Normal Modes Characterizing Magnetoelastic Plane Waves*†

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The general theory of magnetohydrodynamic waves in an ideal conducting fluid embedded in a uniform field of magnetic induction, and the application of the theory to the systematic analysis of the various modes of propagation in incompressible and compressible fluids have been presented by the author in two earlier papers. The techniques developed there have now been extended to the study of homogeneous plane waves in an unbounded conducting solid. It is found that, in the presence of a uniform static magnetic field, the elastic medium sustains five distinct modes of propagation as already pointed out by Knopoff. The first two modes are pure *shear* waves, one of which is a slightly attenuated shear mode with phase velocity modified anisotropically by magnetoelastic coupling, whereas the companion shear mode is highly attenuated and exhibits a propagation constant which is essentially that of an electromagnetic wave of the same frequency. The remaining three modes are *shear-compression* waves of which one mode exhibits a phase velocity intermediate between the phase velocity of shear and compressional waves, the second mode has a phase velocity exceeding that of compressional waves, and the third shear-compression mode is again highly attenuated with the propagation characteristics of an electromagnetic wave.

1. INTRODUCTION

THE advent of magnetohydrodynamics, which dates from Alfvén's classical paper,[1] has led to the re-examination of various physical phenomena in which a moving conducting medium interacts with a magnetic field. As an example of magnetoelastic interactions on a geophysical scale, we have Cagniard's recent suggestion[2] that the observed change in the velocities of seismic waves in going from the mantle to the core of the earth, as well as the fact that only one mode seems capable of traversing the core, could be explained not on the basis of widely different elastic properties for the mantle and the core, as in the current theory which assumes a fluid core, but on the basis of magnetoelastic interactions in the presence of the earth's interior magnetic field. He then argues that the only seismic wave capable of traversing the core is a shear mode with particle velocity lying in the plane of the magnetic field and the wave normal, implying that the other two normal modes are attenuated beyond detection in traversing the core.

Another example of magnetoelastic interactions, this time of laboratory scale, is found in a paper by Robey[3] in which he reports on the dispersive effect of a magnetic field on the phase velocity of longitudinal or compressional waves in a conducting plate of uniform cross section. More recently, Knopoff[4] has presented a paper dealing with the interaction between elastic waves and a magnetic field in electrical conductors and has examined more closely the geophysical implications of the theory, having reached the conclusion that magnetoelastic interactions are not a significant mechanism in the earth's core. Thus, in order to throw more light on these and related questions, we present in this paper a systematic study of the characteristics of propagation of magnetoelastic plane waves.

In two earlier papers by the author,[5,6] hereinafter referred to as I and II, respectively, we presented the general theory of magnetohydrodynamic waves in an ideal conducting fluid embedded in a uniform static magnetic field, and then we proceeded to apply the general theory to the systematic study of the various modes of propagation in incompressible and compressible fluids. The techniques employed in these two papers have now been extended to the study of homogeneous magnetoelastic plane waves in an unbounded conducting solid which is fully characterized by its Lamé moduli and by the rigorously constant macroscopic parameters μ, ϵ, and σ. It is found that, in the presence of a uniform static magnetic field, the medium sustains five distinct modes of propagation as already pointed out by Knopoff.[4] The first two modes are pure *shear* waves, one of which is a slightly attenuated shear mode with phase velocity modified anisotropically by magnetoelastic interaction, whereas the companion shear mode is highly attenuated and exhibits essentially the propagation characteristics of an electromagnetic wave of the same frequency. The remaining three modes are *shear-compression* waves, except when the propagation takes place along certain preferred directions which are discussed here. The three modes in question can be characterized briefly by pointing out that one mode exhibits a phase velocity intermediate between the phase velocities of shear and compressional waves, the second mode has a phase velocity which exceeds that of compressional waves, and the last mode is again a highly

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[3] D. H. Robey, J. Acoust. Soc. Am. **25**, 603 (1953). See also, R. A. Alpher and R. J. Rubin, J. Acoust. Soc. Am. **26**, 452 (1954).

[4] L. Knopoff, J. Geophys. Research **60**, 441 (1955).

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[6] A. Baños, Jr., Proc. Roy. Soc. (London) **A233**, 350–367 (1955).

attenuated wave with propagation characteristics almost identical to those of an electromagnetic wave in the medium.

To apply the techniques mentioned earlier, we first refer to the linearized hydromagnetic wave equation (I–66) which, as written, applies quite generallly to incompressible and compressible fluids as well as to elastic conducting solids, depending merely on the definition of the vector **F**. In the present instance, for time-harmonic magnetoelastic waves in a homogeneous and isotropic solid conductor, we have

$$i\omega \mathbf{F}=\omega^2\rho\mathbf{v}+\rho(V_p^2-V_s^2)\nabla\nabla\cdot\mathbf{v}+\rho V_s^2\nabla^2\mathbf{v}, \quad (1)$$

where **v** is the particle velocity, ω is the angular frequency of time-harmonic oscillations, ρ is the density of the medium, and where

$$V_p=[(\lambda+2\mu)/\rho]^{\frac{1}{2}} \quad\text{and}\quad V_s=(\mu/\rho)^{\frac{1}{2}} \quad (2)$$

are the phase velocities of compressional and shear waves, respectively, for an elastic solid characterized by its Lamé moduli λ and μ.

In order to analyze (I–66), it proves convenient to reduce it to three scalar equations by taking its z component, its divergence, and the z component of its curl. In this way, after substituting for **F** the explicit form (1), we obtain

$$k_p^2(\nabla^2+k_s^2)v_z=-(k_s^2-k_p^2)(\partial/\partial z)(\nabla\cdot\mathbf{v}), \quad (3)$$

$$\{(\nabla^2+k_c^2)[(k_s^2-k_p^2)(\partial^2/\partial z^2)+(\nabla^2+k_s^2)(k_p^2-ia(\nabla^2+k_p^2))]+k_a^2(\nabla^2+k_p^2)(\nabla^2+k_s^2)\}(\nabla\cdot\mathbf{v})=0, \quad (4)$$

$$\{(\nabla^2+k_c^2)[k_s^2-ia(\nabla^2+k_s^2)]+k_a^2(\nabla^2+k_s^2)-k_s^2(1-ib)^{-1}\nabla_t^2\}[\hat{e}_z\cdot(\nabla\times\mathbf{v})]=0, \quad (5)$$

where k_a, k_c, k_p, and k_s are the wave numbers associated respectively with Alfvén's phase velocity, with the velocity of light, and with the compressional and shear phase velocities of the medium,

$$k_a=\omega/V_a=\omega(\mu\rho)^{\frac{1}{2}}/B_0, \quad k_c=\omega(\mu\epsilon)^{\frac{1}{2}}, \quad k_p=\omega/V_p, \quad k_s=\omega/V_s, \quad (6)$$

and where we have introduced the dimensionless parameters

$$a=\omega\rho/\sigma B_0^2 \quad\text{and}\quad b=\omega\epsilon/\sigma, \quad (7)$$

which vanish in the limit of infinite conductivity. Here a is a measure of magnetoelastic coupling, while b is the ratio of the displacement to the conduction currents in the medium. The present set of scalar equations (3), (4), and (5) reduce to (II-7), (II-8), and (II-4), respectively, which refer to a compressible fluid, by merely putting $V_s=0$ and by identifying the velocity of compressional waves V_p with the velocity of sound in the fluid. Thus, we note that the family of magneto-acoustic modes discussed in II can be regarded as the limiting form of the present magnetoelastic modes as the velocity of shear waves is made to vanish.

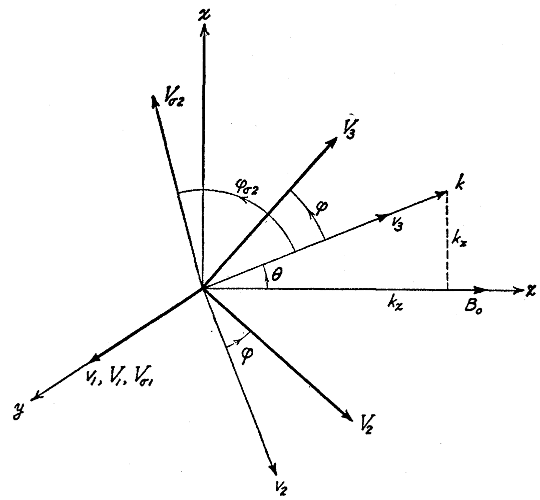

FIG. 1. Velocity vectors characterizing the five magnetoelastic modes.

2. MAGNETOELASTIC PLANE WAVES

In accordance with Sec. I-3, we take the uniform magnetic field parallel to the z axis, $\mathbf{B}_0=\hat{e}_zB_0$, and we assume that the Cartesian components of all field vectors exhibit the common space-time dependence

$$\psi(\mathbf{r},t)=\exp\{i(\mathbf{k}\cdot\mathbf{r}-\omega t)\}, \quad (\nabla^2+k^2)\psi=0, \quad (8)$$

where **k** is the vector propagation constant. Referring to Fig. 1, we choose **k** as

$$\mathbf{k}=\mathbf{n}k=\hat{e}_xk_x+\hat{e}_zk_z; \quad \mathbf{n}=\hat{e}_z\cos\theta+\hat{e}_x\sin\theta, \quad (9)$$

where θ is the angle between the wave normal and the direction of the applied field $(0\leqslant\theta\leqslant\frac{1}{2}\pi)$, and where k_x and k_z represent, respectively, the transverse and longitudinal wave numbers.

Next, we introduce the elementary vector wave functions whose amplitudes are depicted in Fig. 1 and which are defined by

$$\mathbf{v}_1=\hat{e}_yv_0\psi, \quad \mathbf{v}_2=\mathbf{n}\times\mathbf{v}_1=(\mathbf{n}\times\hat{e}_y)v_0\psi, \quad\text{and}\quad \mathbf{v}_3=\mathbf{n}v_0\psi, \quad (10)$$

in which ψ is given by (8) and v_0 is an arbitrary velocity amplitude which in the present linearized theory is assumed small in comparison with Alfvén's phase velocity V_a. All three velocity vectors (10) are solutions of the vector Helmholtz equation, $(\nabla^2+k^2)\mathbf{v}=0$, and in addition exhibit the following properties:

$$\begin{aligned}
&\mathbf{k}\cdot\mathbf{v}_1=0, && \hat{e}_z\cdot(\mathbf{k}\times\mathbf{v}_1)=kv_0\psi\sin\theta, && v_{z1}=0,\\
&\mathbf{k}\cdot\mathbf{v}_2=0, && \hat{e}_z\cdot(\mathbf{k}\times\mathbf{v}_2)=0, && v_{z2}=v_0\psi\sin\theta, \quad (11)\\
&\mathbf{k}\cdot\mathbf{v}_3=kv_0\psi, && \hat{e}_z\cdot(\mathbf{k}\times\mathbf{v}_3)=0, && v_{z3}=v_0\psi\cos\theta.
\end{aligned}$$

Our present problem involves the computation, for a given direction θ, of the wave numbers k corresponding to all possible modes of propagation, as well as the identification of the corresponding velocity wave functions as suitable linear combinations of the elementary forms (10).

The first set of magnetoelastic modes, which is devoid of compressional oscillations, belongs to the class of *velocity modes* in accordance with the nomenclature adopted in II. The set is generated by selecting the particle velocity

$$\mathbf{v}=\mathbf{v}_1, \tag{12}$$

which, according to (11), automatically satisfies (3) and (4) since it is solenoidal and exhibits no z component. Then, to satisfy (5), assuming for the moment that $\theta>0$, we need only equate the bracket to zero after replacing ∇ by $i\mathbf{k}$ for plane waves. In this manner we obtain the quadratic in k^2:

$$(k_a^2+k_s^2\cos^2\theta)k^2-k_s^2(k_a^2+k_c^2) \\ =-ia[(k^2-k_c^2)(k^2-k_s^2)-i(b/a)(1-ib)^{-1}k_s^2k_x^2], \tag{13}$$

which has two roots and, therefore, yields two distinct modes.

In the case of infinite conductivity ($a=0$), the right-hand side vanishes and (13) yields only one mode, the first *normal* mode, which is a pure shear wave modified anisotropically by magnetoelastic interaction. The companion mode which ensues from (13) only when the conductivity is finite is a strongly attenuated pure shear wave propagating with the characteristics of an electromagnetic wave slightly modified by magnetoelastic interaction.

The second set of magnetoelastic modes, which is characterized by the simultaneous presence of shear and compressional oscillations, belongs to the class of *pressure modes* as described in Sec. II-4 for compressible fluids. The set is generated by selecting from (10) the particular linear combination

$$\mathbf{v}=\mathbf{v}_3\cos\phi-\mathbf{v}_2\sin\phi, \tag{14}$$

which is illustrated in Fig. II-12 and in which ϕ is a convenient angular parameter that measures the relative amounts of solenoidal and irrotational components of velocity needed to obtain a solution of the original hydromagnetic wave equation. Thus, according to (11), we see that (5) is automatically satisfied, while (3) yields an equation which determines ϕ in terms of k, namely

$$\tan\theta\,\tan\phi=k_s^2(k^2-k_p^2)/k_p^2(k^2-k_s^2); \tag{15}$$

and, to comply with (4), we merely equate the bracket to zero after replacing ∇ by $i\mathbf{k}$. In this way we obtain the cubic in k^2:

$$(k^2-k_c^2)\{(k_s^2-k_p^2)k_z^2+(k^2-k_s^2)[k_p^2+ia(k^2-k_p^2)]\} \\ +k_a^2(k^2-k_p^2)(k^2-k_s^2)=0, \tag{16}$$

which has three roots and, therefore, yields three distinct modes.

When the conductivity is infinite, we have $a=0$ and (16) becomes a quadratic in k^2 with two distinct roots corresponding to the two additional *normal* modes which characterize the set of plane homogeneous magnetoelastic modes in a medium of infinite conductivity. One such mode is described by the fact that its phase velocity is intermediate between the phase velocities of shear and compressional waves, while the other mode is distinguished by the fact that its phase velocity exceeds the phase velocity of compressional waves in the medium. The third shear-compression mode which only ensues from (16) when the conductivity is finite is a highly attenuated wave propagating with the characteristics of an electromagnetic wave modified anisotropically by magnetoelastic interaction.

3. SHEAR MODES

Rather than attempting to describe from (13) and (16) the characteristics of all five modes listed above, which become quite involved in the general case, we prefer to consider at once some special cases of physical interest. In particular, we shall assume from now on that the electric displacement current can be altogether neglected in comparison with the conduction current in the medium, which we effect by merely putting $\epsilon=0$ in (13) and (16).

When the conductivity is infinite we have from (7) $a=0$ and $b=0$. In this case, (13) yields the *first* normal mode which is characterized by the purely transverse velocity vector

$$\mathbf{V}_1=\mathbf{v}_1=\hat{e}_y v_0\psi, \quad k=k_1\ (0\leqslant k_1\leqslant k_s), \tag{17}$$

where ψ is given by (8) with wave number $k=k_1$ as deduced from (13) after putting $k_c=0$ and $a=0$, namely

$$k_1^2=k_a^2k_s^2/(k_a^2+k_s^2\cos^2\theta). \tag{18}$$

Putting $k_1=\omega/U_1$, where U_1 is the phase velocity of the first normal mode, we obtain from (18)

$$U_1^2=V_s^2+V_a^2\cos^2\theta, \tag{19}$$

which corresponds to a pure shear mode with phase velocity modified anisotropically by magnetoelastic interaction. If we let $V_s\rightarrow 0$, the mode in question behaves like an Alfvén wave or velocity mode in magnetohydrodynamics.

If the conductivity is finite, yet so large that $a\ll 1$, we can regard the right-hand side of (13) as a small perturbation. In this way we obtain to first order of small quantities, instead of (18), the first root of the quadratic

$$k_1^2\approx\frac{k_a^2k_s^2}{k_a^2+k_s^2\cos^2\theta}\left\{1+\frac{iak_s^4\cos^2\theta}{(k_a^2+k_s^2\cos^2\theta)^2}\right\}, \tag{20}$$

which corresponds to the first normal mode (slightly attenuated). The second root that ensues from (13) can be written as

$$k_{\sigma 1}^2\approx i\omega\mu\sigma\left[1+\frac{k_s^2\cos^2\theta}{k_a^2}\right]\left\{1-\frac{iak_s^4\cos^2\theta}{(k_a^2+k_s^2\cos^2\theta)^2}\right\}, \tag{21}$$

which is frankly dominated by the factor $i\omega\mu\sigma$; that is, the propagation constant $k_{\sigma 1}$ is virtually the propagation constant of an electromagnetic wave penetrating into the metal as governed by its own skin depth. Such modes are often referred to in the literature as "skin" waves. Thus, the companion pure shear mode associated with the first normal mode when the conductivity is finite is a highly attenuated modified "skin" wave. It is to be noted further that, as $\theta \to \frac{1}{2}\pi$ in (20) and (21), we obtain *exactly* $k_1^2 = k_s^2$ and $k_{\sigma 1} = i\omega\mu\sigma$, indicating that the first normal mode has degenerated into a pure elastic (shear) wave, while the companion mode has become a pure electromagnetic wave. This result is a simple consequence of the fact that, when the wave normal is at right angles to the direction of the externally applied field ($\theta = \frac{1}{2}\pi$), the particle oscillations for this pair of modes become parallel to the magnetic field and, hence, there is no magnetoelastic interaction.

Finally, we consider the case in which $a \gg 1$ which, according to (7), corresponds to weak magnetoelastic coupling arising from a vanishingly small magnetic field. In this case we return to (13) after multiplying both sides by i/a and seeking the expansion of the roots in reciprocal powers of a. Thus, we obtain the pair of roots

$$k_1^2 \approx k_s^2 \left\{ 1 - \frac{ik_s^2 \cos^2\theta}{a(i\omega\mu\sigma - k_s^2)} \right\}, \tag{22}$$

$$k_{\sigma 1}^2 \approx i\omega\mu\sigma \left\{ 1 + \frac{ik_s^2 \cos^2\theta}{a(i\omega\mu\sigma - k_s^2)} \right\}, \tag{23}$$

which correspond respectively to the first normal mode and to its companion highly attenuated "skin" wave. Hence, in the case of weak magnetoelastic coupling, the first normal mode is governed essentially by the shear velocity in the medium, whereas the companion mode is governed by the electromagnetic properties of the elastic conductor.

4. SHEAR-COMPRESSION MODES

First we examine the case of infinite conductivity in the absence of electric displacement current. Thus, putting $k_e = 0$ and $a = 0$ in (16), we obtain the quadratic in k^2,

$$k^2[(k_s^2 - k_p^2)k_z^2 + (k^2 - k_s^2)k_p^2] + k_a^2(k^2 - k_p^2)(k^2 - k_s^2) = 0, \tag{24}$$

which yields two roots, $k = k_2$ and $k = k_3$, corresponding to the two remaining *normal* modes which, according to (14), are characterized by the velocity vectors

$$\mathbf{V}_2 = \mathbf{v}_2 \cos\phi + \mathbf{v}_3 \sin\phi, \quad k = k_2 \ (k_p \leqslant k_2 \leqslant k_s); \tag{25}$$

$$\mathbf{V}_3 = -\mathbf{v}_2 \sin\phi + \mathbf{v}_3 \cos\phi, \ k = k_3 \ (0 \leqslant k_3 \leqslant k_p), \tag{26}$$

where the angular parameter ϕ $(0 \leqslant \phi \leqslant \frac{1}{2}\pi)$ is determined from (15) either by inserting $k = k_3$,

$$\tan\theta \tan\phi = k_s^2(k_p^2 - k_3^2)/k_p^2(k_s^2 - k_3^2), \tag{27}$$

or else by putting $k = k_2$ and replacing ϕ by $(\phi - \frac{1}{2}\pi)$,

$$\tan\theta \cot\phi = k_s^2(k_2^2 - k_p^2)/k_p^2(k_s^2 - k_2^2), \tag{28}$$

which are equivalent expressions leading to an identity readily verified by using the original quadratic (24).

The situation for all three normal modes, (17), (25), and (26), is illustrated in Fig. 1 in which the angle ϕ appears as the angle of rotation about the y axis which the elementary vector wave functions $\mathbf{v}_2$ and $\mathbf{v}_3$ must undergo in order to generate the corresponding normal modes $\mathbf{V}_2$ and $\mathbf{V}_3$. In the figure the vector $\mathbf{k}$ denotes the direction of the wave normal at a given angle θ for all three normal modes; that is, $\mathbf{k}$ stands for $\mathbf{k}_1$, $\mathbf{k}_2$, or $\mathbf{k}_3$, all of which have the same direction but, of course, different magnitudes in accordance with (17), (25), or (26).

The determination of the roots of (24) is a matter of elementary algebra, but the resulting expressions are much too involved to be of practical use. Instead, we prefer to examine graphically the behavior of the two roots k_2 and k_3 as the magnetic field varies from zero to infinity for fixed values of the angle θ $(0 \leqslant \theta \leqslant \frac{1}{2}\pi)$. To this end we first reduce the number of independent parameters by restricting our attention to an ideal elastic solid satisfying Cauchy's relations,[7] according to which Poisson's ratio becomes $\sigma = \frac{1}{4}$. In this case the Lamé moduli become equal to each other, $\lambda = \mu$, and according to (2) we have a material for which $k_s^2 = 3k_p^2$. Furthermore, we define the dimensionless parameter[8]

$$r = k_a^2(k_s^2 - k_p^2)/k_s^2 k_p^2, \tag{29}$$

which becomes in the present instance $r = 2k_a^2/3k_p^2$ and which provides a measure of the magnetic field since k_a depends on the magnetic field in accordance with (6). Next, we plot in Fig. 2 as functions of $\log r$, the dimensionless roots $x_2 = k_2^2/k_p^2$ and $x_3 = k_3^2/k_p^2$ as deduced numerically from (24) for vaious values fo the angle θ. Thus, we see that the two families of curves for the two normal modes (25) and (26) are distinct, $0 \leqslant k_3 \leqslant k_p$ and $k_p \leqslant k_2 \leqslant k_s$, except at the common point $\theta = 0$ and $r = 1$ where the two modes become degenerate, $k_2 = k_3 = k_p$. It is seen that, for very weak magnetic fields ($B_0 \to 0$), we have independently of the direction of propagation $k_2 \to k_s$ and $k_3 \to k_p$, indicating that in this limit the two normal modes become respectively a pure shear mode and a pure compressional wave. On the other hand, for very strong magnetic fields ($B_0 \to \infty$), we note that $k_3 \to 0$ independently of the direction of propagation, whereas k_2 approaches a limit which depends on the direction of propagation.

The behavior of the angular parameter ϕ as a function of $\log r$ can likewise be obtained numerically from either (27) or (28). In this manner we obtain the parametric

[7] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover Publications, New York, 1944), fourth edition, p. 104, item 70(d).

[8] It is clear that, when $r = 1$, we have $k_a^2 = k_s^2 k_p^2/(k_s^2 - k_p^2)$ or $V_a^2 = V_p^2 - V_s^2$, which fixes the value of Alfvén's phase velocity V_a in relation to the elastic phase velocities V_s and V_p.

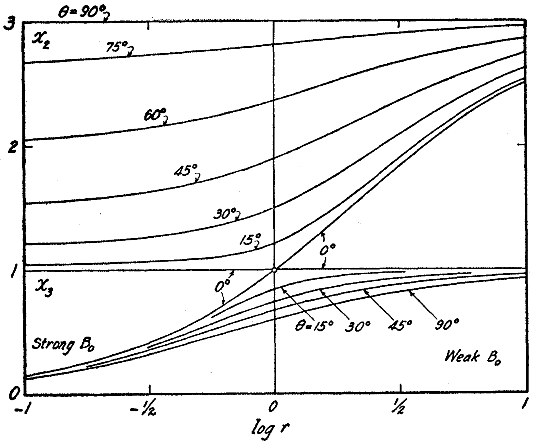

FIG. 2. Dispersive effect of the magnetic field upon the propagation constants of the two normal shear-compression modes.

family of curves depicted in Fig. 3 corresponding to specific values of θ. Particular attention should be paid to the curve $\theta=0$ which behaves discontinuously as r goes through 1, when both normal modes become degenerate. We note that, for $r>1$, all values of ϕ are less than $\frac{1}{4}\pi$ for all θ and, in fact, tend to vanish as $B_0\to 0$, indicating again that in this limit one mode becomes a pure shear wave and the other a pure compressional wave, whatever the direction of propagation,

Next, we consider the behavior of shear-compression modes when the conductivity is finite but $a\ll 1$. In this case we return to the cubic (16) and, after putting $k_c=0$, we expand the three roots to first order in a. We note that the two normal modes (25) and (26) now become slightly attenuated. Thus, putting $k_2=\beta_2+i\alpha_2$ ($\alpha_2\ll\beta_2$) and $k_3=\beta_3+i\alpha_3$ ($\alpha_3\ll\beta_3$), we observe that, to first order of small quantities, the phase constants β_2 and β_3 are still given by the two roots of (24), whereas the corresponding attenuation factors α_2 and α_3 can be deduced from ($\beta_2\neq\beta_3$):

$$\frac{2\alpha_2}{\beta_2}\approx\frac{a(k_s^2-\beta_2^2)(\beta_2^2-k_p^2)}{(\beta_2^2-\beta_3^2)(k_a^2+k_s^2\cos^2\theta+k_p^2\sin^2\theta)}\xrightarrow[\beta_2\to k_s]{}0, \tag{30}$$

$$\frac{2\alpha_3}{\beta_3}\approx\frac{a(k_s^2-\beta_3^2)(k_p^2-\beta_3^2)}{(\beta_2^2-\beta_3^2)(k_a^2+k_s^2\cos^2\theta+k_p^2\sin^2\theta)}\xrightarrow[\beta_3\to k_p]{}0, \tag{31}$$

where the limiting forms correspond to the case in which one mode becomes a pure shear wave while the other becomes a pure compressional wave ($B_0\to 0$). The companion mode which ensues from (16) only when the conductivity is finite is a highly attenuated "skin" wave with propagation constant

$$k_{\sigma 2}^2\approx i\omega\mu\sigma\left[1+\frac{k_s^2\cos^2\theta+k_p^2\sin^2\theta}{k_2^2}\right]\times\left\{1-\frac{ia(k_s^4\cos^2\theta+k_p^4\sin^2\theta)}{(k_a^2+k_s^2\cos^2\theta+k_p^2\sin^2\theta)^2}\right\}, \tag{32}$$

which should be compared with (21). We see from (32) that, for this mode, the propagation characteristics are dominated by the electromagnetic properties of the medium as modified anisotropically by magnetoelastic interaction.

The angular parameters corresponding to the three shear-compression modes discussed above are determined as follows: We note that, for the slightly attenuated normal modes $\mathbf{V}_2$ and $\mathbf{V}_3$, as defined by (25) and (26), the corresponding value of ϕ is still given by (27) or by (28) upon inserting the exact complex root $k_3=\beta_3+i\alpha_3$ or $k_2=\beta_2+i\alpha_2$, respectively, which merely leads to complex values of the angular parameter with relatively small imaginary part. For the companion highly attenuated mode, we note that its velocity vector $\mathbf{V}_{\sigma 2}$ is of the form (14) with $k=k_{\sigma 2}$ as given by (32) and with an angular parameter $\phi=\phi_{\sigma 2}$ which we deduce from (15), upon inserting $k=k_{2\sigma}$ and noting that $|k_{\sigma 2}|\gg\beta_2$, to obtain approximately

$$\tan\theta\,\tan\phi_{\sigma 2}\approx k_s^2/k_p^2, \tag{33}$$

from which it is readily established that $(\theta+\phi_{\sigma 2})\geqslant\frac{1}{2}\pi$ as illustrated in Fig. 1.

Finally, we consider the behavior of the three shear-compression modes in the case of weak magnetoelastic coupling ($a\gg 1$), which in a sense exhibits more clearly the individual character of each mode. Thus, we return to (16) and, after putting $k_c=0$, we multiply both sides of the equation by i/a to yield the cubic in k^2:

$$(k^2-k_s^2)(k^2-k_p^2)(k^2-i\omega\mu\sigma)=(i/a)k^2[(k_s^2\cos^2\theta+k_p^2\sin^2\theta)k^2-k_s^2k_p^2]. \tag{34}$$

First, upon letting $a\to\infty$, we deduce from this equation the three roots $k=k_s$, $k=k_p$, and $k=i\omega\mu\sigma$, which correspond respectively to the uncoupled shear, compression, and electromagnetic waves. Thus, regarding the right-hand side of (34) as a small perturbation and expanding the three roots to first order in a^{-1}, we obtain first a *modified shear wave* corresponding to the second normal mode (25),

$$k_2^2\approx k_s^2\left\{1-\frac{ik_s^2\cos^2\theta}{a(i\omega\mu\sigma-k_s^2)}\right\}, \tag{35}$$

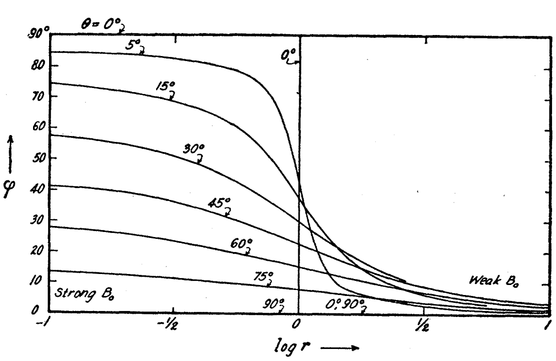

FIG. 3. Dependence of the angular parameter ϕ on the magnetic field.

which turns out to be identical to (22); then a *modified compressional wave* corresponding to the third normal mode (26),

$$k_3^2 \approx k_p^2 \left\{ 1 - \frac{ik_p^2 \sin^2\theta}{a(i\omega\mu\sigma - k_p^2)} \right\}; \quad (36)$$

and lastly a *modified "skin" wave* which corresponds to the companion mode (32),

$$k_{\sigma 2}^2 \approx i\omega\mu\sigma \left\{ 1 - \frac{(k_s^2 \cos^2\theta + k_p^2 \sin^2\theta)\omega\mu\sigma + ik_s^2 k_p^2}{a(i\omega\mu\sigma - k_s^2)(i\omega\mu\sigma - k_p^2)} \right\}. \quad (37)$$

The corresponding angular parameters are readily deduced from (15) upon introducing the respective wave numbers. In this manner we establish that, for the two normal modes (35) and (36), their common angular parameter ϕ, as defined by (25) and (26), is given by

$$\tan\phi \approx \frac{k_s^2 k_p^2 \sin\theta \cos\theta}{k_a^2(k_s^2 - k_p^2)}, \quad (38)$$

in which we have assumed that $\omega\mu\sigma \gg k_s^2$. It is interesting to observe that, according to (38), $\phi = 0$ whenever $\theta = 0$ or $\theta = \frac{1}{2}\pi$, which is also illustrated in Fig. 3. Thus, when $\theta = 0$ the direction of propagation is parallel to the magnetic field and, in this case, $k_3 = k_p$ indicating a pure elastic (compressional) wave, while k_2 as given by (35) corresponds to a shear wave modified slightly by the electromagnetic properties of the medium. Conversely, when $\theta = \frac{1}{2}\pi$, the direction of propagation is at right angles to the magnetic field and, this time, $k_2 = k_s$ indicating a pure elastic shear wave, whereas k_3 as given by (36) corresponds to a compressional wave slightly modified by the electromagnetic properties of the medium. Finally, the companion highly attenuated mode (37) exhibits in this case an angular parameter $\phi_{\sigma 2}$ which is still given by (33) if we assume that $\omega\mu\sigma \gg k_s^2$.

5. DISCUSSION

It now becomes of interest to examine some typical numerical results in the light of the foregoing theory. For the purpose, let us imagine an unbounded, homogeneous and isotropic, conducting solid with the elastic and electromagnetic properties of copper. According to Knopoff's measurements,[9] copper exhibits the phase velocities $V_p = 4.68$ km/sec for compressional waves and $V_s = 2.26$ km/sec for shear waves and a density $\rho = 8.90 \times 10^3$ kg/meter³. If we assume that the medium is subjected to a uniform static field of magnetic induction $B_0 = 1$ weber/meter² (10 000 gauss), we compute for Alfvén's phase velocity (6) the value $V_a = 9.46$ meters/sec. Finally, making use of (29) we compute the parameter $r = 1.77 \times 10^5$, which reflects quite clearly the fact that, in this instance, $V_a^2 \ll (V_p^2 - V_s^2)$.

This fact means, according to Eq. (19) and to Figs. 2 and 3, that in the case of infinite conductivity the three normal elastic modes are but slightly modified by magnetoelastic interactions independently of the frequency (provided, of course, we adhere to a purely macroscopic theory). Then, in the case of finite conductivity, taking $\sigma = 5.8 \times 10^7$ mhos/meter for copper, we compute for the coupling parameter (7) the values $a = 9.6 \times 10^{-4}$ for a frequency of 1 cps (seismic waves) and $a = 96$ for a frequency of 10^5 cps (laboratory experiments). In either case, we find from the equations given here that the normal magnetoelastic modes are but slightly attenuated, whatever the frequency, in the presence of finite conductivity. For example, if the frequency is 10^5 cps, we find from (36) a fractional change in phase velocity $(U_3 - V_p)/V_p \approx 0.51 \times 10^{-6}$ accompanied by an attenuation factor $\alpha_3 \approx 0.27 \times 10^{-7}$ meter^{-1}, with similar results for the other two normal modes, which gives an idea of the smallness of magnetoelastic interactions at all frequencies and under all physically realizable magnetic fields.

In consequence, returning to the laboratory experiment reported by Robey,[3] we find that the observed increase of 1.4% in the resonant frequency of his composite crystal which vibrates essentially in a longitudinal or compressional mode, as brought about by a transverse magnetic field, seems to be due to other causes than magnetoelastic interactions. Finally, in the case of seismic waves which attain velocities of the order of 10 km/sec, and for which the interior of the earth exhibits an Alfvén's phase velocity that cannot exceed 10 meters/sec under any conceivable circumstances, we conclude with Knopoff[4] that magnetoelastic interactions cannot be a significant phenomenon in the interior of the earth as far as seismic waves are concerned and that, therefore, Cagniard's suggestion[2] seems to be without foundation.

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[9] L. Knopoff (private communication).