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Growing Electric Space-Charge Waves

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Some familiar aspects of the theory of growing space-charge waves are restated as a comment on recent criticism of this theory by J. H. Piddington.

IT has been recently asserted by Piddington[1] that most of the work done in the last few years on the propagation of small disturbances in drifting ion streams is invalid. In particular, the proposed explanation for the operation of several amplifying devices is said to be illusory. Piddington's position appears to be that the familiar treatment rests upon "substitution analysis"; that the growing waves sometimes encountered in the analysis are "ephemeral" and cannot be described by a process involving genuine energy transfer between ions and field; and, finally, since it is necessary to look elsewhere for an explanation of the growth observed, that this may lie in the trapping of electrons in moving potential minima. It seems to be desirable to comment on these claims by reiterating some pertinent aspects of the usual theory.

For the purposes of the present discussion it will be sufficient to suppose that the dc properties of the ion streams are uniform in space and time and that all velocities are in the same (z) direction. Temporarily, it will be assumed that one is dealing with a finite number of discrete streams; later some remarks will be made about the more subtle case of continuous velocity distributions. The basic nonlinear equations are the equations of motion, the equation of continuity and an equation relating the fields to the driving currents. For small disturbances it is agreed that one may linearize these equations, obtaining a set of linear partial differential equations, which in the present case have constant coefficients. In any physically interesting case these equations are complemented by definite boundary conditions and one has then a sensibly posed mathematical problem to solve.

One adequate method for the solution of many such problems is that of Laplace transforms.[2–5] Another is by superposition of particular solutions of the differential equations. Clearly, these solutions, for any specific problem, may be drawn from the class of solutions of the form $\exp j(\omega t+\Gamma z)$, where ω and Γ are, in general, complex and satisfy a relation of the form, $f(\omega,\Gamma)=0$, determined solely by the differential equations.[6] This relation between ω and Γ, the dispersion relation, is obviously significant for the problem on hand since it describes an intrinsic property of the ion streams and may, of itself, give some valuable information. But no definitive conclusions can be drawn from it alone about any specific problem, since only the boundary conditions can say what part of the ω-Γ spectrum is excited and to what extent.

A simple, typical example of a problem soluble by either of these methods is that of two beams of electrons in free space having different dc velocities (both of the same sign) which pass through a pair of closely spaced grids to which a small ac voltage is applied.[7,8] One may examine the ac signal level on the mixed beams at any point downstream. This is found to increase roughly exponentially with distance as, in fact, it does in practice. Examination of the solution shows that it consists in part of a term of the form, $\exp j(\omega t+\Gamma z)$, for which $\mathrm{Im}\Gamma<0$. It is this term which causes the solution to increase with z and one may call this a "growing" wave. The analysis of traveling-wave tube operation, which involves a single beam velocity and an external circuit, also gives a solution containing such a growing wave.[9] Clearly, where one growing wave is present, there will be approximately exponential gain at large distances where this term dominates the solution. Such a growing wave is not, however, necessary for gain. In the case of the backward-wave amplifier such a term does not exist and the observable gain is attributable to the interference of waves of uniform amplitude.[4,10,11]

In another class of problems such as the backward-wave oscillator, one is concerned with the time stability of a given ion flow subject to fixed boundary conditions.[4,12] Here the boundary conditions are all spatial and the criterion of stability is the presence or absence in the solution of a term which has a complex frequency. The equations being real, complex frequencies occur in pairs, one of which means growth. The irrelevance of

[1] J. H. Piddington, Phys. Rev. **101**, 14 (1954).
[2] J. R. Pierce, Proc. Inst. Radio Engrs. **40**, 1675 (1952).
[3] M. Scotto and P. Parzen, J. Appl. Phys. **27**, 375 (1956).
[4] L. R. Walker, J. Appl. Phys. **24**, 854 (1953).
[5] R. Q. Twiss, Phys. Rev. **88**, 1392 (1952). This paper discusses the shortcomings of substitution methods and the importance of boundary conditions.
[6] There will be some degenerate cases arising from multiple roots of $f(\omega,\Gamma)=0$ where the solution becomes, for example, $z\exp j(\omega t+\Gamma z)$. See J. R. Pierce, J. Appl. Phys. **15**, 721 (1944); J. Appl. Phys. **21**, 1063 (1950).
[7] A. V. Haeff, Phys. Rev. **74**, 1532 (1948); Proc. Inst. Radio Engrs. **37**, 4 (1949).
[8] J. R. Pierce and W. B. Hebenstreit, Bell System Tech. J. **28**, 33 (1949).
[9] J. R. Pierce, *Traveling Wave Tubes* (D. Van Nostrand Company, Inc., New York, 1950).
[10] See reference 9, Chap. XI.
[11] R. Kompfner and N. T. Williams, Proc. Inst. Radio Engrs. **41**, 602 (1953).
[12] See the references in 6.

the dispersion equation to the nature of the complete solution in certain problems is illustrated by the flow of a single ion stream between short-circuited grids. Here all space charge waves are unattenuated at real frequencies or, in the language of the dispersion relation, Γ is real when ω is; nevertheless complex frequencies arise for the system as a whole when the boundaries are included.

The power flow and energy storage of ac modulated beams is sufficiently well understood[13,14] for a fairly clear picture to be formed of the processes which lead to spatially growing waves of the types encountered in multiple-stream flow and in traveling-wave tubes. A fast space-charge wave (an unattenuated wave moving faster than the electrons) or an electromagnetic circuit-wave stores positive ac energy; to increase the signal level ac energy must be fed in. A slow space charge wave, in which the electrons move faster than the wave, has a negative energy storage and negative power flow; if ac energy is extracted the amplitude increases. If two such waves now interact by means of some mutual coupling of sufficient strength a pair of mixed waves arises, one growing and one decreasing exponentially with distance. The total power flow for each of these new waves is zero since it is at the same time independent of position and proportional to $\exp[2\mathrm{Im}\Gamma]$. It may be considered then that in the growing wave the negative power flow component is continually transferring energy to the positive flow component.[15]

In the case of increasing waves of this type, which are due to coupling between a negative-energy unattenuated wave and a nearly synchronous positive energy unattenuated wave, it is possible to see from the dispersion equation that the increasing wave can result in amplification. If the group velocity of each of the waves, when uncoupled, is in the direction of electron flow, as it is in the cases discussed above, then we may argue that we can excite both of the unattenuated waves so that they will travel away from the point of excitation in the direction of electron flow. If we then introduce coupling between the two unattenuated waves beyond the point of excitation, we will find that we have excited a mixed wave which grows in the direction of electron flow. In general, a wave which decays in the direction of electron flow will also be excited in this procedure.

When the velocity distribution of the ion stream has a continuous range, new features arise in the discussion of propagation. Objections have sometimes been raised to the linearizing procedure here, because of the possibility that waves may occur whose velocity is exactly that of some ion stream. There would then be no signal level small enough for linearization to be valid. However, the theory is actually self-consistent in its linear form because if it ever predicts real propagation constants at real frequencies the resultant wave velocity automatically lies outside the continuous part of the velocity distribution. This particular point is not raised by Piddington.

In the case of discrete velocities it is known that the dispersion relation yields for real frequencies twice as many solutions for Γ as there are streams. It might, therefore, be supposed that in the case of a continuous distribution there would be an infinite number of roots of the dispersion equation. This is not the case; there are only a finite number of collective modes of propagation and this number may be zero.[16] Approximations to the dispersion equation such as Piddington uses in discussing the Maxwellian ion cloud, which are based upon an expansion in moments of the velocity distribution function, may alter the number of collective modes found. Examination of the complete solution of a problem with boundary conditions shows that the part of the ac term in the distribution function which is not accounted for by the collective terms consists of a superposition of disturbances moving with the velocities of the individual streams. These are, in effect, convected disturbances. Clearly the dispersion equation in this problem provides an inadequate description of the motion.

The nonlinear trapping process which Piddington has put forward as a possible mechanism for growing waves can have very little relevance for observed amplification. At low levels the electrons in a double beam tube, for example, have velocities relative to the wave great enough to enable them to over-ride any existing potential trough arising from the wave.

[13] W. H. Louisell and J. R. Pierce, Proc. Inst. Radio Engrs. **43**, 425–427 (1955).

[14] L. R. Walker, J. Appl. Phys. **26**, 1031 (1955); J. Appl. Phys. **25**, 615 (1954).

[15] J. R. Pierce, Bell System Tech. J. **33**, 1343 (1954).

[16] L. R. Walker, J. Appl. Phys. **25**, 131 (1954).