distribution is $(2.0\pm0.4)\times10^{-13}$ cm. Within the accuracy of the Born approximation, which is expected to be good for most carbon calculations, the independent-particle shell model using an infinite harmonic well has given the best agreement with experiment for the ground state and the first excited state scattering. An estimate of the size of the matrix element for the transition from the ground state to the 7.65-Mev state has been made. The transition to the 9.61-Mev level is probably electric and if this is the case, the possible spin and parity assignments for this state have been restricted to 0^+ and 2^+.

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Theory of *S*-Wave Pion Scattering and Photoproduction at Low Energies*†

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A fixed-source analysis of the *s*-wave pion-nucleon interaction is constructed along the lines of the Chew-Low-Wick formalism. A bilinear *s*-wave interaction of the form $\lambda_0\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}+\lambda\boldsymbol{\tau}\cdot(\boldsymbol{\varphi}\times\boldsymbol{\pi})$ is added to the usual *p*-wave coupling $(4\pi)^{\frac{1}{2}}(f/\mu)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\,\boldsymbol{\tau}\cdot\boldsymbol{\varphi}$. Scattering equations are developed and solved in the one-meson approximation. Values for the renormalized coupling parameters λ_0 and λ are determined which give reasonable agreement with the *s*-wave phase shifts up to ~100-Mev pion kinetic energy. This *s*-wave interaction with the parameters fixed by the scattering analysis is then applied to the discussion of the π^+ and π^0 photo-production cross sections. A Kroll-Ruderman theorem is proved for the above nonlocal interaction and it is shown that the contributions to *s*-wave neutral and charged photoproduction are consistent with experiment. Other experimental implications, in particular as to the possible role of π-π forces, are discussed.

I. INTRODUCTION

Chew and Low[1] have shown recently that a simple fixed-source theory of the *p*-wave pion-nucleon interaction is quite powerful in correlating low-energy pion scattering and photoproduction data. With a formalism based on a nonrelativistic approximation (which neglects antinucleons and recoil) to the equations of Low,[2,3] they have especially emphasized important conclusions in their work which are independent of the details of their model. We report here a fixed-source analysis of *s*-wave pion-nucleon interactions constructed along similar lines. In particular we study the elastic scattering of *s*-wave pions at low kinetic energy (≤ 100 Mev) and the *s*-wave photoproduction of low-energy charged and neutral pions.

In C-L, the *p*-wave pion-nucleon coupling is taken to be

$$H_p'=(4\pi)^{\frac{1}{2}}\frac{f^0}{\mu}\int\{\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\boldsymbol{\tau}\cdot\boldsymbol{\varphi}(\mathbf{x})\}s(\mathbf{x})d^3\mathbf{x}, \qquad (1)$$

with a source density

$$s(\mathbf{x})=\int v(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{x})\frac{d^3\boldsymbol{\kappa}}{(2\pi)^3}.$$

On the basis of Eq. (1) and in the "one-meson approximation" a low-energy effective-range theory of the *p*-wave scattering phase shifts is developed. The (3,3) phase shift ($T=\frac{3}{2}$, $J=\frac{3}{2}$) emerges as the dominant one.

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† The term "pion" is used in discussion of the physical and experimental aspects of the scattering and photoproduction. In the more formal and theoretical developments we prefer the word "meson" for the nuclear field quantum. It is not intended that this duality of terms convey a basic reservation on our part as to the identity of these two.

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1 G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570, 1579 (1956); hereafter referred to as C-L. We use the units $\hbar=c=1$. Unless specifically displayed, the pion rest mass $\mu=1$.

2 F. E. Low, Phys. Rev. **97**, 1392 (1955).

3 G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).

C-L also apply the interaction in Eq. (1) to a discussion of single-pion photoproduction. Electromagnetic gauge invariance for a point source theory requires the substitution

$$\nabla \rightarrow \nabla - ie\mathbf{A},$$

when operating on the charged pion field φ in Eq. (1). This added term gives rise to photoproduction of s-wave charged pions which is the dominant contribution to the cross section for the process $\gamma+p\rightarrow n+\pi^+$ at low energy. Neutral-pion photoproduction occurs only in the p state.

In this paper we introduce a specific interaction to describe the s-wave pion scattering and study its contribution to photopion production cross sections. The possibility of charge exchange pion scattering leads to a prediction for the cross section of neutral-pion photoproduction in the s state.

It is clear from considerations of parity that a pion-nucleon s-wave interaction in a static source theory must be at least quadratic in the pion field. Guided by the form which is obtained upon application of the Dyson-Foldy[4] transformation to the relativistic theory, we add to the p-wave coupling the two terms

$$H_s' = \int \{\lambda_0{}^0 \boldsymbol{\varphi}\cdot\boldsymbol{\varphi} + \lambda^0 \boldsymbol{\tau}\cdot(\boldsymbol{\varphi}\times\boldsymbol{\pi})\} s(\mathbf{x}) d^3\mathbf{x}, \quad \text{(2a)}$$

with π the canonical momentum of the pion field.

The terms of this form in the Dyson-Foldy transformed Hamiltonian appear with definite coefficients in relation with Eq. (1):

$$\lambda_0{}^0 = 4\pi(f^0)^2(2M/\mu^2), \quad \lambda^0 = 4\pi(f^0)^2/\mu^2. \quad (3)$$

Nevertheless we treat these constants as free parameters in this paper. Our dual motivations for this are

(*a*) Equation (2a) is considered to be an approximate phenomenological representation of the low-energy s-wave pion interactions contained in a relativistic γ_5-theory, whereas Eqs. (3) are relations valid[4] only to order $(f^0)^2$.

(*b*) Equations (3) express relations between the unrenormalized coupling constants. These ratios are changed in an unknown way by the different coupling-constant renormalizations.

As written, Eq. (2a) includes scattering of waves of all angular momenta. In the interest of simplicity of calculation, we replace it by a separable source version:

$$H_s' = \lambda_0{}^0 \bar{\boldsymbol{\varphi}}\cdot\bar{\boldsymbol{\varphi}} + \lambda^0 \boldsymbol{\tau}\cdot(\bar{\boldsymbol{\varphi}}\times\bar{\boldsymbol{\pi}}), \quad (2)$$

with

$$\bar{\varphi} \equiv \int \varphi(\mathbf{x}) s(\mathbf{x}) d^3\mathbf{x}.$$

The separation of the individual pion absorption and emission points serves to suppress all but s-wave interactions.[5]

Our first aim is to fit the low-energy s-wave scattering phase shifts for the $T=\frac{1}{2}$ and $\frac{3}{2}$ isotopic spin states. A close fit to the data for pion kinetic energies up to $\sim$100 Mev is achieved in the one-meson approximation with suitable choices of the three parameters λ_0 and λ (the two renormalized coupling constants) and the cut-off energy $\omega_{\max} = (\kappa_{\max}{}^2 + \mu^2)^{\frac{1}{2}}$. The experimental phase shifts are taken from Orear's[6] analysis and can be approximately represented at low energies as linear functions of the pion momentum

$$\delta_1 = 0.16(\kappa/\mu c), \quad \delta_3 = -0.11(\kappa/\mu c). \quad (3')$$

In this connection, we remark that an additional contribution to δ_α arises when nucleon recoil is taken into account. However, the recoil contribution to the scattering matrix elements and phase shifts varies as κ^2 and κ^3, respectively for $\kappa\rightarrow 0$, and can thus be ignored.

We also note that both interaction terms in Eq. (2) must be present. This statement is proved in Appendix II in the framework of the one-meson approximation. We can make it plausible here by observing that the first term (λ_0) gives no isotopic spin dependence ($\delta_1=\delta_3$), and the second one (λ) in Born approximation gives $\delta_1 = -2\delta_3$. A more accurate treatment of the λ term increases δ_1 and decreases δ_3, because the former is an attractive and the latter a repulsive phase shift. Hence both terms are necessary to fit the observed ratio.

It is also clear that, in the one-meson approximation, the s- and p-wave scattering problems can be solved independently because of the opposite parities of the states involved. Their mutual influence lies solely in the definitions of the renormalized coupling constants.

We consider next the s-wave pion photoproduction cross sections with the coupling parameters in Eq. (2) fixed by the scattering analysis. The two questions of primary importance are: Does a Kroll-Ruderman[7] theorem hold, and what is the effect of s-wave rescattering of the photoproduced pions on the charged and neutral photoproduction cross sections?

The Kroll-Ruderman theorem states that the zero-total-energy limit of the matrix element for pion photoproduction has the same form as the Born-approximation result,

$$\text{M.E.}(\gamma\rightarrow\pi^+,\pi^-,\pi^0) = (4\pi)^{\frac{1}{2}}\left(\frac{ief}{\mu}\right)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}}{(2k)^{\frac{1}{2}}(2\omega_p)^{\frac{1}{2}}}(\tau^-, -\tau^+, 0)\sqrt{2}, \quad (4)$$

with the renormalized pion-nucleon coupling constant,

[4] S. D. Drell and E. M. Henley, Phys. Rev. 88, 1053 (1952).

[5] In both the weak- and strong-coupling limits, the scattering solutions for the scalar pair term with and without the separability assumption are identical. See G. Wentzel, Phys. Rev. 86, 802, (1952) and reference 4.

[6] Jay Orear, Phys. Rev. 100, 288 (1955).

[7] N. M. Kroll and M. A. Ruderman, Phys. Rev. 93, 233 (1954).

f, being the same as that operating in the p-wave scattering. This theorem has been proved in a variety of ways for the relativistic local pseudoscalar theory, an especially simple proof appearing in Low's[2] work which makes use of the charge-current continuity equation. It is well known that there exists a continuity equation

$$\nabla\cdot\mathbf{j}-i[\rho,H]=0 \qquad (5)$$

in a gauge-invariant formulation of electromagnetic processes. In a covariant local theory of pion-nucleon processes, the gauge-invariant introduction of electromagnetic interactions is simply effected by replacing

$$\mathbf{p}\rightarrow\mathbf{p}-e\mathbf{A}, \qquad (6)$$

when operating on the charged fields.

In a cutoff, nonlocal model of the pion-nucleon interaction, however, the prescription, Eq. (6) is not gauge-invariant and therefore the continuity equation fails within the source. Various methods of remedying this situation have been discussed, especially by Sachs,[8,9] and collaborators who introduce current threads to provide instantaneous charge transfer within the source functions as required by Eq. (5). We analyze this situation in Sec. III where we prove that, for arbitrary nonlocal interactions H, the theorem expressed in Eq. (4) follows as a direct consequence of the existence of a continuity equation alone. This proof is an extension of previous ones which used Eq. (6) to introduce electromagnetic interactions. In the work of C-L, the emergence of a Kroll-Ruderman theorem follows from the fact that the photoproduced pions at threshold do not rescatter because of the absence of any s-wave interactions.

We feel that it is of importance to establish the validity of the Kroll-Ruderman theorem in our work, since we consider the added s-wave interaction Eq. (2) to be an approximate low-energy representation of s-wave pion processes as contained in the relativistic γ_5 theory, to which the Kroll-Ruderman theorem applies.

The photoproduction cross sections calculated in this work differ from the results of C-L in two ways. First of all, rescattering of the s-wave photoproduced pions serves to increase the cross section for π^+ photoproduction by 15% above the C-L prediction. Secondly, charge exchange scattering leads to an s-wave neutral-pion photoproduction which contributes 3% of the charged-pion cross section near threshold. The data[10] are not sufficiently precise to support or discourage these predictions.

The following sections of this paper present a further discussion of the experimental and theoretical numbers.

Two other cross sections involving low-energy s-wave pions can also be studied in connection with Eq. (2) and the work reported here. These are the double s-wave pion photoproduction by magnetic dipole gamma rays and the inelastic scattering of an incident p-wave pion into two s-wave pions: near threshold these processes should dominate over those involving slow outgoing p-wave pions, because of phase space factors. Accurate cross sections for these processes have not yet been measured. They have been calculated on the basis of Eq. (2) by Bincer,[11] whose work will be reported shortly.

The success or failure of Eq. (2) to account for this class of processes can serve as a basis for arguing the possible role of short-range pion-pion forces such as have been conjectured in other connections.[12]

II. SCATTERING

The Hamiltonian is written as the sum of two terms:

$$H=H_0+H', \qquad (7)$$

where

$$H_0=\sum_\kappa \omega_\kappa a_\kappa{}^\dagger a_\kappa \qquad (8)$$

is the Hamiltonian of the free-meson field, and H' describes the interaction between meson and nucleons. In Eq. (8), κ labels both the momentum and isotopic spin state of the meson. The source is restricted to being fixed in position, but otherwise H' is left completely unspecified. The scattering equations are then developed in a manner completely analogous to the C-L treatment.

The starting point is the scattering matrix

$$\tilde{S}_{fi}=\langle\Psi_f{}^{(-)}|\Psi_i{}^{(+)}\rangle, \qquad (9)$$

where $\Psi_i{}^{(+)}$ is an eigenfunction of the total Hamiltonian H. It represents a plane-wave meson incident on a physical nucleon (the two being in a state characterized by i), plus outgoing scattered waves. Similarly, $\Psi_f{}^{(-)}$ represents a plane-wave meson and a nucleon in the state f, plus ingoing scattered waves. Thus

$$H\Psi_i{}^{(+)}=\omega_{\kappa i}\Psi_i{}^{(+)}, \qquad (10)$$

where in writing Eq. (10) the physical nucleon has been chosen to have zero energy. Representing the physical nucleon by ψ_0 (and suppressing the spin and isotopic spin indices) we note that

$$H\psi_0=0. \qquad (11)$$

Thus, in complete analogy to C-L, we find

$$\Psi_{\kappa i}{}^{(+)}=a_{\kappa i}{}^\dagger\psi_0-\frac{1}{H-\omega_{\kappa i}-i\epsilon}[H',a_{\kappa i}{}^\dagger]\psi_0. \qquad (12)$$

[8] R. H. Capps and R. G. Sachs, Phys. Rev. **96**, 540 (1954).
[9] R. H. Capps and W. G. Holladay, Phys. Rev. **99**, 931 (1955).
[10] Goldschmidt-Clermont, Osborne, and Scott, Phys. Rev. **97**, 188 (1955); F. E. Mills and L. J. Koester, Jr., Phys. Rev. **98**, 210 (1955).
[11] A. M. Bincer, Ph.D. thesis, Massachusetts Institute of Technology Physics Department, 1956 (unpublished).
[12] Marc Ross, Phys. Rev. **95**, 1687 (1954); F. J. Dyson, Phys. Rev. **99**, 1037 (1955); Gyo Takeda, Phys. Rev. **100**, 440 (1955).

Similarly, we obtain

$$a_{\kappa i}\psi_0=\frac{1}{H+\omega_{\kappa i}}[H',a_{\kappa i}]\psi_0. \tag{13}$$

This is identical to the C-L form if we set

$$V_{\kappa i}=[H',a_{\kappa i}^\dagger]. \tag{14}$$

Here, however, $V_{\kappa i}$ is a function of meson field operators in addition to the nucleon spin and isotopic spin matrices. Substituting into (9) for $\Psi_i^{(+)}$, and proceeding in the manner of C-L, we find

$$\tilde{S}_{fi}=\delta_{fi}-2\pi i\delta(E_f-E_i)T_{fi}, \tag{15}$$

where

$$T_{fi}=\langle\Psi_f^{(-)}|[H',a_{\kappa i}^\dagger]|\psi_0\rangle=\langle\Psi_f^{(-)}|V_{\kappa i}|\psi_0\rangle, \tag{16}$$

and on the energy shell is the conventional transition amplitude. We now insert (12) into (16) and, using closure, introduce the complete set of states $\Psi_n^{(-)}$ (the index n characterizes the state of the mesons as well as the number present). The equation for T is

$$T_{fi}=\langle\psi_0|[a_{\kappa f},V_{\kappa i}]|\psi_0\rangle -\sum_n\left\{\frac{\langle\psi_0|V_{\kappa i}|\Psi_n^{(-)}\rangle\langle\Psi_n^{(-)}|V_{\kappa f}^\dagger|\psi_0\rangle}{E_n+\omega_f} +\frac{\langle\psi_0|V_{\kappa f}^\dagger|\Psi_n^{(-)}\rangle\langle\Psi_n^{(-)}|V_{\kappa i}|\psi_0\rangle}{E_n-\omega_f-i\epsilon}\right\}. \tag{17}$$

It is convenient to use the following notation:

$$\begin{aligned} T_{ni}&=\langle\Psi_n^{(-)}|V_{\kappa i}|\psi_0\rangle=\langle\Psi_n^{(-)}|[H',a_{\kappa i}^\dagger]|\psi_0\rangle,\\ S_{ni}&=-\langle\Psi_n^{(-)}|V_{\kappa i}^\dagger|\psi_0\rangle=\langle\Psi_n^{(-)}|[H',a_{\kappa i}]|\psi_0\rangle. \end{aligned} \tag{18}$$

Thus

$$T_{fi}=\langle\psi_0|[a_{\kappa f},V_{\kappa i}]|\psi_0\rangle -\sum_n\left\{\frac{T_{nf}^*T_{ni}}{E_n-\omega_f-i\epsilon}+\frac{S_{ni}^*S_{nf}}{E_n+\omega_f}\right\}. \tag{19}$$

If H' is just the p-wave interaction $\boldsymbol{\sigma}\cdot\nabla\,\boldsymbol{\tau}\cdot\boldsymbol{\varphi}(\mathbf{x})$, the inhomogeneous term vanishes, $S_{ni}=T_{ni}$, and Eq. (19) reduces to that considered by C-L. However, for the general case T_{ni} and S_{ni} are distinct, and it is necessary to develop an equation for S_{fi}. This is done in the very same way as the development of the T_{fi} equation, giving

$$S_{fi}=-\langle\psi_0|[a_{\kappa f},V_{\kappa i}^*]|\psi_0\rangle -\sum_n\left\{\frac{T_{nf}^*S_{ni}}{E_n-\omega_f-i\epsilon}+\frac{T_{ni}^*S_{nf}}{E_n+\omega_f}\right\}. \tag{20}$$

Equations (19) and (20) for T_{fi} and S_{fi} have the following simple interpretation. T_{fi} scatters a meson from the state i to the state f, while T_{fi}^* scatters one from f to i. On the other hand, $S_{fi}(S_{fi}^*)$ is the amplitude for creating (annihilating) two mesons in the states i and f. Diagrams for the amplitudes T_{fi} and S_{fi} are given in Fig. 1. Figure 2 shows the one-meson approximation to Eqs. (19) and (20).

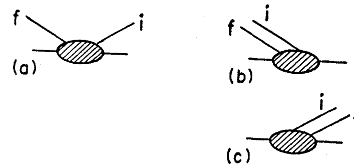

FIG. 1. Diagrams representing T_{fi}, S_{fi} and S_{fi}^* respectively. The blob means the complete physical interaction.

In order to proceed further, it becomes necessary to specify the interaction H'. We take

$$H'=H_s'+H_p', \tag{21}$$

as given in Eqs. (1) and (2).

In line with this choice, $V_{\kappa i}$ may be divided into two parts

$$V_{\kappa i}=V_{\kappa i}^s+V_{\kappa i}^p, \tag{22}$$

where $V_{\kappa i}^s=[H_s',a_{\kappa i}^\dagger]$ and $V_{\kappa i}^p=[H_p',a_{\kappa i}^\dagger]$. The s-wave contribution to T_{0i} and S_{0i} then vanishes; that is,

$$\begin{aligned} T_{0i}&=\langle\psi_0|V_{\kappa i}|\psi_0\rangle=\langle\psi_0|V_{\kappa i}^p|\psi_0\rangle,\\ &\text{and}\\ S_{0i}&=-\langle\psi_0|V_{\kappa i}^\dagger|\psi_0\rangle=-\langle\psi_0|V_{\kappa i}^{p\dagger}|\psi_0\rangle. \end{aligned} \tag{23}$$

T_{0i} represents the amplitude to emit (or absorb if $V_{\kappa i}^\dagger$) a meson from a physical nucleon. Since the meson is pseudoscalar it must be in a p state to conserve parity. (A detailed proof of this is given in Appendix I.) We thus obtain $T_{0i}=S_{0i}$, and Eqs. (19) and (20) thus become

$$T_{fi}=\langle\psi_0|[a_{\kappa f},V_{\kappa i}^s]|\psi_0\rangle+\frac{T_{0f}^*T_{0i}-T_{0i}^*T_{0f}}{\omega_f} -\sum_{n\geq1}\left\{\frac{T_{nf}^*T_{ni}}{E_n-\omega_f-i\epsilon}+\frac{S_{ni}^*S_{nf}}{E_n+\omega_f}\right\}, \tag{24a}$$

$$S_{fi}=-\langle\psi_0|[a_{\kappa f},V_{\kappa i}^{s\dagger}]|\psi_0\rangle+\frac{T_{0f}^*T_{0i}-T_{0i}^*T_{0f}}{\omega_f} -\sum_{n\geq1}\left\{\frac{T_{nf}^*S_{ni}}{E_n-\omega_f-v\epsilon}+\frac{T_{ni}^*S_{nf}}{E_n+\omega_f}\right\}. \tag{24b}$$

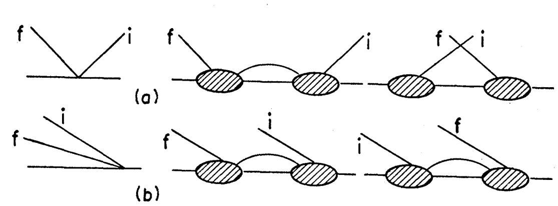

FIG. 2. Diagrams of the one-meson approximations to the T_{fi} and S_{fi} equations. The first pictures (with no blob) represent the first Born approximation, the second pictures represent terms $T_{nf}^*T_{ni}$ and $S_{ni}^*S_{nf}$ in the one-meson approximation to T_{fi}, and $T_{nf}^*S_{ni}$ and $T_{ni}^*S_{nf}$ in the one-meson approximation to S_{fi}.

We shall show that the first term on the right of (24a) is the renormalized s-wave scattering in first Born approximation. The second term is identical to the inhomogeneous term of the C-L equation and is just the renormalized p-wave scattering in the first Born approximation.[13]

We now make the one-meson approximation[14] to Eqs. (24) by dropping all but the one-meson states ($n=1$). This restriction gives two coupled integral equations for T_{fi} and S_{fi} and has the simplifying feature of completely separating s- and p-wave parts. Interference effects are present only in higher approximations in which odd-parity states (relative to the physical nucleon) can be constructed with an odd number of s-wave and any number of p-wave mesons.

In terms of creation and absorption operators, the s-wave interaction is (restoring the isotopic spin indices)

$$V_{\kappa\gamma}{}^{s}=[H_s{}',a_{\kappa\gamma}{}^{\dagger}]$$
$$=\lambda_0{}^0\delta_{\gamma\alpha}\sum_{\kappa'}\frac{v(\kappa)v(\kappa')}{(\omega_\kappa\omega_{\kappa'})^{\frac{1}{2}}}\{a_{\kappa'\alpha}+a_{\kappa'\alpha}{}^{\dagger}\}$$
$$+\lambda^0\frac{i\epsilon_{\gamma\alpha\beta}\tau_\alpha}{2}\sum_{\kappa'}v(\kappa)v(\kappa')\left\{\left[\left(\frac{\omega_{\kappa'}}{\omega_\kappa}\right)^{\frac{1}{2}}-\left(\frac{\omega_\kappa}{\omega_{\kappa'}}\right)^{\frac{1}{2}}\right]a_{\kappa'\beta}\right.$$
$$\left.-\left[\left(\frac{\omega_{\kappa'}}{\omega_\kappa}\right)^{\frac{1}{2}}+\left(\frac{\omega_\kappa}{\omega_{\kappa'}}\right)^{\frac{1}{2}}\right]a_{\kappa'\beta}{}^{\dagger}\right\}. \quad (25)$$

Repeated indices are to be summed. $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric isotropic tensor equal to $(+1,-1)$ for (even, odd) permutations of it indices. Introducing the projection operators for the isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ states,

$$(Q_{\frac{1}{2}})_{fi}=\tfrac{1}{3}\tau_f\tau_i,$$
$$(Q_{\frac{3}{2}})_{fi}=\delta_{fi}-\tfrac{1}{3}\tau_f\tau_i, \quad (26)$$

we write

$$\delta_{fi}=(Q_{\frac{1}{2}})_{fi}+(Q_{\frac{3}{2}})_{fi},$$

and

$$i\epsilon_{i\alpha f}\tau_\alpha=2(Q_{\frac{1}{2}})_{fi}-(Q_{\frac{3}{2}})_{fi}. \quad (27)$$

The renormalization of coupling constants is effected in (24) with the definitions[13]

$$\lambda\langle u_0|\tau_\alpha|u_0\rangle=\lambda^0\langle\psi_0|\tau_\alpha|\psi_0\rangle,$$
$$\lambda_0\langle u_0|u_0\rangle=\lambda_0{}^0\langle\psi_0|\psi_0\rangle;\quad(\lambda_0=\lambda_0{}^0). \quad (28)$$

These definitions are consistent with the fact that u_0 and ψ_0 both transform in the same way under rotations in (isotopic) spin space. With this notation, we find that the inhomogeneous term of the T_{fi} equation becomes (omitting "bare" spinors u_0 in the right side)

$$\langle\psi_0|[a_{\kappa f},V_{\kappa_i}{}^{s}]|\psi_0\rangle$$
$$=(Q_{\frac{1}{2}})_{fi}\left\{v(\kappa_i)v(\kappa_f)\left[\frac{\lambda_0}{(\omega_i\omega_f)^{\frac{1}{2}}}\right.\right.$$
$$\left.\left.-\lambda\left(\left(\frac{\omega_f}{\omega_i}\right)^{\frac{1}{2}}+\left(\frac{\omega_i}{\omega_f}\right)^{\frac{1}{2}}\right)\right]\right\}+(Q_{\frac{3}{2}})_{fi}\left\{v(\kappa_i)v(\kappa_f)\right.$$
$$\left.\times\left[\frac{\lambda_0}{(\omega_i\omega_f)^{\frac{1}{2}}}+\frac{\lambda}{2}\left(\left(\frac{\omega_f}{\omega_i}\right)^{\frac{1}{2}}+\left(\frac{\omega_i}{\omega_f}\right)^{\frac{1}{2}}\right)\right]\right\}. \quad (29)$$

As previously noted, this is just the renormalized first Born approximation. The inhomogeneous term in the S_{fi} equation is

$$-\langle\psi_0|[a_{\kappa f},V_{\kappa_i}{}^{s\dagger}]|\psi_0\rangle$$
$$=(Q_{\frac{1}{2}})_{fi}\left\{v(\kappa_i)v(\kappa_f)\left[-\frac{\lambda_0}{(\omega_i\omega_f)^{\frac{1}{2}}}\right.\right.$$
$$\left.\left.+\lambda\left(\left(\frac{\omega_f}{\omega_i}\right)^{\frac{1}{2}}-\left(\frac{\omega_i}{\omega_f}\right)^{\frac{1}{2}}\right)\right]\right\}+(Q_{\frac{3}{2}})_{fi}\left\{v(\kappa_i)v(\kappa_f)\right.$$
$$\left.\times\left[-\frac{\lambda_0}{(\omega_i\omega_f)^{\frac{1}{2}}}-\frac{\lambda}{2}\left(\left(\frac{\omega_f}{\omega_i}\right)^{\frac{1}{2}}-\left(\frac{\omega_i}{\omega_f}\right)^{\frac{1}{2}}\right)\right]\right\}. \quad (30)$$

It is now useful to carry out a series of manipulations with the aim of reducing the one-meson approximation to (24) to convenient form for solving. First we split T and S into their two isotopic spin parts:

$$T_{fi}=\sum_\alpha\frac{2\pi}{(\omega_i\omega_f)^{\frac{1}{2}}}t^\alpha(\omega_f,\omega_i)(Q_\alpha)_{fi}v(\kappa_f)v(\kappa_i),$$
$$S_{fi}=\sum_\alpha\frac{2\pi}{(\omega_i\omega_f)^{\frac{1}{2}}}s^\alpha(\omega_f,\omega_i)(Q_\alpha)_{fi}v(\kappa_f)v(\kappa_i). \quad (31)$$

Here the summation index α takes on the values $\frac{1}{2}$ and $\frac{3}{2}$.

We see in Eqs. (29) and (30) that t^α and s^α are nonfactorable functions of the energies ω_f and ω_i, in contrast to the p-wave case. In order to get around this difficulty, we construct the combinations $T_{ni}+S_{ni}$ and $T_{ni}-S_{ni}$, which depend on ω_i in a trivial manner. Explicitly, introducing the functions

$$h_\alpha{}^{+}(\omega_f)=(\omega_f/\omega_i)[t^\alpha(\omega_f,\omega_i)+s^\alpha(\omega_f,\omega_i)]$$

and

$$h_\alpha{}^{-}(\omega_f)=[t^\alpha(\omega_f,\omega_i)-s^\alpha(\omega_f,\omega_i)], \quad (32)$$

---

[13] As in C-L, the renormalized p-wave coupling constant f is defined by

$$f^0\langle\psi_0|\sigma_i\tau_\alpha|\psi_0\rangle=f\langle u_0|\sigma_i\tau_\alpha|u_0\rangle,$$

where u_0 is the bare spinor corresponding to ψ_0. Though formally identical with the ratio given in C-L, our f/f^0 is different from theirs because of the presence of s-wave mesons in ψ_0.

[14] There is no quantitative criterion to justify the one-meson approximation. It is hoped that the larger energy denominators in matrix elements for low-energy processes involving more than one "intermediate" meson make them unimportant. We have just received a preprint of a paper by Dr. Earle Lomon (to be published) which presents an exact scattering solution of Eq. (2) with different ranges of coupling constants and energy cutoffs than used here.

and inserting (29), (30), and (31) into (24), it follows that:

$$h_\alpha^+(\omega_f)=\frac{\lambda}{\pi}\Gamma_\alpha\omega_f-\frac{\omega_f}{2\pi}\int\frac{\kappa d\omega}{\omega}v^2(\kappa)\times\left\{\frac{h_\alpha^-(\omega)+(\omega_f/\omega)h_\alpha^+(\omega)]^*h_\alpha^+(\omega)}{\omega-\omega_f-i\epsilon}-A_{\alpha\beta}\frac{h_\beta^+(\omega)^*[h_\beta^-(\omega)-(\omega_f/\omega)h_\beta^+(\omega)]}{\omega+\omega_f}\right\},\tag{33}$$

$$h_\alpha^{(-)}(\omega_f)=\frac{\lambda^0}{\pi}+\frac{\lambda}{\pi}\Gamma_\alpha\omega_f-\frac{1}{2\pi}\int\kappa d\omega v^2(\kappa)\times\left\{\frac{[h_\alpha^-(\omega)+(\omega_f/\omega)h_\alpha^+(\omega)]^*h_\alpha^-(\omega)}{\omega-\omega_f-i\epsilon}+A_{\alpha\beta}\frac{h_\beta^-(\omega)^*[h_\beta^-(\omega)-(\omega_f/\omega)h_\beta^+(\omega)]}{\omega+\omega_f}\right\}.$$

The indices for $A_{\alpha\beta}$ are such that the first row and first column refer to the $\frac{1}{2}$ state, while the second row and second column refer to the $\frac{3}{2}$ state. Correspondingly, $\Gamma_{\frac{1}{2}}=-1$, and $\Gamma_{\frac{3}{2}}=\frac{1}{2}$. $A_{\alpha\beta}$ is the crossing matrix and plays the same role as the analogous quantity in the p-wave theory. It exhibits similar properties:

$$\begin{gathered}\sum_\beta A_{\alpha\beta}A_{\beta\gamma}=\delta_{\alpha\gamma},\\ \sum_\beta A_{\alpha\beta}\Gamma_\beta=-\Gamma_\alpha,\\ \sum_\beta A_{\alpha\beta}=1.\end{gathered}\tag{34}$$

Our final step is to construct linear combinations of h^+ and h^-, one of which reduces to the scattering amplitude on the energy shell and thereby obeys a unitarity relation.

Define

$$a_\alpha(\omega)=\tfrac{1}{2}[h_\alpha^+(\omega)+h_\alpha^-(\omega)],$$

and

$$b_\alpha(\omega)=\tfrac{1}{2}[h_\alpha^+(\omega)-h_\alpha^-(\omega)].\tag{35}$$

In terms of t and s,

$$a_\alpha(\omega_f)=\frac{1}{2}\left[\left(\frac{\omega_f}{\omega_i}+1\right)t^\alpha(\omega_f,\omega_i)+\left(\frac{\omega_f}{\omega_i}-1\right)s^\alpha(\omega_f,\omega_i)\right].\tag{36}$$

Thus $a_\alpha(\omega_f)=t^\alpha(\omega_f,\omega_f)$, which is precisely the quantity of interest. In terms of a and b, Eqs. (24), or (33), become

$$a_\alpha(\omega_f)=C_0+C_1\Gamma_\alpha\omega_f-\frac{\omega_f^2}{\pi}\int\frac{\kappa d\omega}{\omega^2}\left\{\frac{|a_\alpha(\omega)|^2}{\omega-\omega_f-i\epsilon}+A_{\alpha\beta}\frac{|a_\beta(\omega)|^2}{\omega+\omega_f}\right\}v^2(\kappa),\tag{37a}$$

and

$$b_\alpha(\omega_f)=-C_0+\omega_fD_\alpha-\frac{\omega_f^2}{\pi}\int\frac{\kappa d\omega}{\omega^2}\left\{\frac{a_\alpha(\omega)^*b_\alpha(\omega)}{\omega-\omega_f-i\epsilon}+A_{\alpha\beta}\frac{a_\beta(\omega)b_\beta(\omega)^*}{\omega+\omega_f}\right\}v^2(\kappa),\tag{37b}$$

where the identity

$$\frac{\omega}{\omega_f}\left(\frac{1}{\omega-\omega_f}\right)=\frac{1}{\omega_f}+\frac{1}{\omega-\omega_f}\tag{38}$$

has been used.

In Eqs. (37), C_0, $C_1\Gamma_\alpha$, and D_α are defined by

$$C_0\equiv\frac{\lambda_0}{2\pi}-\frac{1}{\pi}\int\frac{\kappa d\omega}{\omega}(\delta_{\alpha\beta}+A_{\alpha\beta})|\tfrac{1}{2}h_\beta^-|^2v^2(\kappa),\tag{39}$$

$$C_1\Gamma_\alpha\equiv\frac{\lambda}{\pi}\Gamma_\alpha-\frac{1}{\pi}\int\frac{\kappa d\omega}{\omega^2}(\delta_{\alpha\beta}-A_{\alpha\beta})\{|\tfrac{1}{2}h_\beta^-|^2+\tfrac{1}{4}(h_\beta^{-*}h_\beta^++h_\beta^-h_\beta^{+*})\}v^2(\kappa),\tag{40}$$

$$D_\alpha\equiv\frac{1}{\pi}\int\frac{\kappa d\omega}{\omega^2}\{(\delta_{\alpha\beta}-A_{\alpha\beta})|\tfrac{1}{2}h_\beta^-|^2-(\delta_{\alpha\beta}+A_{\alpha\beta})\times\tfrac{1}{4}(h_\beta^{-*}h_\beta^+-h_\beta^-h_\beta^{+*})\}v^2(\kappa).\tag{41}$$

The first of Eqs. (37) represents our final result. It gives a single integral equation for the quantity $a_\alpha(\omega)$, which is related to the phase shift δ_α by

$$\operatorname{Im}a_\alpha(\omega)=-\kappa v^2(\kappa)|a_\alpha(\omega)|^2,\tag{42}$$

so that

$$a_\alpha(\omega)=-\frac{\sin\delta_\alpha(\omega)\exp[i\delta_\alpha(\omega)]}{\kappa v^2(\kappa)}.\tag{43}$$

Equation (37a) is identical to the dispersion-theoretic result of Goldberger,[15] obtained on more general grounds. With our specific model of the s-wave interactions, we have in addition the equation (37b) for b_α and the relations (39)–(41), the solutions to which relate the parameters C_0 and C_1 to the coupling constants λ_0 and λ. With λ_0 and λ and the cut-off energy $\omega_{\max}$ fixed by the scattering analysis, the interaction Hamiltonian of the model may be applied to a discussion of other low-energy processes, and thus its specific predictions may be compared with experiment.

A solution of the nonlinear integral Eq. (37a) for a_α gives the behavior of the scattering phase shifts in terms of the three parameters C_0, C_1, and $\omega_{\max}$. The first two terms of this equation describe the zero-total energy limiting behavior of the phase shifts. This limit differs from that in the p-wave theory in several important respects. First of all, Eqs. (37) and (43) show that

$$\delta_\alpha(\kappa)/\kappa\rightarrow C_0\tag{44}$$

[15] M. L. Goldberger (private communication).

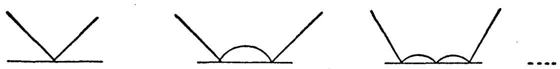

FIG. 3. Diagrams contributing to the aero-energy limit of $a_\alpha(\omega)$.

in the limit $\omega\to 0$. That the phase shift is independent of isotopic spin at zero total energy is as required by the result of Deser, Goldberger, and Thirring.[16]

However, in contrast with the C-L case, Eqs. (39)–(41) show that this low-energy limit does not directly measure the renormalized coupling constants. Because of the pair nature of the interaction, the ω^{-1} singularity as ${}^1\omega\to 0$ which is responsible for this very powerful result in C-L is not present here, and all orders in λ_0 and λ contribute to the zero-energy limit, as illustrated in Fig. 3.

Another important new feature of the low-energy behavior of the solutions for the s-wave phase shifts is apparent in the comparison of Eqs. (3′) and (44). Whereas the phase shifts are independent of isotopic spin in the limit $\omega=0$, the experimental values for low kinetic energy are of opposite sign for the $\frac{1}{2}$ and $\frac{3}{2}$ states, indicating either a zero or a discontinuity in one of the phase shifts between $\omega=0$ and $\omega=1$.

As a first step in constructing a solution to (37a), we very primitively approximate a_α by

$$a_\alpha(\omega)=C_0+C_1\Gamma_\alpha\omega. \tag{45}$$

This solution provides a satisfactory fit to the data at low energies (≤ 100 Mev, with the parameters) $C_0=0.04$ and $C_1=0.14$. These were chosen to match the experimental data[6] at 60-Mev kinetic energy.

It should be noted that (45) is not the first Born approximation. All orders of rescattering contribute to the integrals which relate the renormalized coupling constants λ_0 and λ to C_0 and C_1. Equation (45) is to be used as a guide in constructing approximate solutions to (37a) which satisfy more of the formal properties required by this equation.

We now examine Eq. (37a) further with the methods discussed in C-L. Consider $a_\alpha(\omega)$ to be a function of the complex variable z, with

$$a_\alpha(\omega)=\lim_{z\to\omega+i\epsilon} a_\alpha(z),$$

and list the properties of $a_\alpha(z)$ as given by its integral equation:

I: (i) as $z\to 0$, $a_\alpha(z)$ is regular;
(ii) as $z\to\infty$, $a_\alpha(z)$ diverges with z;
(iii) $a_\alpha(z)$ has branch lines from ± 1 to $\pm\infty$ on the positive and negative real axis;
(iv) Reality: $a_\alpha(z^*)=a_\alpha(z)^*$;
(v) Unitarity: $a_\alpha(\omega+i\epsilon)-a_\alpha(\omega-i\epsilon)=-2i\kappa|a_\alpha(\omega)|^2$;
(vi) Crossing: $a_\alpha(-z)=A_{\alpha\beta}a_\beta(z)$.

[16] Deser, Goldberger, and Thirring, Phys. Rev. **94**, 711 (1954). The result is proved here for a covariant local pseudoscalar renormalized theory.

Let us define $p_\alpha(z)=1/a_\alpha(z)$. Then

II: (i) as $z\to 0$, $p_\alpha(z)$ is regular;
(ii) as $z\to\infty$, $p_\alpha(z)\to 0$ like $1/z$;
(iii) $p_\alpha(z)$ has branches from ± 1 to $\pm\infty$;
(iv) Reality: $p_\alpha(z)^*=p_\alpha(z^*)$;
(v) Unitarity: $p_\alpha(\omega+i\epsilon)-p_\alpha(\omega-i\epsilon)=2i\kappa$;
(vi) Crossing: $[p_\alpha(-z)]^{-1}=A_{\alpha\beta}[p_\beta(z)]^{-1}$.

In addition, $p_\alpha(z)$ has a pole at every point in the complex plane at which $a_\alpha(z)$ has a zero.

As C-L have shown, the function $p_\alpha(z)$ satisfying the above properties may be written in the form

$$p_\alpha(z)=\frac{1}{\pi}\int \kappa d\omega v^2(\kappa)\times\left(\frac{1}{\omega-z}+\frac{1}{\omega+z}\sum_\beta A_{\alpha\beta}\left|\frac{p_\alpha(-\omega)}{p_\beta(\omega)}\right|^2\right)+M_\alpha(z), \tag{46}$$

where the added meromorphic function $M_\alpha(z)$ represents the contribution to $p_\alpha(z)$ coming from zeros in $a_\alpha(z)$.[17] The arbitrariness of this extra term results from the fact that the conditions II are not sufficient to uniquely specify $p_\alpha(z)$. In order to determine $M_\alpha(z)$, the zeros in $a_\alpha(z)$ implied by Eq. (42) must be known.

In the p-wave theory, the requirement was made that the solution reduce to that obtained from perturbation theory, in the limit $f\to 0$. This condition is replaced here by the demand that the solution reduce to the one obtained by applying perturbation theory to C_0 and C_1, namely Eq. (45). We thus require that

$$p_\alpha(z)\to\frac{1}{C_0+C_1\Gamma_\alpha z},$$

as C_0, $C_1\to 0$. This means that $a_\alpha(z)$ vanishes at $z=-C_0/C_1\Gamma_\alpha$, so that $M_\alpha(z)$ cannot be dropped. In fact it can be seen from (37a) that $a_\alpha(0)$ is independent of α, and $a_\alpha(\infty)\sim\Gamma_\alpha z$, so that either $a_{\frac{1}{2}}$ or $a_{\frac{3}{2}}$ vanishes somewhere between $z=0$ and $z=\infty$, or else a resonance appears.

The simplest choice which reduces to the perturbation limit, then, is to assume that $a_\alpha(z)$ has only one zero between $z=0$ and $z=\infty$; thus

$$M_\alpha(z)=A_\alpha/(z-B_\alpha).$$

The constants A_α and B_α introduced here are not entirely arbitrary, but must be chosen to be consistent with conditions II. It should be noted, however, that they contain the coupling constants, and therefore all the information of the theory. Without $M_\alpha(z)$, Eq. (46) is a self-contained equation for $p_\alpha(z)$ which does not involve λ_0 and λ.

[17] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).

Equation (46) may be rewritten in the form

$$p_\alpha(z)=\frac{A_\alpha}{z-B_\alpha}+\frac{1}{\pi}\int \kappa d\omega\left(\frac{1}{\omega-z}+\frac{1}{\omega+z}\right)$$
$$+\begin{cases}-\dfrac{1}{\pi}\displaystyle\int \frac{\kappa d\omega}{\omega+z}\frac{4|p_1-p_3|^2}{|4p_1-p_3|^2}, & \alpha=1\\ +\dfrac{1}{\pi}\displaystyle\int \frac{\kappa d\omega}{\omega+z}\frac{2|p_1-p_3|^2}{|2p_3+p_1|^2}, & \alpha=3.\end{cases} \quad (47)$$

As a first approximation, we drop the last parts of Eq. (47) and set $A_\alpha=A/\Gamma_\alpha$, $B_\alpha=B/\Gamma_\alpha$, in order to satisfy conditions II (i)–(v). This corresponds to putting the crossing terms equal to unity. Corrections to this solution from the last parts of (47) are calculated to be less than 10% for $\omega\lesssim 2$ and are included in the curves of Fig. 4. A reasonable fit to the data is achieved with the parameter $A=6.8$, $B=-0.57$, and $\omega_{\max}=4.5$. As is evident in Fig. 4, the crude low-energy approximation, Eq. (45), is a fairly good representation of the solution at low energies.

For larger $\omega(\omega\gtrsim 2)$, the deviation of the crossing term from unity begins to alter the p_1 (and a_1) solution considerably. As can be seen from (47), the corrections are such as to decrease a_1. We therefore expect the exact solution to (47) to look very much like the one drawn in Fig. 4 but with an effectively smaller cutoff which will serve to prevent a_1 from growing too large. The parameters A and B for the curve in Fig. 4 were chosen to fit the scattering data for pions in the 40–60 Mev kinetic energy interval. This gives a threshold value of $(\delta_1-\delta_3)=0.19\eta$ in agreement with the Panofsky experiment.[18] Because of a partial cancellation of terms in the solution for a_1, the high-energy $(\omega\sim 4)$ contributions in the integrals in Eq. (47) affect the behavior of δ_1 at the high-energy end of the curve in Fig. 4. Therefore the large value of δ_1 at $\eta=1.5$ reflects the cutoff and the one-meson approximation as well as the physics in the model being discussed. This is not the case for δ_3.

By using Eq. (45) for a_α, approximate solutions to the equations for b_α, λ_0, and λ may be constructed. The results are given here for a cutoff of $\omega_{\max}=4.5$ and are found to be very insensitive to this choice. To a good approximation, $b_\alpha(z)$ is a constant at low energies:

$$b_\alpha(z)=-0.04+(<0.01\Gamma_\alpha z), \quad (48)$$

and the coupling constants are

$$\lambda_0=0.4/\mu,\quad \lambda=0.4/\mu^2.$$

[18] B. T. Feld, "Meson physics," lecture notes, Massachusetts Institute of Technology, 1955 (unpublished); H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), Vol. II, Sec. 33. Both these sources contain comprehensive discussions of the low-energy *s*-phase shifts.

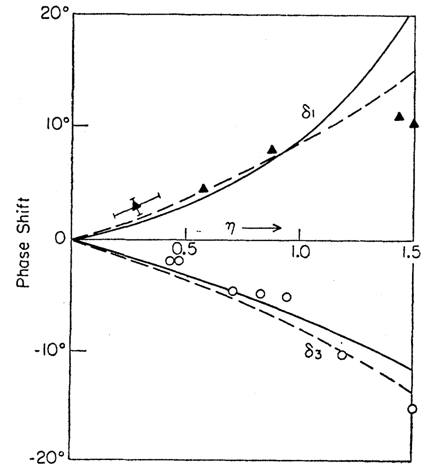

FIG. 4. *S*-wave phase shifts $\delta_{\frac{1}{2}}$ and $\delta_{\frac{3}{2}}$ as functions of meson momentum $\eta=\kappa/\mu c$. The experimental points are taken from Orear (reference 6). The dashed curve represents the approximate solution, Eq. (45). The solid curve is the solution constructed with Eq. (47).

These values change by less than 10% when $\omega_{\max}$ varies between 4 and 5 μ and are unaffected by the weak z dependence in b_α.

III. PHOTOPRODUCTION

As in the case of pion scattering discussed above, equations for pion photoproduction may be derived in an analogous manner to that of C-L. We add to the Hamiltonian (7) a term $-\int \mathbf{j}\cdot\mathbf{A}d^3\mathbf{x}$, describing the coupling of the entire pion-nucleon current $\mathbf{j}$ to the electromagnetic field. The quantity of interest is the transition amplitude, to first order in $\mathbf{A}$, from a state of a free photon of momentum $\mathbf{k}$ and polarization of $\boldsymbol{\varepsilon}$ together with a physical nucleon, to a one-meson final state. This transition amplitude is

$$\mathfrak{M}_{\mathbf{k}}(p)=\left\langle \Psi_p^{(-)}\middle| -\int \mathbf{j}\cdot\mathbf{A}_{\mathbf{k}}d^3\mathbf{x}\middle|\psi_0\right\rangle, \quad (49)$$

where $\mathbf{A}_{\mathbf{k}}=(2k)^{-\frac{1}{2}}\boldsymbol{\varepsilon}\exp(i\mathbf{k}\cdot\mathbf{x})$. Using (12), we have

$$\mathfrak{M}_{\mathbf{k}}(p)=\left\langle\psi_0\middle|\left[a_p, -\int \mathbf{j}\cdot\mathbf{A}_{\mathbf{k}}d^3\mathbf{x}\right]\middle|\psi_0\right\rangle$$
$$-\sum_n\left\{\frac{T_{np}^*\mathfrak{M}_{\mathbf{k}}(n)}{E_n-\omega_p-i\epsilon}-\frac{\mathfrak{M}_{\mathbf{k}}(n)^*S_{np}}{E_n+\omega_p}\right\}. \quad (50)$$

The current $\mathbf{j}$ appearing in these equations may be most conveniently defined as the coefficient of $-\mathbf{A}$ in an expansion of the Hamiltonian density in powers of the electromagnetic field. The vector potential $\mathbf{A}$ is inserted into the original meson-nucleon Hamiltonian (7) in a local theory with the prescription Eq. (6); this automatically insures gauge invariance. In a nonlocal theory, however, such as a finite cutoff gives us, this replacement is not sufficient. The nonlocal property of the coupling allows a meson to be created anywhere in an extended region, and the above prescription does not provide a current describing the transfer of charge from the nucleon core at the center to the point of creation of the meson. This prescription then gives gauge-dependent results as expressed by the violation of the continuity equation within the finite source.

If the static theory is viewed as the replacement of the nucleon fields $\bar{\psi}(\mathbf{x})\cdots\psi(\mathbf{x})$ in a local relativistic theory by a source density $s(\mathbf{x})$, then the transformation properties of the nucleon at the point $\mathbf{x}$ under a gauge transformation are lost. These transformation properties may be imitated by replacing $\bar{\psi}\cdots\psi$ by

$$\bar{\psi}(0)\exp\left(-ie\frac{1+\tau_3}{2}\int_0^{\mathbf{x}}\mathbf{A}\cdot d\mathbf{s}\right)\cdots \times\psi(0)\exp\left(ie\frac{1+\tau_3}{2}\int_0^{\mathbf{x}}\mathbf{A}\cdot d\mathbf{s}\right)$$

times $s(\mathbf{x})$, instead of just $s(\mathbf{x})$.[8,9] Equivalently, we may replace $\bar{\psi}(\mathbf{x})\cdots\psi(\mathbf{x})$ by $\bar{\psi}(0)\cdots\psi(0)s(\mathbf{x})$ if we multiply the meson fields $[\varphi(\mathbf{x}),\varphi^*(\mathbf{x}),\varphi_3(\mathbf{x})]$ by the factors $\exp[(-1,+1,0)e\int_0^{\mathbf{x}}\mathbf{A}\cdot d\mathbf{s}]$.

With this modification, then, the Hamiltonian has been made gauge-invariant. The presence of such line integrals means an extra contribution to the current beyond that appearing in a local theory. In the φ^2 coupling terms of (2), such factors would cancel out except for the fact that a separable source is being used.

These extra currents represent exactly the propagation of charge from the origin to the point $\mathbf{x}$ at which the coupling to the electromagnetic field occurs, the charge traveling along the path along which the line integral is evaluated. The line integral $\int_0^{\mathbf{x}}\mathbf{A}\cdot d\mathbf{s}$ is independent of path only in the limit of zero photon energy, where $\nabla\times\mathbf{A}\rightarrow 0$; thus only in this limit is such a method of constructing a gauge-invariant theory unique.

The use of a gauge-invariant theory, with the consequent introduction of such extra line currents, gives us an equation of continuity $\nabla\cdot\mathbf{j}+i[H,\rho]=0$ (the operators appear in the Schrödinger representation throughout) holding everywhere, inside as well as outside the source. Furthermore, since the extra currents represent the instantaneous transfer of charge, their presence does not affect the charge density. The quantity $\rho(\mathbf{x})$, therefore, is still given by $e[(1+\tau_3)/2]\delta(\mathbf{x})+ie(\pi^*\varphi^*-\pi\varphi)$, as in a local theory. (For further discussion of this, see Appendix III.)

Once an equation of continuity is established everywhere, the Kroll-Ruderman theorem follows. It is essential that this theorem hold, since we should like to view the static cutoff theory as an approximation to a complete local relativistic theory. The static theory should therefore imitate such properties of a covariant theory as are known, and the Kroll-Ruderman theorem holds for any gauge-invariant, local, covariant, and renormalizable pseudoscalar field theory.[7]

We now prove the Kroll-Ruderman theorem, rewriting Eq. (49) as

$$\mathfrak{M}_{\mathbf{k}}(p)=-\frac{1}{(2k)^{\frac{1}{2}}}\left\langle\Psi_p^{(-)}\middle|\int\mathbf{j}\cdot\boldsymbol{\varepsilon}e^{i\mathbf{k}\cdot\mathbf{x}}d^3\mathbf{x}\middle|\psi_0\right\rangle, \quad (51)$$

and studying its k, $\omega_p\rightarrow 0$ limit.

Using the identity[19]

$$\boldsymbol{\varepsilon}e^{i\mathbf{k}\cdot\mathbf{x}}=\nabla\left\{\boldsymbol{\varepsilon}\cdot\mathbf{x}\int_0^1 d\lambda e^{i\lambda\mathbf{k}\cdot\mathbf{x}}\right\} -i[(\boldsymbol{\varepsilon}\times\mathbf{k})\times\mathbf{x}]\int_0^1\lambda d\lambda e^{i\lambda\mathbf{k}\cdot\mathbf{x}}, \quad (52)$$

and the equation of continuity, we obtain, in the limit $k\rightarrow 0$:

$$\begin{aligned}\mathfrak{M}_{\mathbf{k}}(p)&=\frac{1}{(2k)^{\frac{1}{2}}}\left\langle\Psi_p^{(-)}\middle|-\int d^3\mathbf{x}\mathbf{j}(\mathbf{x})\cdot\nabla\left\{\boldsymbol{\varepsilon}\cdot\mathbf{x}\int_0^1 d\lambda e^{i\lambda\mathbf{k}\cdot\mathbf{x}}\right\}\middle|\psi_0\right\rangle\\&=\frac{i}{(2k)^{\frac{1}{2}}}\left\langle\Psi_p^{(-)}\middle|\int d^3\mathbf{x}[H,\rho(\mathbf{x})]\boldsymbol{\varepsilon}\cdot\mathbf{x}\int_0^1 d\lambda e^{i\lambda\mathbf{k}\cdot\mathbf{x}}\middle|\psi_0\right\rangle\\&=\frac{i}{(2k)^{\frac{1}{2}}}\int d^3\mathbf{x}\boldsymbol{\varepsilon}\cdot\mathbf{x}\int_0^1 d\lambda e^{i\lambda\mathbf{k}\cdot\mathbf{x}} \quad (53)\\&\times\left\langle\psi_0\middle|[a_p,[H,\rho(\mathbf{x})]]-[H',a_p^\dagger]^\dagger\frac{1}{H-\omega_p-i\epsilon}[H,\rho(\mathbf{x})]+[H,\rho(\mathbf{x})]\frac{1}{H+\omega_p}[H',a_p]\middle|\psi_0\right\rangle. \quad (54)\end{aligned}$$

Taking next the limit $\omega_p\rightarrow 0$ and using $H|\psi_0\rangle=0$, there results

$$\begin{aligned}\mathfrak{M}_{\mathbf{k}}(p)&=\frac{i}{(2k)^{\frac{1}{2}}}\int d^3\mathbf{x}\boldsymbol{\varepsilon}\cdot\mathbf{x}\int_0^1 d\lambda e^{i\lambda\mathbf{k}\cdot\mathbf{x}}\\&\times\langle\psi_0|[a_p[H,\rho(\mathbf{x})]]+[[H',a_p],\rho(\mathbf{x})]|\psi_0\rangle \quad (55)\\&=\frac{i}{(2k)^{\frac{1}{2}}}\int d^3\mathbf{x}\boldsymbol{\varepsilon}\cdot\mathbf{x}\int_0^1 d\lambda e^{i\lambda\mathbf{k}\cdot\mathbf{x}}\\&\times\langle\psi_0|[a_p,[H_0,\rho(\mathbf{x})]]+[H',[a_p,\rho(\mathbf{x})]]|\psi_0\rangle. \quad (56)\end{aligned}$$

The first term here is proportional to $\mathbf{k}$, and vanishes in the limit $k\rightarrow 0$. The second term may easily be

[19] R. H. Capps, Phys. Rev. **99**, 926 (1955).

calculated directly, using

$$\rho(\mathbf{x})=e[(1+\tau_3)/2]\delta(\mathbf{x})+ie[\pi^*(\mathbf{x})\varphi^*(\mathbf{x})-\pi(\mathbf{x})\varphi(\mathbf{x})]. \tag{57}$$

Only the second term here contributes to the commutator $[a_p,\rho(\mathbf{x})]$. Since ρ is bilinear in the meson field, the commutator is proportional to a_p with an amplitude $+1$, -1, or 0 for positive, negative and neutral mesons, respectively. Then, from the identity

$$\langle\psi_0|[H',a_p]|\psi_0\rangle=-(4\pi)^{\frac{1}{2}}\left(\frac{if}{\mu}\right)\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{(2\omega_p)^{\frac{1}{2}}}\tau_p, \tag{58}$$

where f is the renormalized p-wave coupling constant, the Kroll-Ruderman theorem follows. We find, finally, by performing the integral over $d^3\mathbf{x}$ and letting $k\to 0$,

$$\lim_{k,\,\omega_p\to 0}\mathfrak{M}_{\mathbf{k}}(p)=(4\pi)^{\frac{1}{2}}\left(\frac{ief}{\mu}\right)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}}{(4k\omega_p)^{\frac{1}{2}}}\left(\frac{\tau_p\tau_3-\tau_3\tau_p}{2}\right),$$

which agrees with Eq. (4).

Let us next look at the photoproduction equation obtained from the Hamiltonian [Eqs. (1), (2), (7), (8)]. The inhomogeneous term is easily calculated:

$$\begin{aligned}\left\langle\psi_0\left|\left[a_p,-\int\mathbf{j}\cdot\mathbf{A}_{\mathbf{k}}d^3\mathbf{x}\right]\right|\psi_0\right\rangle
&=(4\pi)^{\frac{1}{2}}\left(\frac{ief}{\mu}\right)\frac{1}{(4k\omega_p)^{\frac{1}{2}}}\left(\frac{\tau_p\tau_3-\tau_3\tau_p}{2}\right)\Bigg\{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}v(|\mathbf{p}-\mathbf{k}|)\\
&\quad-\frac{2\boldsymbol{\sigma}\cdot(\mathbf{p}-\mathbf{k})\boldsymbol{\varepsilon}\cdot\mathbf{p}}{\omega_{\mathbf{p}-\mathbf{k}}^2}v(|\mathbf{p}-\mathbf{k}|)+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\varepsilon}\cdot\mathbf{p}}{p^2}pv'(p)\\
&\quad+\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}v(p)\left(-\frac{\lambda_0}{3\pi^2}\int\frac{\kappa^2d\omega}{\omega}v(\kappa)v'(\kappa)\right)\\
&\quad+\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}v(p)\left(\omega_p\frac{\lambda}{6\pi^2}\int\frac{\kappa^2d\omega}{\omega}v(\kappa)v'(\kappa)\right)\Bigg\}\\
&\quad+(4\pi)^{\frac{1}{2}}\left(\frac{ief}{\mu}\right)\frac{1}{(4k\omega_p)^{\frac{1}{2}}}\delta_{p,3}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\\
&\quad\times v(p)\left(\omega_p\frac{\lambda}{3\pi^2}\int\frac{\kappa^2d\omega}{\omega}v'(\kappa)v(\kappa)\right).\end{aligned} \tag{59}$$

The first two terms in (59) have been discussed by C-L in connection with the p-wave theory; the remaining terms come from the line integrals used in making the theory gauge-invariant. They arise, respectively, from the p-wave term, the $\lambda_0\varphi^2$ term, and the last two from the $\lambda\boldsymbol{\tau}\cdot(\boldsymbol{\varphi}\times\boldsymbol{\pi})$ term. Using the values of λ_0 and λ determined from the scattering theory, we see that the last three gauge terms are quite small for any reasonable choice of the cutoff. The contribution of the p-wave gauge term is proportional to $v'(p)$; since the cutoff should be approximately flat up to fairly large momenta, this term will also be small in the region of interest. We therefore drop the gauge terms. It should be emphasized, therefore, that the importance of the gauge terms lies in the fact that their presence is required for an equation of continuity, and hence a Kroll-Ruderman theorem.

We now specialize to the one-meson approximation. As in the scattering problem, the equation then separates into s- and p-wave parts, and the photoproduction amplitude may be written as the sum of an s-wave and a p-wave contribution. C-L have discussed the p-wave term, and treated the s-wave term in perturbation theory. Since only s-wave mesons will be of interest here, T and S in Eq. (50) will be the quantities determined from the s-wave scattering, and we may pick out from the inhomogeneous term only those parts contributing to s-wave photoproduction. These will all be of the form $\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}$ multiplied by a function $f(k^2,p^2)$.

The s-wave scattering amplitudes T and S are spin-independent, and therefore we may write

$$\mathfrak{M}_{\mathbf{k}}(p)=(4\pi)^{\frac{1}{2}}\left(\frac{ief}{\mu}\right)\frac{1}{(4k\omega_p)^{\frac{1}{2}}}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\times\sum_\alpha 2\pi M_\alpha(p,k)(p|Q_\alpha|3), \tag{60}$$

where

$$\begin{aligned}M_\alpha(p,k)=&-f(k^2,p^2)\frac{\Gamma_\alpha}{\pi}\\
&-\frac{1}{\pi}\int\kappa d\omega v^2(\kappa)\left\{\frac{t^\alpha(\omega,\omega_p)M_\alpha(\omega,k)}{\omega-\omega_p-i\epsilon}+A_{\alpha\beta}\frac{M_\beta(\omega,k)^*s^\beta(\omega,\omega_p)}{\omega+\omega_p}\right\}.\end{aligned} \tag{61}$$

If the gauge terms are neglected, $f(k^2,p^2)$ is just the s-wave part of the $\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}$ and meson current contributions; thus, with $\gamma\equiv(\omega_p^2+k^2)/2pk$,

$$f(k^2,p^2)=1-\frac{p}{2k}\left(\gamma-\frac{\gamma^2-1}{2}\log\frac{\gamma+1}{\gamma-1}\right); \tag{62}$$

at low energies ($k\approx 1$),

$$f(k^2,p^2)\approx 1-\frac{2}{3}\left(\frac{p^2}{\omega_p^2+k^2}\right). \tag{63}$$

Since the s-wave scattering is small, a perturbation expansion of Eq. (61), using the values of t and s determined from the scattering analysis, may be expected to be valid. One iteration of Eq. (61) gives

$$M_\alpha(p,1)=-\frac{1}{\pi}\Gamma_\alpha+\frac{1}{\pi}(-0.07\Gamma_\alpha+0.13), \tag{64}$$

the first term in the right-hand side being the lowest order result. The correction is very insensitive to the effective cutoff and is calculated for the parameters in the discussion of scattering. The validity of the iteration procedure may be checked by the use of the following theorem. We write $M_\alpha(p,k)=M_\alpha^{(1)}(p,k)+M_\alpha^{(2)}(p,k)$ corresponding to the two inhomogeneous terms in Eqs. (61), (63). The first arises from the interaction current and the second from the meson current. The following identity relates the first term with the scattering calculation:

$$M_\alpha^{(1)}(p,k)=M_\alpha^{(1)}(p)=-\frac{1}{\omega_p}\frac{a_\alpha(\omega_p)+b_\alpha(\omega_p)}{\lambda}. \quad (65)$$

This is easily seen by comparing the equation for $M_\alpha^{(1)}$ with Eqs. (37) for a_α and b_α. Near threshold the contribution of the interaction term dominates, the meson current contribution vanishing with p^2 in Eq. (63). Thus if $M_\alpha^{(2)}$ can be neglected, we can calculate $M_\alpha(p,k)$ directly in terms of a_α, b_α, and λ. Doing this gives results in reasonable agreement with the iteration solution,

$$M_\alpha^{(1)}(p)=-\frac{1.1}{\pi}\Gamma_\alpha. \quad (66)$$

A more accurate solution for a_α introduces an α-independent contribution which corresponds to π^0 photoproduction.

The experimental implications of the above photoproduction calculation are evident with Eq. (64) exhibited in terms of pion charge states:

$$M_{+,-,0}(p,1)=+\frac{1}{(2^{\frac{1}{2}}\pi)}(1.07\tau_-, -1.07\tau_+, 0.18). \quad (67)$$

First of all, there is a 15% increase in the charged-pion photoproduction cross section at low energies as a result of the s-wave rescattering. This means a 15% reduction in the magnitude of the renormalized p-wave coupling constant, f^2, which is assigned to fit the low-energy photoproduction data. In C-L, the effective-range approximation to the p-wave scattering gives[20] $f^2=0.076$, and the analysis of charged pion photoproduction[21] gives $f^2=0.073\pm0.007$. When s-wave scattering is now included, this latter number drops 15% to $f^2=0.064\pm0.006$, and the agreement between the scattering and photoproduction analyses is less striking. However, we feel that it would be premature to interpret these numbers as indicating a disagreement. In particular, the value for f^2 in the scattering analysis is uncertain by $\sim$20% due to the extrapolation procedure.[20] It also remains for a criterion to be given for the quantitative validity of the "one-meson approximation" which underlies this discussion.

The neutral-pion photoproduction predicted in Eq. (67) arises from s-wave charge exchange scattering and has a low-energy cross section of 3% of that for π^+ production, or

$$(d\sigma_{\pi^0})_s=(0.03)\left(\frac{2f^2}{137}\right)\left(\frac{p_\pi}{k_\gamma}\right)d\Omega. \quad (68)$$

This contribution is evaluated to be 1μb ($1\mu\text{b}=10^{-30}$ cm^2) for $(E_\gamma)_{\text{lab}}=150$ Mev (5 Mev above threshold), rising to 2μb for $(E_\gamma)_{\text{lab}}=160$ Mev, and to 3μb for $(E_\gamma)_{\text{lab}}=180$ Mev. In their analysis of the threshold π^0 production experiments of Mills and Koester,[10] Bethe and de Hoffmann[22] give $1\pm1\mu$b as a rough estimate of $(\sigma_{\pi^0})_s$ near threshold. They arrive at this number by extrapolating the cross sections to threshold with the characteristic p-wave energy variation. In order to compare Eq. (68) directly with π^0 production data in the energy range 160–180 Mev, it is necessary to know the p-wave contribution as calculated in C-L; this is $\sim2.5\mu$b for $(E_\gamma)_{\text{lab}}=160$ Mev and 10μb for $(E_\gamma)_{\text{lab}}=180$ Mev. However, these numbers from C-L for the p-wave contribution are obtained by using the full static magnetic moments for the neutron and proton and may be a significant overestimate.[23]

All that one can say at this time, then, is that Eq. (68) is not inconsistent with experiment. It would be of great value to measure accurately the excitation function for the π^0 production cross section in the energy range from threshold [$(E_\gamma)_{\text{lab}}=145$ Mev] up to $\sim$180 Mev. With such information the s-wave contribution can be determined independently of the major uncertainties in the p-wave analysis. It is an elusive feature of such s-wave effects that they are visible only at energies near enough to threshold so that the generally stronger p-wave interactions are suppressed by phase space. At such low energies the cross sections are usually very small and accurate quantitative data exceedingly hard to obtain.

IV. CONCLUSION

To summarize briefly, a fixed-source description of the s-wave pion-nucleon interaction has been constructed along similar lines to the Chew-Low[1] work. With the bilinear s-wave interaction of Eq. (2), it proved possible to reproduce the low-energy scattering phase shifts δ_1 and δ_3. This interaction, with the (renormalized) coupling parameters λ_0 and λ determined in the scattering analysis, was then applied to a discussion of the photoproduction. First of all, a Kroll-Ruderman theorem was established. The proof of this theorem was based upon two facts, the pseudoscalar property of the mesons and the possibility of introducing electromagnetic interactions into a nonlocal field theory in a gauge-invariant way. Secondly, it was shown that the contributions to the photoproduction cross sections

[20] S. J. Lindenbaum and L. C. L. Yuan, Phys. Rev. **100**, 306 (1955). A similar number is obtained by U. Haber-Schaim (to be published) from the dispersion relations of Goldberger *et al.*

[21] G. Bernardini as quoted in reference 1.

[22] Reference 18, Sec. 36.

[23] See discussion on p. 1586 of reference 1 and reference there to earlier results.

at low energies due to s-wave rescattering are consistent with experiment. However, more precise data very near threshold are required before it is possible to say whether these s-wave contributions aid or injure the agreement.

In that we have constructed a fixed-source theory, there are no nucleon recoil currents in the above considerations. Hence the plus-to-minus ratio in the two processes

$$\gamma+p\rightarrow n+\pi^+,$$
$$\gamma+n\rightarrow p+\pi^-,$$

is predicted to be unity, in disagreement with experiment. A study of the plus-to-minus ratio does not lie within the scope of a no-recoil theory.[24]

By way of orientation, it is interesting to compare the values of λ_0 and λ given in Eqs. (48) with the unrenormalized perturbation relations, Eqs. (3), as derived from the Hamiltonian of a relativistic γ_5 theory. The definitions of the renormalized coupling constants in Eq. (28) and footnote 13 differ for the three constants λ_0, λ, and f. In the absence of any relation between the unrenormalized and renormalized constants, λ_0, λ, and f cannot be compared with λ_0^0, λ^0, and f^0 in a consistent way. Ignoring this fact, we have that λ_0 is smaller by a factor ~ 0.04 and λ is smaller by a factor of ~ 0.4 than the values in Eq. (3) for λ_0^0 and λ^0. It is thus seen that the ratio λ_0/λ must be about one-tenth the ratio of λ_0^0/λ^0 given in Eq. (3) in order to reproduce the observed isotopic spin dependence of the s-wave phase shifts.

Finally we note that the static theory developed in this paper can, in principle at least, be confronted with two additional cross sections involving low-energy s-wave pions. These are the magnetic dipole photoproduction of two s-wave pions[25] and the inelastic scattering of an incident p-wave into two s-wave pions.[11] Comparison with measured cross sections for these processes very near threshold will help define the possible role of a pion-pion interaction. In particular, if such an interaction exists and is characterized by a short range, it will be most effective in processes with two s-wave pions.

APPENDIX I

We wish to prove that $\langle\psi_0|V_{\kappa,\alpha}{}^s|\psi_0\rangle=0$. To do this we formalize the previous parity argument. Let I be the inversion operator for a pseudoscalar field:

$$I=\exp\left\{-\frac{i\pi}{4}\sum_{\kappa,\alpha}(a_{\kappa,\alpha}{}^\dagger+a_{-\kappa,\alpha}{}^\dagger)(a_{\kappa,\alpha}+a_{-\kappa,\alpha})\right\}.$$

Then

$$\begin{aligned}&II^\dagger=I^\dagger I=1,\\ &Ia_{\kappa,\alpha}I^\dagger=-a_{-\kappa,\alpha},\\ &Ia_{\kappa,\alpha}{}^\dagger I^\dagger=-a_{-\kappa,\alpha}{}^\dagger,\\ &I\phi(\mathbf{x})I^\dagger=-\phi(-\mathbf{x});\\ &I\pi(\mathbf{x})I^\dagger=-\pi(-\mathbf{x}),\\ &[H_s',I]=[H_p',I]=[H_0,I]=0.\end{aligned}\tag{I-1}$$

The above conditions express the fact that we deal with a pseudoscalar meson field. We now note that if $|u_0\rangle$ represents a bare nucleon and $|\psi_0\rangle$ a physical nucleon, then

$$|\psi_0\rangle=S|u_0\rangle,\tag{I-2}$$

where S is a scalar operator; that is, it is a function of H_0, H_s' and H_p' only, so that

$$[I,S]=0.\tag{I-3}$$

Since

$$\begin{aligned}I|u_0\rangle&=|u_0\rangle,\\ I|\psi_0\rangle&=IS|u_0\rangle=S|u_0\rangle=|\psi_0\rangle.\end{aligned}\tag{I-4}$$

Therefore,

$$\begin{aligned}\langle\psi_0|V_{\kappa,\alpha}{}^s|\psi_0\rangle&=\langle\psi_0|IV_{\kappa,\alpha}{}^sI^\dagger|\psi_0\rangle\\ &=\langle\psi_0|I[H_s',a_{\kappa,\alpha}]I^\dagger|\psi_0\rangle\\ &=\langle\psi_0|[H_s'Ia_{\kappa,\alpha}I^\dagger]|\psi_0\rangle\\ &=-\langle\psi_0|[H_s',a_{-\kappa,\alpha}]|\psi_0\rangle\\ &=-\langle\psi_0|V_{-\kappa,\alpha}{}^s|\psi_0\rangle.\end{aligned}\tag{I-5}$$

But,

$$V_{\kappa,\alpha}{}^s=V_{-\kappa,\alpha}{}^s\tag{I-6}$$

as can be seen by referring to Eq. (25) for $V_{\kappa,\alpha}{}^s$. Hence, the required result follows.

APPENDIX II

We wish to verify the statement made in Sec. I that both the λ_0 and the λ terms are necessary to fit (3′).

Consider the function

$$a_\alpha{}^+(\omega)=\tfrac{1}{2}[a_\alpha(\omega)+a_\alpha(-\omega)]\tag{II-1}$$

$$=\tfrac{1}{2}[a_\alpha(\omega)+A_{\alpha\beta}a_\beta(\omega)]\tag{II-2}$$

$$=\tfrac{1}{2}[a_{\frac{1}{2}}(\omega)+2a_{\frac{3}{2}}(\omega)].\tag{II-3}$$

Then from (37a),

$$a_\alpha{}^+(\omega_f)=C_0-\frac{\omega_f^2}{\pi}\int\frac{\kappa d\omega}{\omega}v^2(\kappa)\frac{1}{\omega^2-\omega_f^2-i\epsilon}\times\{|a_\alpha|^2+A_{\alpha\beta}|a_\beta|^2\}.\tag{II-4}$$

Referring to (39), we note that if $\lambda_0=0$ then $C_0\leq 0$. This together with Eq. (II-4) show that $a_\alpha{}^+(1)\leq 0$.

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[24] A study of the π^+/π^- ratio and of the entire s-wave photoproduction problem based on relativistic dispersion relations has been carried out by Chew, Goldberger, Low, and Nambu (to be published); see also *Proceedings of The Sixth Annual Rochester High-Energy Conference* (to be published). In this work relativistic features of the theory lead also to s-wave π^0 production.

[25] This is to be contrasted with the predominantly electric dipole photoproduction of one s- and one p-wave pion discussed by R. Cutkosky and F. Zachariasen [Phys. Rev. **103**, 1108 (1956)]. In analogy with the magnetic and electric dipole cross sections for the deuteron photodisintegration, the double s-wave production will be significant at threshold.

However, from (II-3), (43), and (3′) we note that

$$a_\alpha^+(1) = -\tfrac{1}{2}\{0.16-0.22\} \geq 0$$

is required by experiment. Hence we conclude that in the one-meson approximation we must have both $\lambda_0 \neq 0$ and $\lambda \neq 0$.

APPENDIX III

A gauge transformation, $\mathbf{A}\to\mathbf{A}+\nabla\chi$, in any field-theoretic Hamiltonian $H(\mathbf{A})=\int \mathcal{H}(\mathbf{A})d^3\mathbf{x}$, is equivalent[19] to the similarity transformation $\exp(iD)H(\mathbf{A}) \times\exp(-iD)$, with $D=\int\rho(\mathbf{x})\chi(\mathbf{x})d^3\mathbf{x}$; $\rho(\mathbf{x})$ is the charge density operator.

Consider first an infinitesimal transformation $\epsilon\chi$. We have

$$\int d^3\mathbf{x}\mathcal{H}(\mathbf{A}+\epsilon\nabla\chi)=H(\mathbf{A})-\epsilon\int d^3\mathbf{x}\nabla\chi\cdot\mathbf{j}, \quad \text{(III-1)}$$

where by definition, $\mathbf{j}$ is the total current operator. Furthermore,

$$e^{i\epsilon D}H(\mathbf{A})e^{-i\epsilon D}=H(\mathbf{A})+i\epsilon[D,H(\mathbf{A})], \quad \text{(III-2)}$$

and hence

$$-\int d^3\mathbf{x}\nabla\chi\cdot\mathbf{j}=i\int d^3\mathbf{x}\chi(\mathbf{x})[\rho(\mathbf{x}),H]. \quad \text{(III-3)}$$

Integrating by parts, and observing that χ is arbitrary, we obtain

$$\nabla\cdot\mathbf{j}(\mathbf{x})+i[H,\rho(\mathbf{x})]=0. \quad \text{(III-4)}$$

Thus, if the Hamiltonian is gauge-invariant, an equation of continuity holds everywhere.

It remains to determine the form of $\rho(\mathbf{x})$ required for static cut-off theories of the type considered here. If a $\rho(\mathbf{x})$ can be constructed for which the similarity transformation

$$\exp(iD)H(\mathbf{A})\exp(-iD)$$

produces the required gauge transformation, then this $\rho(\mathbf{x})$ must satisfy the continuity equation with the total current.

We first observe that for a local theory

$$\rho(\mathbf{x})=e[(1+\tau_3)/2]\delta(\mathbf{x})+ie(\pi^*\varphi^*-\pi\varphi) \quad \text{(III-5)}$$

is the required charge density. For a nonlocal theory, the only modification needed is to multiply each field φ, π^* in $H(\mathbf{A})$ by $\exp(-ie\int_0^{\mathbf{x}}\mathbf{A}\cdot d\mathbf{s})$ and each field φ^*, π by $\exp(ie\int_0^{\mathbf{x}}\mathbf{A}\cdot d\mathbf{s})$. The charge density (III-5) will still be correct if the similarity transformation produces the required gauge transformation on the line integrals.

By charge conservation, the fields φ and π always occur in combinations of the form $\tau_+\varphi$, $\varphi\varphi^*$, $\tau_-\varphi^*$, etc. For any Hamiltonian $H(\mathbf{A})$, then, which is a polynomial in φ and π we have, writing $H=f(\tau_+\varphi,\cdots)$:

$$e^{iD}H(\mathbf{A})e^{-iD} = f\left(e^{iD}\tau_+\varphi\exp\left(-ie\int\mathbf{A}\cdot d\mathbf{s}\right)e^{-iD},\cdots\right). \quad \text{(III-6)}$$

We wish to verify that this is

$$f\left(\tau_+\varphi\exp\left(-ie\int\mathbf{A}\cdot d\mathbf{s}\right)\exp\left(-ie\int\nabla\chi\cdot d\mathbf{s}\right),\cdots\right).$$

Now, using the charge density (III-5) in D,

$$e^{iD}\varphi(\mathbf{x})e^{-iD}=e^{-ie\chi(\mathbf{x})}\varphi(\mathbf{x}), \quad \text{(III-7)}$$

and

$$e^{iD}\tau_+e^{-iD}=e^{ie\chi(0)}\tau_+. \quad \text{(III-8)}$$

We thus have

$$e^{iD}\tau_+\exp\left(-ie\int\mathbf{A}\cdot d\mathbf{s}\right)\varphi e^{-iD} = \tau_+\varphi\exp\left[-ie\int(\mathbf{A}+\nabla\chi)\cdot d\mathbf{s}\right]. \quad \text{(III-9)}$$

The charge density (III-5) therefore produces the required gauge transformation in the Hamiltonian even with the presence of the extra line integrals. It is easily verified that the remaining combinations of $\boldsymbol{\tau}$, φ, and π also transform properly.

The local charge density, then, satisfies the equation of continuity in a nonlocal theory. However, the current density is modified by the added source terms.