Li's value of 9.332 Mev for the energy difference of $Ne^{20}+\alpha-Mg^{24}$ a systematic error is observed between the two sets of resonances. The energies as obtained by the $Ne^{20}+\alpha$ reaction are consistently 14 kev higher. If this 14-kev correction is made, resonances 6, 11, 12, 13, and 15 coincide perfectly between the two experiments, both in energy and in spin and parity. Only one resonance in the region of overlap has not yet been obtained by the $Na^{23}+p$ reaction. This is a 2+ resonance which should fall at a proton energy of 280 kev. It seems likely that when elastic scattering of protons is extended to this region that it too will be found.

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Approximation for Deuteron Stripping Reactions on Heavy Target Nuclei*

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The (d,p) stripping reaction for $l_n=0$ is discussed for the case where Coulomb effects predominate, employing a zero-range neutron-proton interaction and neglecting the finite nuclear size. The angular distribution of the emergent proton is shown to change drastically from forward to backward peaking as the Sommerfeld number η increases. The case $l_n\neq 0$ is discussed.

INTRODUCTION AND SUMMARY

THE characteristic feature of the deuteron stripping process, involving discrete levels for the residual nuclei, is an angular distribution that shows pronounced forward-to-backward asymmetry, with associated maxima and minima. The Butler discussion of the stripping process,[1] and the Born approximation treatment[2] as well, give an adequate explanation of this phenomenon, subject to certain approximations. Among these approximations is the neglect of the effect of Coulomb forces on the incident deuteron and the emergent proton for (d,p) reactions. There have been subsequent treatments which have taken the Coulomb forces into account. The most comprehensive has been that of Tobocman and Kalos.[3] These authors took into account not only Coulomb effects but also nuclear effects on the incident and emergent particles. Such a treatment necessitates a partial wave expansion and rather extensive calculations tailored to each particular reaction under consideration, but, compensating for this difficulty, the results agree much better with the experiments considered. Somewhat earlier, Butler and Austern[4] had discussed the Coulomb effects by means of a numerical example of $l=2(d,p)$ stripping on $Z=15$. Qualitatively, it is clear that when the Coulomb forces can be considered small, the effect should be primarily a smearing out of the otherwise well-defined incident and emergent momenta. Principally, then, one would expect a smoothing out of the distribution and a filling in of the minima, just as observed.

It is the purpose of this note to consider the opposite limiting case,[5] namely the situation where the Coulomb effects dominate, that is, for large values of the parameters η_d and η_p. This case shows a great many simplifications over the usual situation. Because of the large Coulomb repulsion, nuclear effects on the incident and emergent particles are minimized. As a result, the partial wave expansion, which the nuclear effects would require, can be avoided. Moreover, the nuclear radius which enters as a parameter in the usual theory, is seen by the same argument to be of slight concern.

Even this case is, of course, intractable without further assumptions. We shall assume that the n-p interaction for the deuteron has zero range. For the usual stripping development, this assumption is of minor effect. Furthermore, we shall employ only the first Born approximation, neglecting the interaction interior to the nucleus. This, as discussed in many papers, is equivalent to the (perhaps more convincing) Butler approach. Finally we shall, for reasons that will

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[1] S. T. Butler, Phys. Rev. **80**, 1095 (1950); Proc. Roy. Soc. (London) **A208** (1951); Phys. Rev. **88**, 685 (1952). A more complete list of references will be found in the reviews by R. Huby, Progr. Nuclear Phys. **3**, 177 (1953) and by W. Tobocman, Naval Research Laboratory Report (unpublished).
[2] Bhatia, Huang, Huby, and Newns, Phil. Mag. **43**, 485 (1952); R. Huby, Proc. Roy. Soc. (London) **A215**, 385 (1952); Fujimoto, Hayakawa, and Nishijima, Progr. Theoret. Phys. (Japan) **10**, 113 (1953); F. L. Friedman and W. Tobocman, Phys. Rev. **92**, 93 (1953); P. B. Daitch and J. B. French, Phys. Rev. **87**, 900 (1952).
[3] W. Tobocman and M. H. Kalos, Phys. Rev. **97**, 132 (1955). Recently I. P. Grant, Proc. Phys. Soc. (London) **A68**, 244 (1955) has discussed Coulomb effects in a detailed manner similar to Tobocman and Kalos, employing, however, an approximate form for the deuteron Coulomb wave function that is not well adapted to heavy target nuclei.
[4] S. T. Butler and N. Austern, Phys. Rev. **93**, 355 (1954).
[5] Before the write-up of our results was completed, it came to our attention that K. A. Ter-Martirosyan, Zhur. Eksptl. i Teort. Fiz. **29**, 713, ff. (1955) has also discussed stripping in this approximation, and arrives at similar conclusions. We have accordingly abbreviated our work, in the overlapping discussion of the approximation for very large η.

be clear later, restrict detailed attention to the $l_n=0$ case.

The approach we are considering has already been applied to the related problem of the electric disintegration of the deuteron by Landau and Lifshitz[6] and by Goldberger and French.[7]

The principal result of this investigation is to show that as the Coulomb effects become dominant the character of the angular distribution changes drastically, from a predominantly forward distribution to a predominantly backward distribution which becomes more peaked as the Coulomb parameters η_p and η_d increase. An approximate expression valid for large η for this angular distribution is developed and the "exact" form is shown in curves for a typical example.

FORMAL DEVELOPMENT

Consider the process whereby a nucleus of charge Z (of sufficiently large mass that it may be considered as fixed in position) is bombarded by a Coulomb "plane" wave, ψ_d, of deuterons (momentum $\mathbf{k}_d$), and a Coulomb "plane" wave, ψ_p, of protons (momentum $\mathbf{k}_p$) is observed to emerge.[8] The first Born approximation amplitude for this stripping process is given by

$$A=\int d\tau_p d\tau_n d\xi \chi_F^*(\xi,r_n)\phi(\mathbf{r}_n-\mathbf{r}_p)\chi_I(\xi)\psi_p^*\psi_d V_{np}. \quad (1)$$

The initial state of the nucleus is given by the wave function $\chi_I(\xi)$ and the final state of the nucleus plus captured ($l_n=0$) neutron is given by the wave function $\chi_F(\xi,r_n)$. The integral over the nuclear coordinates, ξ, takes the following form for $|\mathbf{r}_n|>R$ (the nuclear radius):

$$\int d\xi \chi_F^* \chi_I=\left(\frac{2M\gamma}{4\pi\hbar^2 R}\right)^{\frac{1}{2}} \frac{h_0(ik_n r_n)}{h_0(ik_n R)}. \quad (2)$$

Here M denotes the neutron mass, and γ the reduced width for the neutron capture.

For $|\mathbf{r}_n|<R$, we take the integral to be zero.

If we now take the n-p potential, V_{np}, to have zero range, the integral over the relative coordinate $\boldsymbol{\rho}=\mathbf{r}_n-\mathbf{r}_p$ may be carried out. That is,

$$\int d^3\rho V_{np}(\rho)\phi(\rho)=(2\alpha)^{\frac{1}{2}}\hbar^2/M, \quad (3)$$

where $\alpha=(ME_b/\hbar^2)^{\frac{1}{2}}$ is the reciprocal radius of the deuteron, $\alpha^{-1}=4.32\times10^{-13}$ cm.

[6] L. Landau and E. Lifshitz, J. Exptl. Theoret. Phys. (U.S.S.R.) **18**, 750 (1948).

[7] M. L. Goldberger and J. B. French (unpublished manuscript circulated in 1948). A part of this manuscript has appeared [J. B. French and M. L. Goldberger, Phys. Rev. **87**, 899 (1952)]. The authors would like to acknowledge their indebtedness to Professor Goldberger and Professor French.

[8] Our treatment of this process closely parallels the standard treatment as given, for example, by W. Tobocman (unpublished Naval Research Laboratory review).

The $l_n=0(d,p)$ stripping amplitude thus takes the form

$$A=a\bar{I},$$

where

$$a\equiv\left(\frac{\hbar^2\gamma\alpha}{\pi MR}\right)^{\frac{1}{2}}[h_0(ik_nR)]^{-1}, \quad (4a)$$

and

$$\bar{I}=\int_R^\infty r^2dr\int d\Omega\psi_d\psi_p^*h_0(ik_nr). \quad (4b)$$

This result for A yields the usual Butler stripping amplitude for $l_n=0$, if the waves are inserted for ψ_d and ψ_p. As it stands, for Coulomb waves, the integral cannot be treated in closed form. However, the region for $r<R$ makes a small contribution for large values of η_p and η_d, so that it is reasonable to extend the integral to include this range. The error in this is worst for $\eta_d=\eta_p=0$, but even here the result is qualitatively correct, showing a forward maximum without, however, any other maxima or minima.

With this modification the integral can be given in closed form, using a result due to Sommerfeld.[9] That is,

$$\begin{aligned}I=8\pi^2&\left[\frac{\eta_p\eta_d}{(e^{2\pi\eta_p}-1)(e^{2\pi\eta_d}-1)}\right]^{\frac{1}{2}}\\&\times[2(k_p-\tfrac{1}{2}k_d)^2+k_0^2+4k_pk_d\sin^2(\theta/2)]^{-1-i\eta_p-i\eta_d}\\&\times[2k_p^2-\tfrac{3}{2}k_d^2+k_0^2+2ik_d(k_p^2+k_0^2-\tfrac{1}{2}k_d^2)^{\frac{1}{2}}]^{i\eta_p}\\&\times[\tfrac{1}{2}k_d^2+k_0^2+2ik_p(k_p^2+k_0^2-\tfrac{1}{2}k_d^2)^{\frac{1}{2}}]^{i\eta_d}\\&\times{}_2F_1\left(-i\eta_p,-i\eta_d,1;-\frac{4k_pk_d\sin^2(\theta/2)}{k_0^2+2(k_p-\frac{1}{2}k_d)^2}\right). \quad (5)\end{aligned}$$

The values of k_p, k_d, and k_n are related by energy conservation, i.e., $\frac{1}{2}k_d^2=k_p^2+k_0^2-k_n^2$, where $k_0^2=2\alpha^2$.

For the case with $l_n\neq0$, the corresponding integral contains the function $h_{l_n}^{(1)}(ik_nr_n)Y_{lm}(\vartheta,\varphi)$. This more general integral apparently cannot be evaluated in closed form, and although the evaluation in terms of angular momentum contributions can be carried out, the result is rather complicated, and will not be presented here.

EVALUATION OF THE ANGULAR DISTRIBUTION

The angular distribution of the emergent protons is proportional to the square of the absolute value of I, Eq. (5). Since we are concerned only with the shape of the distribution, all angle-independent factors can be omitted. Thus one finds

$$W(\theta)=[2(k_p-\tfrac{1}{2}k_d)^2+k_0^2+4k_pk_d\sin^2(\theta/2)]^{-2}|{}_2F_1|^2. \quad (6)$$

The first factor in this formula for $W(\theta)$ is responsible for the broad forward peak in the usual case (with $R=0$) where η_p and η_d are taken to be zero. The second

[9] A. Sommerfeld, *Wellenmechanik* (Frederick Ungar Publishing Company, New York, 1953), Chap. 7.

factor is responsible for the Coulomb-produced changes in the angular distribution.

To obtain a qualitative feeling for the type of change produced by the Coulomb effects, it is useful to consider first the limit of large nuclear charge (Z), or small velocities, or both. That is, we take $\eta_d, \eta_p \gg 1$.

A convenient approximation for the hypergeometric function (which gives a good account of its behavior over the entire range from the turning point to small values of the argument), is afforded by a modification of the JWKB method,[10] employing the Bessel functions of order $\frac{1}{3}$. The result has the form

$$ {}_2F_1(-i\eta_p, -i\eta_d, 1; x) \cong B x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}(\eta_p+\eta_d)} \times (\varphi/g^{\frac{1}{2}})^{\frac{1}{2}} h_{\frac{1}{3}}(\varphi). \quad (7) $$

Here B is a constant independent of x, that is adjusted for best fit. Since we are concerned only with the x dependence, B can be set equal to unity.

The function g is rather complicated in general, but for large values of η we have

$$ \begin{aligned} g &= x^{-1}(1-x)^{-2}(x-x_0)(\eta_p-\eta_d)^2, \\ x_0 &= -4k_pk_d/(k_p-\tfrac{1}{2}k_d)^2. \end{aligned} \quad (8) $$

The function φ is the phase function,

$$ \begin{aligned} \varphi &= \int_{x_0}^{x} g^{\frac{1}{2}} dx \\ &= i(\eta_d-\eta_p)\cos^{-1}\left[\frac{k_0^2+2(k_p-\frac{1}{2}k_d)^2\cos\theta}{k_0^2+2(k_p-\frac{1}{2}k_d)^2}\right] \\ &\quad + i(\eta_p+\eta_d)\cos^{-1} \\ &\qquad \times\left[\frac{k_0^2+2(k_p^2+\frac{1}{4}k_d^2)\cos\theta-2k_pk_d}{k_0^2+2(k_p^2+\frac{1}{4}k_d^2)-2k_pk_d}\right]. \end{aligned} \quad (9) $$

For typical values of the parameters (k_p, k_d not too large compared to k_0), the phase function φ is large enough so that the $h_{\frac{1}{3}}(\varphi)$ becomes approximately exponential. If one considers all factors except the exponential as slowly varying with angle, then an approximate asymptotic form for $W(\theta)$ is obtained:

$$ W(\theta) \propto \exp(-2|\phi|). \quad (10) $$

From Eq. (9) it is seen that $|\varphi|$ is smallest for $\theta=\pi$, and rapidly increases as θ departs from π. Letting $(\pi-\theta)$ be a small quantity, we obtain the approximate result

$$ \begin{aligned} W(\theta) &= \exp[-a^2(\pi-\theta)^2], \\ a^2 &= \sqrt{2}|\eta_d-\eta_p|\frac{k_0}{|k_p-\frac{1}{2}k_d|}\left(\frac{2(k_p-\frac{1}{2}k_d)^2+k_0^2}{2(k_p+\frac{1}{2}k_d)^2+k_0^2}\right). \end{aligned} \quad (11) $$

Thus we find that for large values of η_p and η_d, the angular distribution becomes Gaussian about the backward direction, with a width that decreases as η increases. This result is similar, of course, to the result already given in reference 6, for the electric disintegration of the deuteron.

A better than qualitative result can also be obtained from the approximation of Eq. (7). This approximation is, however, based essentially upon an expansion in η^{-1} and can be quite inaccurate at, say, $\eta\sim 3$, which is a not untypical value. It is unnecessary to attempt to improve the approximation actually since the ${}_2F_1$ function that enters has been rather extensively investigated in connection with bremsstrahlung[11,12] and Coulomb excitation[11] problems where it also occurs.

We illustrate these more accurate results, for a typical case, in Fig. 1. Since the results are not very sensitive to the binding energy of the capturing level, we have arbitrarily selected the neutron to be bound with an energy of 2.23 Mev. The angular distribution of the emergent protons, normalized to unity at the maximum, is shown for various deuteron bombarding energies with uranium as the target element. The curves show a pronounced change in the angular distribution, from forward to backward peaking as the energy decreases. Perhaps the most noticeable feature is the high energy required ($\sim$300 Mev) before the distribution assumes the familiar Butler shape.

Similar results are as easily obtained for a variety of experimental conditions, but will not be given here.

It is of interest to inquire as to the effect of taking $l_n\neq 0$. For low values of $\eta(\sim 3/2)$, the results of Fig. 1 and of reference 4, show that the Butler peaks are displaced slightly to larger angles, and hence l_n is a critical parameter. As mentioned earlier, a closed form for the $l_n\neq 0$ case apparently cannot be obtained. Although one may obtain a formal result by employing an angular momentum expansion (for which the radial integrals can be obtained fairly readily), this is clearly not a feasible procedure since the result yields an angular distribution in the form of a Legendre series rather poorly adapted to sharply peaked functions.

Some insight can be obtained from this series, however, in the limit of large η. This limit insures the regime of classical orbits, and, as will be shown in detail elsewhere, the classical identification of the angle of scattering with the orbit eccentricity [i.e., $\sin(\theta/2) = \epsilon^{-1} = (1+l^2/\eta_d\eta_p)^{-\frac{1}{2}}$] results independently of the specific radial interaction, provided only that a classical limit for the radial integrals exists (in the sense that the radial integrals vary slowly with eccentricity).

For the case at hand, there does not exist a true classical limit since, for η large, v_d does not necessarily approach v_p. As a result, the radial integrals no longer vary slowly, but decrease exponentially with $\eta\epsilon$. This

[10] R. E. Langer, Phys. Rev. **51**, 669 (1937); P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, Chap. 9.

[11] Thaler, Goldstein, McHale, and Biedenharn, Phys. Rev. **102**, 1567 (1956).

[12] R. Berger (unpublished).

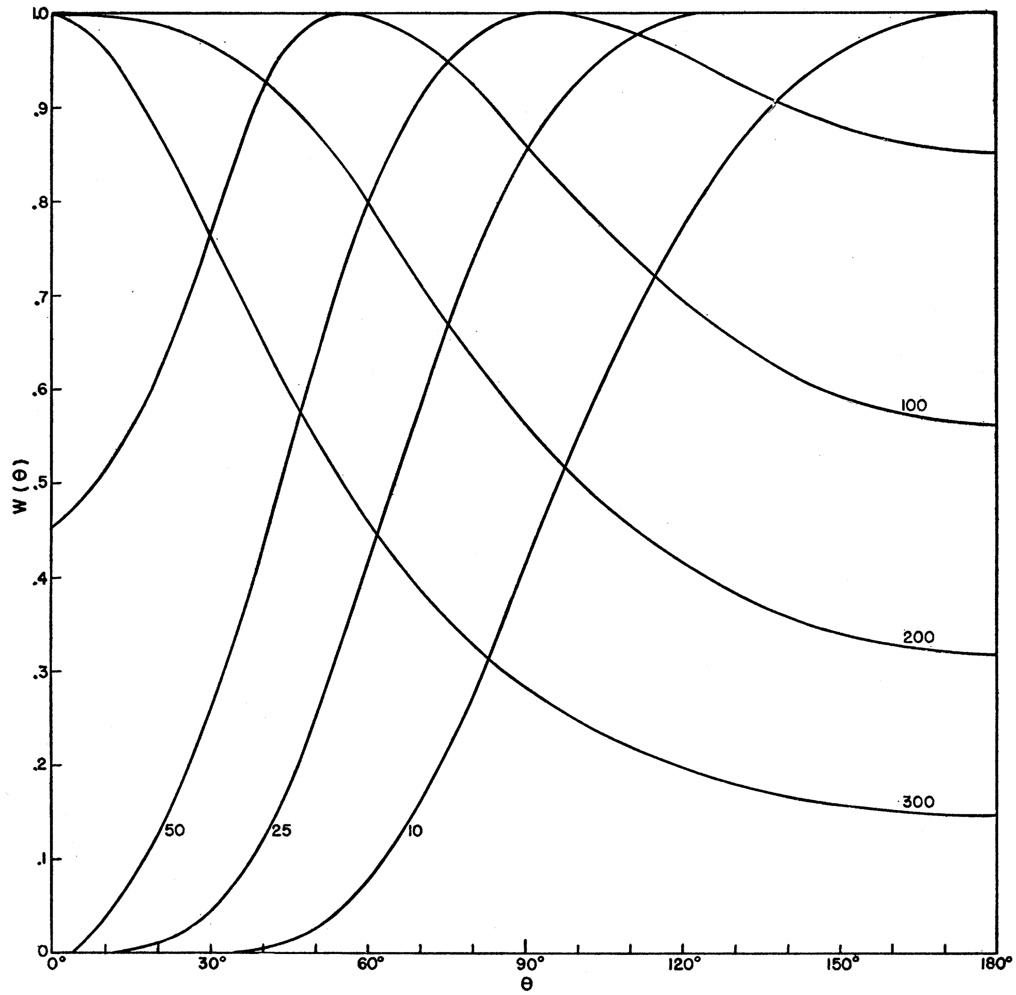

FIG. 1. The angular distribution, $W(\theta)$ *versus* θ, for the emergent protons in a (d,p) reaction on $Z=92$. The curves are normalized to unity at the maximum; the neutron is taken to be captured in a bound level of -2.23 Mev. The energy of the incident deuterons (in Mev) labels the various curves. The Sommerfeld number η_d ranges from $\eta_d=1.3$ at $E_d=300$ Mev to $\eta_d=7.1$ at $E_d=10$ Mev.

exponential behavior, however, does not depend sensitively upon the parameter l_n. If we continue to identify the angle of scattering with the eccentricity, it is clear that small ϵ, and hence $\theta\sim\pi$, is strongly favored. Although the relation $\sin(\theta/2)=\epsilon^{-1}$ no longer holds precisely, this only affects the width of the Gaussian about $\theta=\pi$. Thus one sees that the backward peaking is a characteristic effect of large η, and large momentum change, and does not depend sensitively on l_n.‡ A similar result also holds for the analogous case of inelastically scattered particles in Coulomb excitation, for large values of η and $\xi=\eta_f-\eta_i$.

‡ *Note added in proof.*—This insensitivity to the value of l_n in cases where Coulomb effects are considered of major importance has also been found in some unpublished calculations of Tobocman and Kalos (private communication from W. Tobocman).

CONCLUDING REMARKS

The pronounced backward maximum that occurs for large values of η should be an easily distinguished feature of (d,p) stripping. To date experimental evidence for this effect is lacking. (There have been a few cases where strong backward peaks have been observed,[13] but it is not clear that this should be interpreted as indicated above since η is not large.)

For large values of η, the cross section falls off exponentially and perhaps intensity will be troublesome. An experiment which promises to circumvent such difficulties, the (d,p) reaction on U, using fission coincidences to select the protons, is in progress at this laboratory.

[13] For example, G. C. Phillips, Phys. Rev. 80, 164 (1950).