Theory of Inelastic Electron Scattering by the C^{12} Nucleus*

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The recently measured elastic and inelastic electron scattering cross sections of C^{12} are utilized to test the shell model of light nuclei. Collective effects are important in the excitation of two of the excited states (7.68 and 4.42 Mev), and are acceptably accounted for by the independent-particle approach to collective oscillations recently proposed by the authors. The inelastic scattering data leading to the 9.61 Mev state are shown to identify it as probably 1^-. Some of the results of higher energy scattering experiments on C^{12} and O^{16} are predicted.

I. INTRODUCTION

FREGEAU and Hofstadter[1] have recently measured the elastic and inelastic scattering of high-energy electrons from C^{12}. They find that the cross sections can be analyzed in terms of form factors, indicating that the Born approximation holds.[2] Morpurgo[3] has calculated the form factor for the excitation of the 4.43-Mev 2^+ state, using shell-model wave functions in both L-S and j-j coupling. His results, which we have verified, give too small an inelastic cross section by about a factor of two with L-S coupling, and by a factor of six with j-j coupling. In Sec. III, after extracting as much information about the ground state wave function as is contained in the elastic form factor, we will show that the admixture of a small amount of collective spheroidal oscillation to Morpurgo's wave function brings the theory of the 2^+ level into agreement with experiment. Section III also deals with the description of the 7.65-Mev 0^+ state[4,5] as primarily a collective dilational oscillation,[6] which provides good agreement with the experimental inelastic form factor. The subject of Sec. IV is the interpretation of the scattering to the 9.61-Mev level, which is shown to be probably 1^-. Section V consists mostly of a discussion of the results to be expected from a measurement of electron scattering from O^{16}.

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[1] J. H. Fregeau and R. Hofstadter, Phys. Rev. **99**, 1503 (1955).

[2] L. I. Schiff [Phys. Rev. **96**, 765 (1954)] has discussed multipole transitions in inelastic electron scattering, and justification of some of the formulas we shall use may be found in his paper and in those to which he refers. He has found [Phys. Rev. **103**, 443 (1956)] that the criterion for validity of the Born approximation in electron scattering is $Ze^2/\hbar c \ln(a_0/r_0) \ll 1$, where a_0 is the Bohr radius; and r_0 is the electron wavelength, the nuclear radius, or a_0/Z, whichever is smallest. Thus C^{12} is on the borderline, and one ought not take the Born approximation for granted, but should consider Fregeau and Hofstadter's results as evidence of its validity.

[3] G. Morpurgo, Nuovo cimento 3, 430 (1956).

[4] F. Ajzenberg and T. Lauritsen [Revs. Modern Phys. **27**, 77 (1955)] review evidence leading to tentative identification of this level.

[5] L. I. Schiff [Phys. Rev. **98**, 1281 (1955)] has considered this state to be a two-particle excitation. His results lead to much too small a form factor.

[6] The present authors [R. A. Ferrell and W. M. Visscher, Phys. Rev. **102**, 450 (1956)] have considered the 6.06-Mev 0^+ state in O^{16} as a collective dilational oscillation, and have shown that a certain superposition of radially excited single-particle wave functions gives an excitation energy of 9 Mev, and a monopole matrix element to the ground configuration about twice that deduced from the experimental pair-emission lifetime. By relaxing one of the restrictions on our assumed wave function (namely, letting the s and p shells dilate independently) it has proved possible to reduce the calculated energy to 6.65 Mev, using the same forces as before.

II. STATES OF COLLECTIVE EXCITATION IN C^{12}

Following the notation of reference 6 we write the first excited collective states as derivatives of the ground state with respect to a deformation parameter α.

$$\Psi = C\frac{\partial\Phi}{\partial\alpha}\bigg|_{\alpha=0}. \tag{1}$$

This wave function describes a 0^+ "breathing mode" dilational state if α is defined in the following way:

$$\Phi(\mathbf{r}_i;\alpha)=\Phi(\mathbf{r}_ie^{-\alpha};0)e^{-\frac{3}{2}A\alpha}, \tag{2}$$

and a 2^+ spheroidal oscillation if

$$\Phi(\mathbf{r}_i;\alpha')=\Phi(e^{-\alpha'}x_i,e^{-\alpha'}y_i,e^{+2\alpha'}z_i;0). \tag{3}$$

The normalization constant depends on the specific choice of Φ. If it is chosen to be a Slater determinant of harmonic oscillator single-particle functions, the evaluation of C is particularly easy. It can then be easily verified that

$$\frac{\partial\Phi}{\partial\alpha}\bigg|_{\alpha=0}=\gamma\sum_{i=1}^{A}(r_i^2-\langle r^2\rangle_0)\Phi|_{\alpha=0}, \tag{4}$$

and

$$\frac{\partial\Phi}{\partial\alpha'}\bigg|_{\alpha'=0}=-\gamma\sum_{i=1}^{A}(3z_i^2-r_i^2)\Phi|_{\alpha'=0}, \tag{5}$$

where γ is the parameter appearing in the factor $\exp(-\frac{1}{2}\gamma r^2)$ which is common to all the single-particle harmonic oscillator wave functions, and

$$\langle r^2\rangle_0=(\Phi,r_i^2\Phi)_{\alpha=0}.$$

The requirement that $(\Psi,\Psi)=1$ can then be rewritten

$$\frac{2}{C^2}=\frac{\partial}{\partial\alpha}(\Phi(\mathbf{r}_je^{-\alpha}),\quad \gamma\sum_{j=1}^{A}(r_i^2-\langle r^2\rangle_0)\Phi(\mathbf{r}_je^{-\alpha}))|_{\alpha=0}, \tag{6}$$

which, by a change of variable in the integration implied by the scalar product, can be easily shown to

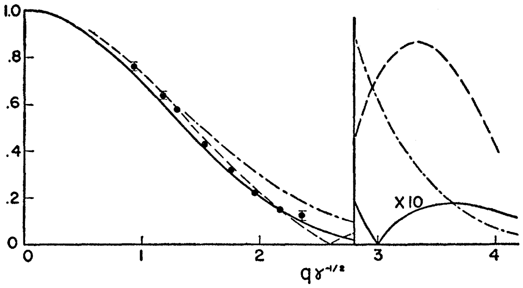

FIG. 1. Charge density as a function of radius for the harmonic-oscillator shell model. The dotted lines are the Gaussian and uniform charge distributions for C^{12} taken from Fregeau and Hofstadter. R is measured in units of the harmonic oscillator wave-function size-parameter $\gamma^{-\frac{1}{2}}=1.63\times 10^{-13}$ cm.

be $2A\gamma\langle r^2\rangle_0$. Therefore, for C^{12}

$$\Psi_{0^+}=\frac{1}{(26)^{\frac{1}{2}}}\frac{\partial\Phi}{\partial\alpha}\bigg|_{\alpha=0}, \tag{7}$$

and a similar procedure for the spheroidal state yields

$$\Psi_{2^+}=\frac{1}{(52)^{\frac{1}{2}}}\frac{\partial\Phi}{\partial\alpha'}\bigg|_{\alpha'=0}. \tag{8}$$

The form factors describing inelastic electron scattering to these states can quickly be found; they are related to the derivatives of the elastic form factor with respect to α and α'. The expression for the elastic form factor

$$F=\left(\Phi,\sum_{j=1}^{A}e^{i\mathbf{q}\cdot\mathbf{r}_j}\left(\frac{1+\tau_3(j)}{A}\right)\Phi\right), \tag{9}$$

(where $\mathbf{q}$ is the momentum transferred from the electron to the nucleus), upon differentiation yields

$$\begin{aligned}\frac{\partial F}{\partial\alpha}\bigg|_{\alpha=0}&=(26)^{\frac{1}{2}}\{F_{0^+}(\mathbf{q})+F_{0^+}{}^*(-\mathbf{q})\},\\ \frac{\partial F}{\partial\alpha'}\bigg|_{\alpha'=0}&=(52)^{\frac{1}{2}}\{F^0{}_{2^+}(\mathbf{q})+F^{0*}{}_{2^+}(-\mathbf{q})\},\end{aligned} \tag{10}$$

where

$$F_{2^+}{}^0=\left(\Psi^0{}_{2^+},\sum_{i=1}^{A}e^{i\mathbf{q}\cdot\mathbf{r}_j}\left(\frac{1+\tau_3(j)}{A}\right)\Phi\right)_{\alpha'=0}$$

is the inelastic form-factor for excitation of the 2^+ collective oscillation which has $M=0$. Since a change of integration variable in Eq. (9) shows us that $F(\mathbf{q},\alpha)=F(\mathbf{q}e^{\alpha},0)$, and[3]

$$F(q,0)=(1-q^2/9\gamma)\exp(-q^2/4\gamma), \tag{11}$$

we see that

$$F_{0^+}(\text{coll.})=\frac{1}{2(26)^{\frac{1}{2}}}\frac{\partial F}{\partial\alpha}\bigg|_{\alpha=0}=-\left(\frac{13}{2}\right)^{\frac{1}{2}}\frac{q^2}{36\gamma}\times(1-q^2/13\gamma)\exp(-q^2/4\gamma), \tag{12}$$

and that

$$F_{2^+}{}^0(\text{coll.})=\frac{1}{2(52)^{\frac{1}{2}}}\frac{\partial F}{\partial\alpha'}\bigg|_{\alpha'=0}=-\sqrt{2}F_{0^+}(\text{coll.}). \tag{13}$$

In Eq. (13) we have chosen the z axis to be in the direction of $\mathbf{q}$. Then the $M=0$ state (8) is the only substate of $\Psi_{2^+}{}^M$ which can be excited, and the sum which would otherwise have to be performed to obtain the cross section reduces to a single term.

III. COMPARISON WITH EXPERIMENT

The harmonic oscillator wave functions fix the charge and density distribution in the nucleus. For C^{12} and O^{16} its prediction is illustrated in Fig. 1. Since the elastic form factor is nothing more than the Fourier transform of the charge distribution, comparison with the experimental data provides a test of the model. Figure 2 shows the theoretical form factor (11), and the experimental points. The size parameter $\gamma^{-\frac{1}{2}}$ has been fixed at 1.630×10^{-13} cm by requiring that the rms radius of the charge distribution agree with that found by Fregeau and Hofstadter, namely $(2.40\pm 0.25)\times 10^{-13}$ cm. Although this curve is not violently sensitive, in the range of q covered by experiment, to the shape of the charge distribution, our model fits the data somewhat better than either the Gaussian or uniform shapes, which are plotted by Fregeau and Hofstadter[7] (their Fig. 11). One feature of the elastic form factor, however, does depend strongly on the shape assumed; namely, some shapes predict a root. Ours predicts one at $q=1.84\times 10^{13}$ cm^{-1}, corresponding to 90° scattering of a 256-Mev electron. The Gaussian shape would give no zero in the form factor, and the uniform shape predicts one at $q=1.58\times 10^{13}$ cm^{-1}, corresponding to

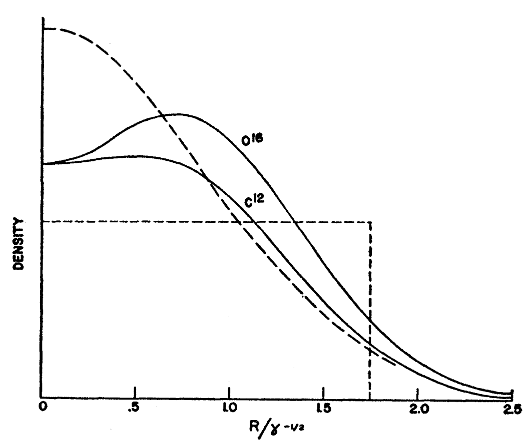

FIG. 2. Elastic form factor. The experimental points are those of reference 1. The theoretical form factors corresponding to the charge distribution of Fig. 1 are plotted, the prediction of the harmonic oscillator model as a solid line. The Gaussian density function gives rise to no zero in the form factor.

[7] Our charge distribution is probably the same as that used by D. G. Ravenhall, whose work (unpublished) is mentioned by Fregeau and Hofstadter.

90° scattering of a 220-Mev electron. The experiments reported in reference 1 do not go higher than 187 Mev. The results of a higher energy experiment would more exactly fix the shape of the charge distribution.

The inelastic scattering to the 2^+ 4.43-Mev state, which is the subject of Morpurgo's paper,[3] can be described in terms of a form factor, and the experimental points of reference 1 are plotted on Fig. 3. Since there is a low-lying 2^+ configuration in the p shell $(p_{\frac{3}{2}})^{-1}p_{\frac{1}{2}}$ in j-j coupling; ${}^{1,1}D_2$ in L-S coupling, one expects the 4.43-Mev state to be composed mostly of this configuration, with a relatively small amount of collective excitation (8) mixed in. (Ψ_{2^+}(coll.) of Eq. (8) is orthogonal to all states within the $1p$ shell. When it is expanded in product wave functions, one sees that it is a linear combination of single-particle excitations from the s and p shells into $1d$, $1f$, and $2p$ states.) An admixture of the order of 10% is not unreasonable, because the energy of Ψ_{2^+}(coll.) will be lowered in the same way the 0^+ state of O^{16} is, as discussed in reference 6. The curves of Fig. 3 include Morpurgo's result for L-S coupling,

$$F_{2^+}(L\text{-}S)=(1/18)(7/5)^{\frac{1}{2}}(q^2/\gamma)\exp(-q^2/4\gamma) \quad (14)$$

[j-j coupling gives the same shape, but a smaller form factor by a factor $(5/14)^{\frac{1}{2}}$] and the form-factor corresponding to a 10% admixture of Ψ_{2^+}(coll.); namely,

$$F_{2^+}(10\%\text{ coll.})=(0.1)^{\frac{1}{2}}F_{2^+}(\text{coll.})+(0.9)^{\frac{1}{2}}F_{2^+}(L\text{-}S).$$

It is possible to fit the data fairly well with any mixture of collective wave function and Morpurgo's L-S wave function from about 10% one to 10% of the other. The best fit obtained by mixing the collective with the j-j wave function drops down too soon, and is 20% low at $q\gamma^{-\frac{1}{2}}=2$. Probably one can fit the experiments with almost any intermediate (between j-j and L-S) coupling, but we have calculated only the two extremes.†

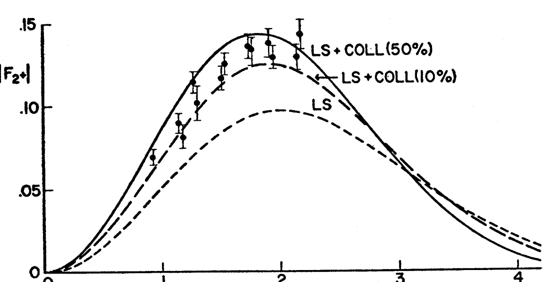

FIG. 3. Inelastic form factor for the excitation of the 2^+ 4.43-Mev level. Experimental points are taken from Fregeau and Hofstadter. The theoretical curves corresponding to pure L-S coupled p-shell excitation, and 10% and 50% admixture of the collective model are plotted.

† *Note added in proof.*—The recent value of $(2.6\pm0.9)\times10^{-14}$ sec for the mean lifetime of the 2^+ state of C^{12} [Devons, Manning, and Towle, Proc. Phys. Soc. (London) **A69**, 173 (1956)] is somewhat too small to be consistent with the electron scattering data. The calculated lifetime varies (in LS coupling) from 12.5×10^{-14} sec to a minimum of 3.7×10^{-14} sec for zero and 70% collective admixture, respectively. Such a large admixture requires the existence of a relatively low-lying 2^+ collective state, as in O^{16}. That this may also be the case in C^{12} is suggested by the recent tentative detection by means of the β decay of B^{12} of a very broad distribution of α particles which may result from the decay of such a state of as yet undetermined spin (only 0^+ and 2^+ would provide allowed β and α decays), in the region of 8–11 Mev, [C. W. Cook and T. Lauritsen (private communication)]. Because of the limited resolution in the electron scattering equipment, such a 2^+ level would produce a spurious contribution to the inelastic cross section assigned to the 9.61-Mev level, thus possibly affecting the conclusion reached in Sec. IV.

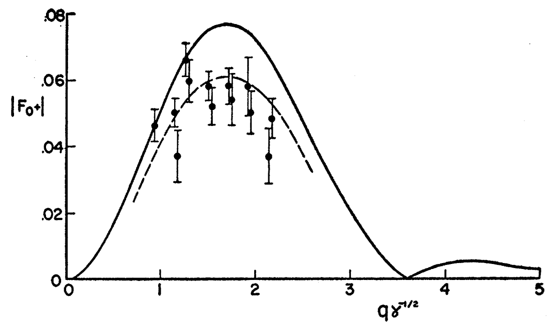

FIG. 4. Inelastic form factor for the excitation of the 7.65-Mev level. The form factor for the 0^+ dilational collective state (12) is plotted as a solid line; the dotted line is the form factor corresponding to 35% particle excitation. Note that a zero is predicted in the scattering cross section for a higher momentum transfer than those reported in reference 1.

The experimental form factor for the 7.65-Mev 0^+ state is given on Fig. 4. There is no low-lying p-shell configuration which has spin 0 [8]; the lowest is a ${}^{3,3}P_0$ state (in L-S coupling), estimated by Inglis to lie at about 17 Mev, and forbidden to electric electron excitation by a spin selection rule ($\Delta S=0$). The theoretical form factor, if the 7.65-Mev state is assumed to be 100% dilational oscillation [for which the wave function is given by Eq. (7)], is given by Eq. (12), and is compared with experiment on Fig. 4. It is about 25% too high. Admixture of the p shell ${}^{3,3}P_0$ (or its j-j analog, or intermediate) function affects $|F_{7.65}|$ only by reducing the amplitude of Ψ_{0^+}(coll.), and if one assumes 35% p shell, 65% collective wave functions, the resulting curve, given by $|F_{7.65}|=(0.65)^{\frac{1}{2}}|F_{0^+}(\text{coll.})|$ fits the data quite well. Comparable amplitudes for the collective and p-shell configurations are what one would expect, because the unperturbed energies for the two states are nearly the same. (The collective energy can be estimated by the methods of reference 6, and Inglis has estimated the energy of the ${}^{3,3}P_0$ state.) Two orthogonal superpositions of these configurations are approximate eigenfunctions of the Hamiltonian. One of them must be lowered by configuration interaction, and is assumed to be the 7.65-Mev state.

IV. INTERPRETATION OF THE 9.61-MEV LEVEL

An attractive conjecture for the 9.61-Mev level, since it yields much the same angular distribution of scattered electrons as the 4.42-Mev 2^+ level, would be

[8] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).

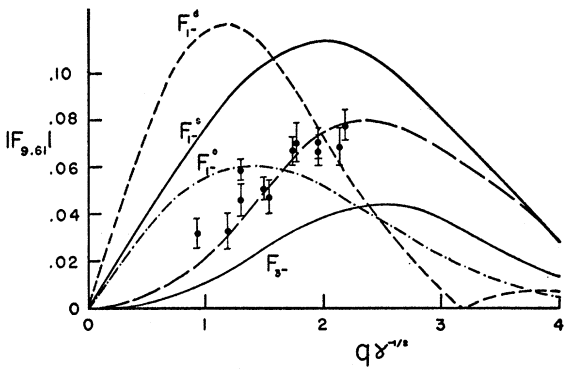

FIG. 5. Inelastic form factor for the excitation of the 9.61-Mev level. The experimental points are those of Fregeau and Hofstadter, and the curves are the theoretical form factors (15), plus an unlabeled curve corresponding to 85% $(1p_{3/2})^{-1}2s_{1/2}$ with 15% admixture of $(1p_{3/2})^{-1}1d_{5/2}$. This curve is within the errors of most of the points; by varying the admixture coefficients one can improve the agreement with experiment, but not in a unique way.

that it, too, is 2^+. This was suggested by Fregeau and Hofstadter. However, the work of Graue[9] on the angular distribution of the neutron group from the $B^{11}(d,n)C^{12}$ reaction which corresponds to the 9.61-Mev state of C^{12} indicates that the proton is captured to a large extent into a d orbital. This definitely fixes the parity as odd, and the spin as $0 \leq J \leq 4$.

The fact that the 9.61-Mev level decays by α-emission to the ground state of Be^8 then requires that the spin be odd, too. We have calculated the inelastic form factors for both 1^- and 3^- states, using j-j coupling. They are

$$
\begin{aligned}
|F_{3^-}| &= \frac{1}{30\sqrt{6}}\left(\frac{q^3}{\gamma^{\frac{3}{2}}}\right)\exp(-q^2/4\gamma),\\
|F_{1^-}{}^0| &= \frac{1}{6\sqrt{6}}\frac{q}{\gamma^{\frac{1}{2}}}\exp(-q^2/4\gamma),\\
|F_{1^-}{}^s| &= \frac{1}{9\sqrt{2}}\frac{q}{\gamma^{\frac{1}{2}}}\exp(-q^2/4\gamma)\left(1+\frac{q^2}{4\gamma}\right),\\
|F_{1^-}{}^d| &= \frac{1}{6}\frac{q}{\gamma^{\frac{1}{2}}}\exp(-q^2/4\gamma)\left(1-\frac{q^2}{10\gamma}\right),
\end{aligned}
\tag{15}
$$

where $F_1{}^{0,s,d}$ describe excitation into the $(1s_{1/2})^{-1}1p_{1/2}$, $(1p_{3/2})^{-1}2s_{1/2}$, and $(1p_{3/2})^{-1}1d_{5/2}$ configurations, respectively. These are plotted on Fig. 5, along with the experimental data. F_{3^-} is much too low. There are, however, many ways in which the 1^- form factors can be combined to give agreement with observation. The dashed curve, for example, is that corresponding to 85% $2s_{1/2}$ excitation, and 15% $1d_{5/2}$.[10]

[9] A. Graue, Phil. Mag. **45**, 1205 (1954). The authors are grateful to Dr. C. L. McGinnis for calling their attention to this work.

[10] That the $2s_{\frac{1}{2}}$ configuration should be favored is also indicated by the fact that the first excited state of C^{13} is $1/2^+$, the $5/2^+$ level lying 770 kev higher, according to Ajzenberg and Lauritsen (reference 4).

Although F_{3^-} should be calculated in L-S and intermediate coupling to more definitely exclude $J=3^-$, we tentatively assign $J=1^-$ to the 9.61-Mev level.[11] An experiment which seems not to have been done, namely a measurement of the angular distribution of the α's in the $C^{12}(\gamma,\alpha)Be^8$ reaction at γ-ray energies near 9.61 Mev, would fix the spin definitely. For $J=1^-$ it should be $\sin^2\theta$, relative to the incoming beam.

V. CONCLUSION

As in the case for the elastic scattering, the 0^+ theoretical cross section vanishes, but at a somewhat higher q, corresponding to 90° scattering at about 500 Mev. The 2^+ form factor, too, might have a root, whose existence and energy depends sensitively on the admixture of Ψ_{2^+}(coll.) in the wave function. Thus there is much to be learned from the extension of the scattering experiments to higher energies.

Since the closed p-shell nucleus O^{16} has no low-lying excited particle configurations of the same symmetry as Ψ_{2^+}(coll.) and Ψ_{0^+}(coll.), one might expect the 0^+ and 2^+ states in its spectrum to be quite pure collective oscillations. Inelastic electron scattering from this nucleus should therefore provide a direct and stringent test of the collective wave functions (7) and (8).[12] For example, if the 0^+ and 2^+ states of O^{16} are accurately described by our wave functions,[13] both $|F_{0^+}|$ and $|F_{2^+}|$ will have roots at the same value of momentum transfer. For O^{16}, one finds the following form factors:

$$
\begin{aligned}
F(q,0) &= (1-q^2/6\gamma)\exp(-q^2/4\gamma),\\
F_{0^+}(\text{coll.}) &= -\frac{5q^2}{72\gamma}\left(1-\frac{q^2}{10\gamma}\right)\exp(-q^2/4\gamma),\\
F_{2^+}(\text{coll.}) &= -\sqrt{2}F_{0^+}(\text{coll.}).
\end{aligned}
\tag{16}
$$

In the above work the recoil of the struck nucleus has consistently been ignored; no differentiation has been made between laboratory and center-of-mass coordinate systems. This neglect has greatly simplified the form-factor calculations. It introduces an error of only a few percent at the energies of the existing experiments. Since the percentage error is proportional to the recoil velocity, however, it will become more serious as the energy increases. The finite size of the proton has also been ignored.

[11] A. E. Glassgold and A. Galonsky [A. E. Glassgold and A. Galonsky, Phys. Rev. **103**, 701 (1956)] have re-examined the α model for C^{12}. They find that it can correlate the first three excited states of C^{12}, and identifies that 9.61 level as 1^- or 2^+. It suffers, however, from the difficulty that it predicts a 3^- state at 5.54 Mev which has not been observed.

[12] The pair lifetime of the 0^+ state discussed in reference 6 is inversely proportional to the square of the inelastic form-factor for small q. Thus Ψ_{0^+}(coll.) has already been tested to some extent for O^{16}, with encouraging results.

[13] Our wave functions can be improved by allowing the s and p shells to oscillate with different phases and amplitudes. The predictions stated here will then be somewhat modified.