho)dx=\Gamma(\rho)^{-1}(\rho x)^{\rho-1}e^{-\rho x}\rho dx, \qquad (1)$$

where $\nu=2\rho$ is a parameter which is referred to in the literature on statistics as the number of degrees of freedom. This terminology is also appropriate for physical discussions, and the distributions under investigation will be referred to as chi-squared distributions with one and two degrees of freedom, respectively. The above expression is recognized as being proportional to the integrand of the integral which defines the gamma function; the gamma function $\Gamma(\rho)$ in (1) thus serves to normalize $P(x)$ to unity. It is evident that $\langle x\rangle_{\text{Av}}=1$ and that the variance of $x(\text{var}x\equiv\langle x^2\rangle_{\text{Av}}-\langle x\rangle_{\text{Av}}{}^2)$ is equal to ρ^{-1} so that the parameter ρ (or, equivalently, the number of degrees of freedom ν) characterizes the width of the distribution, the greater the number the narrower the distribution. Chi-squared distributions for several integral values of ν are drawn in Fig. 1: when $\rho\geqslant 1$, the maxima (most-probable values) appear at $x=1-\rho^{-1}$; when $\rho\leqslant 1$, the function becomes infinite at the origin; and when $\rho=\infty$, it reduces to a delta function at $x=1$. The chi-squared distribution can thus describe a wide variety of distributions, and the object is now to determine the range of ν (considered as a continuous variable) that is reasonably consistent with the data and to test the hypotheses that the "true" distribution has one or that it has two degrees of freedom.[16]

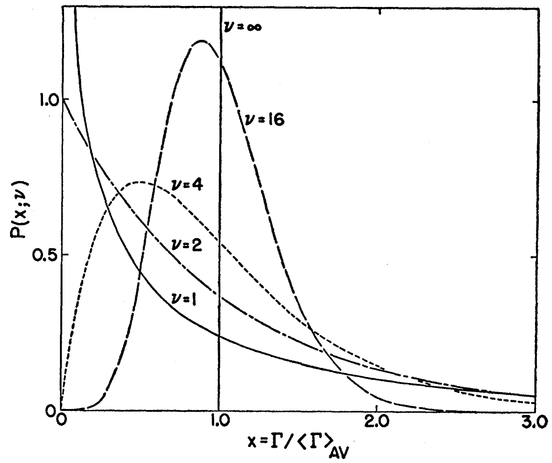

FIG. 1. The chi-squared distribution (1) is plotted for $\nu=1$, 2, 4, 16, and ∞ degrees of freedom. The abscissa x is the ratio of the width Γ to its average $\langle\Gamma\rangle_{\text{Av}}$. Note that $\nu=2\rho$.

A statistically efficient method (that is, one that admits a small uncertainty) for determining the best-fitting value of the parameter ρ is the maximum-likelihood method.[17] According to this method the most-probable value of ρ is the one that maximizes the logarithm of the likelihood function, which is the product of the $P(x_i;\rho)$ for the set of m measurements x_i. In this way it is found that the most probable value of ρ is the

[12] A. Bohr, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy* (Columbia University Press, New York, 1956), Vol. 2, Report P/911.

[13] See, for example, R. G. Thomas, Phys. Rev. **97**, 224 (1955).

[14] See Eq. (45) of reference 13.

[15] Sophie Oleksa (to be published).

[16] The distribution $x^{-\frac{1}{2}}\exp(-x^{\frac{1}{2}})$ was also tested by Hughes and Harvey,[4] and found to be inconsistent with the aggregate of the data. This distribution was observed by Bethe to give a good account of the reduced widths of U^{235} [H. A. Bethe, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy* (Columbia University Press, New York, 1956), Vol. 4, Report P/585]. We will not consider it here because the chi-squared distribution seems general enough, and for the same reason we will not consider any of a large variety of other distributions that might be proposed.

[17] See, for example, M. G. Kendall, *The Advanced Theory of Statistics* (Charles Griffin and Company, Ltd., London, 1946), Vol. II, Chap. 17.

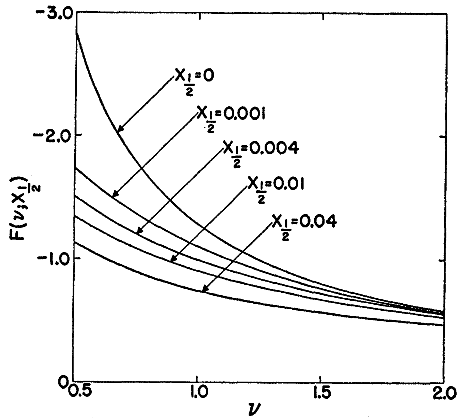

FIG. 2. Plot of the function $F(\rho; x_{\frac{1}{2}})$ of (11) for several values of the parameter $x_{\frac{1}{2}}$, the value of x at which the efficiency for detecting levels is one-half. The most probable value of $x_{\frac{1}{2}}$ is 0.01. When $x_{\frac{1}{2}}=0$ this function reduces to the function defined by (2a). Note that $\nu=2\rho$.

one that satisfies the transcendental equation (the subscript and superscript of Γ_n^0 being dropped)

$$\frac{1}{m}\sum_{i=1}^{m}\ln(\Gamma_i/\langle\Gamma\rangle_{\text{Av}})-F(\rho)=0, \tag{2}$$

where

$$F(\rho)=\psi(\rho)-\ln\rho, \tag{2a}$$

$\psi(\ )$ being the derivative of the logarithm of the gamma function. The curve in Fig. 2 labeled $x_{\frac{1}{2}}=0$ is the function $F(\rho)$, and it may be used for the determination of ρ when the sum in (2) is known. The asymptotic expression for the variance of this determination is

$$\text{var}_s\rho=m^{-1}[\psi'(\rho)-\rho^{-1}]^{-1}, \tag{2b}$$

where ψ' is the derivative of ψ.[18]

For U^{238} there are 11 reduced-width measurements that can be used.[19] By approximating the population average $\langle\Gamma\rangle_{\text{Av}}$ by the sample average,

$$\bar{\Gamma}=\frac{1}{m}\sum_{i=1}^{m}\Gamma_i, \tag{3}$$

a value of -1.22 is obtained for the sum in (2), and referring to the $x_{\frac{1}{2}}=0$ curve of Fig. (2) a value of $\nu=1.04$ degrees of freedom is deduced. According to (2b) the standard deviation of this determination is 0.36 degree of freedom, indicating that the value $\nu=1$ is consistent with the data whereas the value $\nu=2$ is not.

Although the above example does illustrate the main features of the maximum likelihood analysis, several important refinements are called for: 1. The sample sizes available for the various nuclides are not large, and a correction must be made for a possible difference of the sample average from the population average. 2. Samples of reasonable size are available for about 15 nuclides, and an analysis is desired which makes use of all of these for the estimate of the best universal value of ν. 3. The experimental uncertainties of the reported reduced widths should be taken into account. 4. As a consequence of instrumental limitations, levels with small widths may escape detection, and although there may be only a few of these, rather large errors can be introduced into the ν determination, especially when ν is small, as it seems to be. These complications will now be considered.

1. The sample average (3) has a chi-squared distribution with $m\nu$ degrees of freedom[20]:

$$P(\bar{\Gamma};\rho)d\bar{\Gamma}=\frac{1}{\Gamma(m\rho)}\left(\frac{m\rho\bar{\Gamma}}{\langle\Gamma\rangle_{\text{Av}}}\right)^{m\rho-1} \times\exp\left(-\frac{m\rho\bar{\Gamma}}{\langle\Gamma\rangle_{\text{Av}}}\right)d\left(\frac{m\rho\bar{\Gamma}}{\langle\Gamma\rangle_{\text{Av}}}\right). \tag{4}$$

From this distribution it is evident that

$$\langle\bar{\Gamma}\rangle_{\text{Av}}=\langle\Gamma\rangle_{\text{Av}}, \tag{4a}$$

$$\text{var}_s\bar{\Gamma}=\langle\Gamma\rangle_{\text{Av}}^2/m\rho, \tag{4b}$$

$$\langle\ln(\bar{\Gamma}/\langle\Gamma\rangle_{\text{Av}})\rangle=F(m\rho), \tag{4c}$$

and that the most probable value of $\bar{\Gamma}$ is $[1-(m\rho)^{-1}]\times\langle\Gamma\rangle_{\text{Av}}$. Consider the function

$$\Phi=\frac{1}{m}\sum_{i=1}^{m}\ln(\Gamma_i/\bar{\Gamma})+F(m\rho)-F(\rho). \tag{5}$$

Using (4c) it is evident that

$$\langle\Phi\rangle_{\text{Av}}=0, \tag{5a}$$

and it may be shown using (1) and (4) that

$$\text{var}_s\Phi=m^{-1}\psi'(\rho)-\psi'(m\rho). \tag{5b}$$

The best estimate for ρ may be taken as the solution to

$$\Phi(\rho)=0. \tag{5c}$$

According to the central-limit theorem, as the sample size increases the distribution of the function Φ asymptotically approaches normality:

$$P(\Phi)d\Phi\sim(2\pi\ \text{var}\Phi)^{-\frac{1}{2}}\exp(-\Phi^2/2\ \text{var}\Phi)d\Phi. \tag{5d}$$

It will be assumed that the samples are large enough to justify a normal approximation (5d), and the hypotheses may then be tested by consideration of the error

[18] For tables of $\Gamma(\ )$, ψ, and ψ' see Harold T. Davis, *Tables of Higher Mathematical Functions* (The Principia Press, Inc., Bloomington), Vol. I (1933) and Vol. II (1935).

[19] These are the measurements which appear above the dotted line in the table of resonance parameters of heavy nuclei in reference 3. The listing below the dotted line is not considered to be complete.

[20] See M. G. Kendall, *The Advanced Theory of Statistics* (Charles Griffin and Company, Ltd., London, 1946), Vol. I, Example 10.11. This result may readily be verified by noting that its Laplace transform is equal to the mth power of the Laplace transform of the distribution (1), in agreement with the convolution theorem.

function

$$\mathcal{P}=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_{|\alpha|}^{\infty}\exp(-\tfrac{1}{2}z^2)dz, \tag{5e}$$

where

$$\alpha=\Phi/(\mathrm{var}\Phi)^{\frac{1}{2}}.$$

It is evident from (4b) that the variance of an average determination of a sample of widths which obey a $\nu=1$ distribution is twice as large as it is for a $\nu=2$ distribution. In many of the published listings of average reduced widths and of strength functions, the uncertainties were arrived at assuming $\nu=2$; these uncertainties should be increased accordingly if the $\nu=1$ distribution is accepted as the correct one.[21] For the U^{238} data the average of the 11 widths is 2.7 mv (1 mv$\equiv 10^{-3}$ ev), and the standard deviation corresponding to (4b) is 1.2 mv, indicating that the errors of the average extimates from the typical samples available for heavy elements are apt to be rather large.

2. It is a straightforward matter to modify the above formulas for treatment of the composite data. In place of (5), one considers the function

$$\Phi=(1/m)\sum_{ij}\ln(\Gamma_{ij}/\bar{\Gamma}_j)+\sum_j(m_j/m)F(m_j\rho)-F(\rho), \tag{6}$$

where Γ_{ij} is the ith width of the jth nuclide; m_j is the number of the widths of the jth nuclide; and $m=\sum_j m_j$. It may be verified that

$$\langle\Phi\rangle_{\mathrm{Av}}=0, \tag{6a}$$

and that

$$\mathrm{var}_s\Phi=(1/m)\psi'(\rho)-\sum_j(m_j/m)^2\psi'(m_j\rho). \tag{6b}$$

For even the smallest sample which is used ($m_j=3$), it is sufficiently accurate to use for $F(m_j\rho)$ and for $\psi'(m_j\rho)$ just the first few terms of the asymptotic expansions[22]:

$$\psi(z)-\ln z\sim-\tfrac{1}{2}z^{-1}-(1/12)z^{-2}+(1/120)z^{-4}-\cdots, \tag{7a}$$

$$\psi'(z)\sim z^{-1}+\tfrac{1}{2}z^{-2}+\tfrac{1}{6}z^{-3}-(1/30)z^{-5}+\cdots. \tag{7b}$$

3. If the possible values of the errors of the Γ_{ij} in the first sum in (6) are assumed to be small, independent, and normally distributed with zero means and with standard deviations σ_{ij}, the additional contribution to the variance of Φ is found to be

$$\mathrm{var}_e\Phi=m^{-2}\sum_{ij}[\sigma_{ij}(\Gamma_{ij}-\bar{\Gamma}_j)/\Gamma_{ij}\bar{\Gamma}_j]^2. \tag{8}$$

The additional contribution to the variance (4b) of $\bar{\Gamma}_j$ is

$$\mathrm{var}_e\bar{\Gamma}_j=m_j^{-2}\sum_{i=1}^{m_j}\sigma_{ij}^2. \tag{9}$$

For the evaluation of (5e), the contributions (5b) and (8) may be added as though they were independent.

[21] There are several indirect methods for determining the strength function which effectively make use of very large samples, and they are therefore not subject to this uncertainty: see S. E. Darden, Phys. Rev. **99**, 748 (1955); D. J. Hughes and V. E. Pilcher, Phys. Rev. **100**, 1249(A) (1955).

[22] Erdélyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, Sec. 1.18, Eq. (7).

4. Since the instrumental resolution depends upon the neutron energy (as does the average neutron width), the efficiency for detecting levels will also depend on the neutron energy as well as on the average reduced width of the sample. An analysis which accounts for these dependences has not been devised, and we will have to be content with using an over-all efficiency factor $E(x)$ for all energies and for all samples. The "experimental" distribution function will be taken as the product of the "theoretical" factor, which is given by (1), and the "experimental" factor $E(x)$, this product being then renormalized to unity. According to an analysis by Harvey,[23] the over-all efficiency is one-half at a value $x\equiv x_{\frac{1}{2}}=\mathrm{antilog}_{10}(-2.00\pm0.15)$; the efficiency increases from 30 to 70% for a factor 10 increase in x and from 15 to 85% for a factor 100 increase in x; it is uncertain outside of the latter interval. As a function of $\log_{10}x$, the efficiency curve may be approximated closely by an error function with a standard deviation of 0.85. However, for the present considerations it is more convenient to represent it by a function of the form

$$E(x;x_{\frac{1}{2}};\sigma)=1-\exp[-(\kappa x)^\sigma], \tag{10}$$

where

$$\kappa=(\ln 2)^{1/\sigma}/x_{\frac{1}{2}},$$

and a good fit is obtained with a value of 0.53 for the shape parameter σ. The analysis has been carried out with $\sigma=\frac{1}{2}$, 1, and ∞.[24] The results were found to be essentially the same in each case so that it will suffice to present here the details for just the simplest case, $\sigma=1$.

The quantity κ is very large compared with ρ, and it is therefore permissible to neglect $-\rho x$ when it appears together with $-\kappa x$ as the argument of the exponential function; after renormalization the "experimental" distribution is then

$$P(x)dx=(1+q)\Gamma(\rho)^{-1}(\rho x)^{\rho-1}(e^{-\rho x}-e^{-\kappa x})\rho dx, \tag{1'}$$

where

$$q=(\zeta^\rho-1)^{-1},\quad \zeta=\kappa/\rho.$$

We were unable to derive from (1′) the distribution corresponding to (4) for the sample average. However, the low-order moments can readily be derived from (1′); to a high degree of accuracy one finds that

$$\langle\bar{\Gamma}\rangle_{\mathrm{Av}}=(1+q)\langle\Gamma\rangle_{\mathrm{Av}}, \tag{4a'}$$

$$\mathrm{var}_s\bar{\Gamma}=\langle\bar{\Gamma}\rangle_{\mathrm{Av}}^2/\bar{m}\rho, \tag{4b'}$$

where

$$\bar{m}=m(1+q)(1-\rho q)^{-1}.$$

For the final analysis it is sufficiently accurate to approximate the distribution for the sample average by (4) with $\langle\Gamma\rangle_{\mathrm{Av}}$ replaced by $(1+q)\langle\Gamma\rangle_{\mathrm{Av}}$ and with m replaced by $\bar{m}$, so that the correct expectation values (4a′) and (4b′) are obtained. With this approximation,

[23] J. A. Harvey (private communication).

[24] The analysis for $\sigma=\infty$ was actually carried out by truncating the distribution (1) at $x_{\frac{1}{2}}$ and by replacing the normalizing gamma function by the appropriate incomplete gamma function.

one can write

$$\langle \ln[\bar{\Gamma}/(1+q)\langle\Gamma\rangle_{\text{Av}}]\rangle_{\text{Av}} = F(\bar{m}\rho), \quad (4c')$$

and it may be shown that (6a) is very nearly satisfied if in (6) the function $F(m_j\rho)$ is replaced by $F(\bar{m}_j\rho)$ and if the function $F(\rho)$ is replaced by

$$F(\rho; x_{\frac{1}{2}}) = F(\rho) + q \ln\zeta - \ln(1+q). \quad (11)$$

The expression corresponding to (6b) is found to be

$$\text{var}_s\Phi = (1/m)[\psi'(\rho) - q(1+q)\ln^2\zeta] - \sum_j (m_j/m)^2\psi'(\bar{m}_j\rho). \quad (6b')$$

The function $F(\rho; x_{\frac{1}{2}})$ is plotted in Fig. (2) for several half-efficiency values $x_{\frac{1}{2}}$, including the most-probable value $x_{\frac{1}{2}}=0.01$. It is evident that when $\nu=1$ the function is rather sensitive to $x_{\frac{1}{2}}$, but when $\nu=2$ it is not. With $x_{\frac{1}{2}}=0.01$ and $\nu=1$, the factor $(1+q)=1.09$, and (4a′) indicates that to estimate $\langle\Gamma\rangle_{\text{Av}}$ the sample average should be reduced by about 9%, which amount is usually small compared with the statistical uncertainty corresponding to (4b′). The quantity $\bar{m}$ in (4b′) is equal to $1.15m$.

The expressions (5) through (9) with the modifications indicated under item 4 were used to analyze a total of 148 neutron reduced widths for 15 different nuclides. This total includes the 3 recent determinations for manganese[25] in addition to the 145 values which were analyzed by Hughes and Harvey.[4,26] The sum over ij in (6) is found to be -0.795 and the sum over j is -0.091 for $\nu=1$ and -0.050 for $\nu=2$; by using the $x_{\frac{1}{2}}=0.01$ curve of Fig. 2, a value $\nu=1.02$ is obtained as the solution to (5c). The standard deviations corresponding to (6b′) and (8) are 0.062 and 0.020, respectively, for $\nu=1$, giving an over-all standard deviation of 0.065, which is primarily due to statistics. The standard deviation of the ν estimate corresponding to the combined variances of Φ is about 0.13 degree of freedom. The hypothesis $\nu=1$ gives a probability integral $\mathcal{P}$, Eq. (5e), which is close to unity, indicating that this hypothesis is quite consistent with the data, while the hypothesis $\nu=2$ gives a value $\alpha=-6.4$ and an inadmissibly small value of $\mathcal{P}$. The $\nu=1$ hypothesis would be acceptable for any value of $x_{\frac{1}{2}}$ in the range antilog$_{10}$ (-2.00 ± 0.15) specified by Harvey.[23] However, for $x_{\frac{1}{2}}=0$, $\mathcal{P}$ is only 0.006 for $\nu=1$, thus indicating the importance of the efficiency correction; for $\nu=2$ it would still be extremely small.

There is one datum which appears to be at variance with the $\nu=1$ hypothesis. The plotted point for $y\equiv x^{\frac{1}{2}}=0.1$ on Fig. 2 of the paper by Hughes and Harvey[4] falls way below the curve for $\nu=1$ and near to the curve for $\nu=2$. Since this point just corresponds to the most-probable half-efficiency value $x_{\frac{1}{2}}=0.01$, it should be raised by a factor of about two, thus placing it almost within a standard deviation of the $\nu=1$ curve and several standard deviations away from the $\nu=2$ curve.

With the exception of U^{238}, all nuclides of reference 26 can form two different spin states when interacting with low-energy neutrons. It has been assumed here that the two spin states have identical reduced-width distributions. At the time that this analysis was undertaken, this assumption seemed reasonable and was not known to be in contradiction with any data. However, from a recent summary of the data on the angular momentum associated with slow-neutron resonances, Sailor has found indications that the compound nucleus is preferentially formed in the state of higher spin.[27] If the two spin states are equally abundant, this indication implies that the average neutron width is not the same in each state. Unless the widths of the states of the lower spin are unobservably small, our analysis is apt to be biased towards a ν value which is too small. With the existing data it is difficult to estimate the extent of this bias.

Before proceeding with the radiation and fission widths, it is worthwhile to include an explanatory remark on the physical significance of the terminology "degrees of freedom." In Sec. I, arguments were put forth for the plausibility of a Gaussian amplitude distribution, corresponding to a chi-squared width distribution with one degree of freedom. These arguments took account of the fact that the amplitudes could suitably be chosen as real.[8] Now, if the amplitudes had been regarded as complex with independent real and imaginary parts, these arguments would have led to a width distribution with two degrees of freedom, one for the real part and one for the imaginary part.[28] In the next section on radiation widths a situation is illustrated involving effectively a large number of degrees of freedom, and in the following section on fission widths one encounters a distribution with effectively only a few degrees of freedom.

III. RADIATION WIDTHS

The total radiation width Γ which pertains to the neutron resonance region is expected for heavy nuclei to be the sum of a large number n of partial radiation widths Γ_i[29]:

$$\Gamma = \sum_{i=1}^{n} \Gamma_i. \quad (12)$$

We will examine here the consequences of assuming that

[25] Bollinger, Dahlbert, Palmer, and Thomas, Phys. Rev. **100**, 126 (1955).

[26] The actual values and their uncertainties were read from the listing of resonance parameters in reference 3. The nuclei were: Mo^{95}(4); Mo^{97}(4); In^{113}(7); In^{115}(8); Sn^{117}(5); Cs^{133}(12); $Eu^{151,153}$(14); Tb^{159}(16); Ho^{165}(10); Tm^{169}(10); Hf^{177}(12); Hf^{179}(17); Ta^{181}(10); U^{238}(11); the number of widths used for each nucleus is indicated in parentheses. Only those widths appearing above the dotted lines in the listings were used.

[27] V. L. Sailor (to be published).

[28] See J. M. C. Scott, Phil. Mag. **45**, 1322 (1954).

[29] See J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Chap. XII; and the article by B. B. Kinsey, in *Beta- and Gamma-Ray Spectroscopy* (Interscience Publishers, Inc., New York, 1955), pp. 795–822.

the distributions of the individual Γ_i are independent and chi-squared with one degree of freedom, like the neutron widths. As mentioned in Sec. I, the latter assumption seems reasonable, while the former one stands on less certain grounds because some correlations are expected in the distributions of the partial widths for transitions to states of low excitation.

If all of the partial widths had the same average value $\langle\Gamma_i\rangle_{\text{Av}}$, the probability distribution for the total width would be chi squared with n degrees of freedom; that is, it would be given by (4) with $m\bar{\Gamma}$ replaced by Γ and with ρ replaced in $\frac{1}{2}n$. The distribution for the total width would thus become narrower as the number of partial widths becomes larger. For the general case where the $\langle\Gamma_i\rangle_{\text{Av}}$ are unequal, we have succeeded in deriving the distributions for the total only for even values of the number of degrees of freedom ν of the partial-width distribution. However, these total distributions are rather complicated and all we need to consider anyway are the average and the variance; for any value of ν they are [30]

$$\langle\Gamma\rangle_{\text{Av}}=\sum_{i=1}^{n}\langle\Gamma_i\rangle_{\text{Av}}, \tag{13a}$$

$$\langle(\Gamma-\langle\Gamma\rangle_{\text{Av}})^2\rangle_{\text{Av}}=(2/\nu)\sum_{i=1}^{n}\langle\Gamma_i\rangle_{\text{Av}}^2. \tag{13b}$$

These equations show that in the general case the distribution also becomes narrower as the number of partial widths becomes larger. In the following it is assumed that $\nu=1$.

The partial radiation widths are believed to be proportional to E_γ^{2l+1}, where E_γ is the transition energy and l is its multipolarity.[29] By approximating the sum over radiative transitions by an integration using the level density formula

$$\rho(E)=C\exp(E/T), \tag{14}$$

with constant temperature T and a constant coefficient C, one finds that the average of any power m of E_γ is

$$\langle E_\gamma^m\rangle_{\text{Av}}=CT^{m+1}e^r\gamma(m+1,r), \tag{15}$$

where $r=B/T$, B being the neutron binding energy, and

$$\gamma(\alpha,x)\equiv\int_0^x t^{\alpha-1}e^{-t}dt$$

is an incomplete gamma function. For electric dipole radiation, the square of the *coefficient of variation* V is therefore predicted to be

$$\begin{aligned}V^2&\equiv\langle(\Gamma-\langle\Gamma\rangle_{\text{Av}})^2\rangle_{\text{Av}}/\langle\Gamma_{\text{Av}}\rangle_{\text{Av}}^2=2\langle E_\gamma^6\rangle_{\text{Av}}/\langle E_\gamma^3\rangle_{\text{Av}}^2\\&=2n^{-1}(1-e^{-r})[\gamma(7,r)/\gamma^2(4,r)],\end{aligned} \tag{16}$$

where

$$n=\int_0^B\rho(E)dE=CT(e^r-1)$$

is the number of partial widths. Using Pearson's tables,[31] one finds that

$$V^2=\frac{2}{n}\times\begin{cases}2.3, & r=0\\5.3, & r=3\\10.9, & r=6\\20.0, & r=\infty.\end{cases} \tag{17}$$

There are only a few elements for which a sufficient number of radiation widths have been determined for statistical considerations. The use of the above formulas will be illustrated by analyzing the widths for Ta^{181}, which has the largest reported number, although there may be some question as to their reliability; in mv they are: 49±6, 49±11, 50±10, 51±10, 50±15, 40±15.[3] It is apparent that these widths show very little fluctuation other than that which could be ascribed to the indicated experimental uncertainties. The actual distribution must be narrow, and it may therefore be approximated by a normal distribution, the coefficient of variation of which may be compared with (17). As a generalization to the familiar chi-squared test of sample variances, it may be shown that the weighted sample variance has a chi-squared distribution with $m-1$ degrees of freedom, where m is the number of measurements, the weighting factors being equal to the reciprocal of the sum of the population variance and the variance corresponding to the uncertainty of the measurement. Assuming to begin with that the population variance is zero [that is, that n in (17) is infinite], one arrives at a total probability of less than one percent for weighted sample variances which are smaller than the observed one, indicating that there may be some systematic errors in the measurements or in the estimations of the uncertainties. For illustrative purposes, the undertainties are neglected altogether, thus allowing one to state, for example, that there is only a 5% total probability for the weighted sample variances being smaller than observed when a population variance of (8.7 mv)^2 is assumed; this variance corresponds to $n=340$ in (17) with $r=3$. This statement is consistent with the view that there are many independently-contributing partial widths, but it sheds very little light on the question of what fraction of all of the partial widths contribute independently. Thus, the number of partial widths is expected to be of the order of magnitude of the ratio of the nuclear temperature to the mean level spacing, which ratio is about 10^5 for a typical heavy nucleus.

The radiation widths of eleven nuclides for which more than one width are reported have carefully been analyzed for fluctuations by Levin and Hughes.[32] They found rather definite indications of fluctuations in the widths of several of the accurately measured nuclei,

[30] See, for example, S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943), Appendix IV.

[31] K. Pearson, *Tables of the Incomplete Gamma Function* (Cambridge University Press, Cambridge, 1934).

[32] J. S. Levin and D. J. Hughes, Phys. Rev. **101**, 1328 (1956).

notably In^{115} and Eu^{151}.[33] However, for these two it is observed that the radiation widths fall into two groups, each group having a definite but distinct isomeric branching ratio, thus suggesting that each group corresponds to a different spin state of the compound nucleus. With this contingency, they concluded that it was not possible with the existing data to reject the hypothesis that the radiation widths of a particular spin state of a particular nucleus are the same at all of the levels. A more critical testing of this hypothesis should be realized with a zero-spin target nucleus, with which only compound states of a single spin value could be excited with low-energy neutrons. The only such nucleus with a sufficient number of reported widths is U^{238}, these being 24±2, 25±5, 29±9, 17±10 mv.[3] From the chi-squared test of the weighted sample variance, it may be concluded that with a population variance of (12 mv)^2 there is only a 5% probability for a sample variance smaller than the observed one. This population variance corresponds to a value of $n=40$ in (17) with $r=3$. This conclusion is not significantly different from the previous one for Ta^{181}.[34]

IV. FISSION WIDTHS (U^{235})

An analysis has been made of the fluctuations of the 15 fission widths of U^{235} which are provisionally reported in the recent compilation by Hughes and Egelstaff ind in a private communication from Sailor.[35] These widths fluctuate considerably but not as much as do the neutron widths. The solution to Eq. (5c) is $\nu=2.3$ degrees of freedom, the standard deviations being 0.8 degree of freedom from the statistics and 0.3 degree of freedom from the indicated experimental uncertainties. The average width is 71 mv as estimated from the sample average, the standard deviations being 17 mv from the statistics and 5 mv from the indicated experimental uncertainties. To test the hypothesis that there is only one fission channel, that is, that the distribution has only one degree of freedom, the probability integral $\mathcal{P}(\nu=1)$ of (5e) was evaluated and found to be 0.10, which is small but not inadmissible according to most statistical criteria.

Another way to estimate the number of degrees of freedom of the best-fitting chi-squared distribution is to equate the first and second moments of the sample to the corresponding moments of the population. In this way one estimates that[36]

$$\nu=2(1-m^{-1})/V^2, \tag{18}$$

where

$$V^2=[m^{-1}\sum_i(\Gamma_i-\bar{\Gamma})^2]/\bar{\Gamma}^2,$$

the variance being given to order m^{-1} by

$$\text{var}_s\nu=\nu^2(1+4\nu^{-1})m/(m-1)^2. \tag{18a}$$

Using the same data, these give 3.3±1.6 degrees of freedom, which is consistent with the maximum likelihood estimate. The estimate (18) is especially poor when ν is small,[37] but for large ν the variances (2b) and (18a) may be shown, by using (7b), to be asymptotically equivalent.

The number of degrees of freedom will in general be smaller than the actual number of channels if the average widths for the various channels are unequal and if there are correlations in the distributions. Another difficulty in the interpretation is that there are two spin states formed when low-energy neutrons are captured by U^{235}, and these states will not necessarily have the same distributions.[38] The fact that the ratio of the average capture cross section to the average fission cross section at low energies[39] is about equal to the ratio of the average capture width[40] to the average fission width of the low-energy resonances indicates that the average fission widths for the two spin states are equal to within a factor of two. However, there is no way of telling from the existing data whether or not the distributions for the two spin states have the same variances. In spite of these complications, the original qualitative conclusions should remain valid: namely, that there are not very many channels involved in the slow-neutron-induced fission of U^{235}, and the likelihood is small of there being only one channel (for each spin state). The main

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[33] H. H. Landon and V. L. Sailor, Phys. Rev. **98**, 1267 (1955). See also H. H. Landon, Phys. Rev. **100**, 1414 (1955); G. Igo and H. H. Landon, Phys. Rev. **101**, 726 (1956).

[34] Six radiation widths for U^{238} have recently been reported by J. E. Lynn and N. J. Pattenden, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy* (Columbia University Press, New York, 1956), Vol. 4, Report P/423. They are 26.1±1.5, 28.8±2.3, 24.9±4.2, 18.6±2.7, 15.5±5.4, 13.6 ±4.8 mv, thus revealing rather significant fluctuations. With these data, one could state with essentially 95% confidence that n lies in the range from about 30 to 500.

[35] D. J. Hughes and P. A. Egelstaff, *Progress in Nuclear Energy* (Pergamon Press, London, 1956), Vol. 1, Chap. II. The widths in the compilation were used except for those which have recently been determined more accurately by Sailor; the values and their uncertainties are: 99±5, 120±15, 13±3, 110±45, 130±25, 87±15, 3±2, 9.5±5, 10.5±5, 70±16, 42±18, 43±21, 90±26, 200±20, 44±20 mv.

[36] See Sec. 27.7 of reference 7, in particular Eq. (27.7.3). The derivations of Eqs. (18) and (18a) above are very similar to those of the example presented in connection with Eqs. (27.7.10) of this reference. However, here we treat the square of V whereas the example treats V. The expectation value of V^2 has also been evaluated to one higher order in m^{-1} in order to obtain the factor $(1-m^{-1})$ of (18) which makes the estimate unbiased.

[37] See Sec. 17.51 of reference 17.

[38] The spin of U^{235} is 7/2 [K. L. Vander Sluis and J. R. McNally, Jr., J. Opt. Soc. Am. **45**, 65 (1955)] and its parity is presumably even [M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955), p. 81] so that even-parity states of spin 3 and 4 are formed. From the theoretical account of slow-neutron-induced fission which is given in reference 12, it is implied that the average fission width of the 4^+ states will be larger than that of the 3^+ states, because the former type state is contained in the rotational band associated with the lowest nucleonic configuration while the latter type would appear in a band involving an excited configuration. This configuration excitation is estimated to be of the order of an Mev, so that the difference of the average fission widths is expected to be especially large if the slow-neutron excitation energy exceeds the absolute threshold for fission of the 4^+ states by less than about one Mev. However, recent results on the $U^{235}(d,p)$ *fission* reaction indicate that this excess is about $1\frac{1}{2}$ Mev (unpublished experiments of K. Boyer and R. H. Stokes), and it is perhaps not inconsistent with the theory for the difference of the widths to be small.

[39] Kanne, Stewart, and White, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy* (Columbia University Press, New York, 1956), Vol. 4, Report P/595.

[40] V. L. Sailor, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy* (Columbia University Press, New York, 1956), Vol. 4, Report P/586.

reason for seeking the best fitting chi-squared distribution is that this distribution is very convenient to use for calculating the effects of the fission width fluctuations on the averages and variances of fission and competing reaction cross sections.[41]

Interference effects have been noticed by Sailor[40] in the fission cross section of U^{235}. It has also been noticed by him that the occurrence of interference effects in fission and the nonoccurrence of such effects in reactions in which radiative capture dominates are consistent with the view that there are many exit channels in the latter case, the reduced-width amplitudes of which have random signs, whereas there are effectively only a few such channels in the former case.

As a final remark, it is noted that there is no significant correlation in the fission and neutron width distributions of U^{235}. This observation confirms an earlier one by Harvey.[42] It also indicates that no correction to the analysis using (5c) is needed for failure to detect levels having very small neutron widths.[23]

ACKNOWLEDGMENTS

The writers are grateful to J. A. Harvey for supplying the very important data on the efficiency for detecting weak resonances, to D. J. Hughes for a prepublication copy of the fission-width listing, to Max Goldstein for supervising the numerical computations, to V. L. Sailor for his comments on the reliability of the neutron resonance data and for a listing of U^{235} fission widths, and to R. K. Zeigler for suggesting the use of the maximum-likelihood method.

APPENDIX

The relation between the descriptions of the fission process in terms of fission channels and in terms of fission fragment pairs may be illustrated by a few simple equations. The wave functions ψ_c for the fission channels c may be expanded in terms of the wave functions φ_p for the fission fragment pairs p as

$$\psi_c = \Sigma_p \alpha_{cp} \varphi_p, \tag{19}$$

with real coefficients α_{cp}. Both sets of wave functions are assumed to be orthonormal (but not necessarily complete) so that

$$\Sigma_p \alpha_{cp} \alpha_{c'p} = \delta_{cc'}. \tag{20}$$

On the "surface" $\mathfrak{S}$ of the nuclear configuration space, the wave functions X_λ of the levels λ of the compound nucleus may be expressed in terms of the fission-channel states as

$$X_\lambda(\mathfrak{S}) \sim \Sigma_c \gamma_{\lambda c} \psi_c \tag{21a}$$

plus terms associated with other reactions, where the coefficients $\gamma_{\lambda c}$ are the reduced-width amplitudes for the fission channels c. Using (19) the surface expansion may alternatively be expressed as

$$X_\lambda(\mathfrak{S}) \sim \Sigma_p \gamma_{\lambda p} \varphi_p, \tag{21b}$$

plus the other terms, where

$$\gamma_{\lambda p} = \Sigma_c \gamma_{\lambda c} \alpha_{cp} \tag{22}$$

is the reduced-width amplitude associated with the pth fission fragment pair. The total fission width $\Gamma_{\lambda f}$ for the level λ, which is normally expressed as the sum of partial widths for the various fragment pairs, may also be expressed as a sum over the partial widths associated with the various fission channels; using (20), one finds that

$$\Gamma_{\lambda f} = \Sigma_p \gamma_{\lambda p}^2 = \Sigma_c \gamma_{\lambda c}^2. \tag{23}$$

Now if the distributions of the $\gamma_{\lambda c}$ are normal with respect to levels, with zero means, and independent with respect to channels, then the distributions of the $\gamma_{\lambda p}$, as expressed by (22), will also be normal with variances

$$\langle \gamma_{\lambda p}^2 \rangle_{\text{Av}} = \Sigma_c \alpha_{cp}^2 \langle \gamma_{\lambda c}^2 \rangle_{\text{Av}}, \tag{24a}$$

the averages being with respect to the levels, and with covariances (correlation coefficients)

$$\langle \gamma_{\lambda p} \gamma_{\lambda p'} \rangle_{\text{Av}} = \Sigma_c \alpha_{cp} \alpha_{cp'} \langle \gamma_{\lambda c}^2 \rangle_{\text{Av}}. \tag{24b}$$

Near the fission threshold it is expected that only a few fission channels will have appreciable amplitudes $\gamma_{\lambda c}$ so that the covariances will not generally vanish even though the φ_c may constitute a complete set. This means that the distributions of the $\gamma_{\lambda p}$ are expected to be correlated, and therefore to be dependent.[11]

[41] A likelihood analysis of the 15 fission widths for Pu^{239}, which are provisionally reported by L. M. Bollinger, R. E. Coté, J. M. LeBlanc, and G. E. Thomas, gives for the number of degrees of freedom $\nu \lesssim 1.7$ with an uncertainty of 0.5 degree of freedom from the statistics. Only an upper limit to ν can be stated because one of the resonances has no detectable fission width.

[42] J. A. Harvey, Bull. Am. Phys. Soc. Ser. II, 1, 86 (1956).