p-p Scattering in the Bev Range*

W. RARITA†
Department of Physics, Brooklyn College, Brooklyn, New York
(Received April 24, 1956)

We outline some general methods of attacking the p-p scattering problem in the Bev range. We find that a fairly definitive phase shift analysis can be made if the orbital quantum number is limited to three and if j independence is assumed. For the 1-Bev case, the inequality $\sigma(0°) \geqslant (k\sigma_t/4\pi)^2$ (the minimum theorem) plays an essential role and facilitates the calculation. The phase shifts are found to depend on a single finite parameter. Detailed results are tabulated. The ray optical theory and the complex square-well potential are investigated and shown to be generally inadequate to describe the calculated phase shifts. A square-well potential with a central core suggests itself as a possible successful model.

INTRODUCTION

PROTON–PROTON scattering has played a central role in our knowledge of the nucleon-nucleon interaction. The highly precise data available for this process yield in turn our most accurate information of the nuclear interaction. Until recently, before the experiments of Smith, McReynolds, and Snow[1] on the angular distribution and of Shapiro, Leavitt, and Chen[2] on the total cross section for the p-p scattering were performed, our principal information for the p-p system was for the $^1S_0$ and $^3P_0$ states. At Bev energies, the higher orbital states are expected to enter prominently. In fact, although we shall restrict ourselves to states up to F for reasons of simplicity, we have no *a priori* justification for this assumption. The Pauli exclusion principle acts to eliminate a large number of states and brings our analysis within range of feasibility. The additional complication with which we have to contend, of the complex phase shifts now required to explain the large absorption cross section observed, is compensated by the knowledge we acquire of the distribution, excitation, and production of mesons bound to the proton. Even with the minimal condition that only the orbital states, ^1S, ^1D, $^3P_{0,1,2}$, and $^3F_{2,3,4}$, shall enter our discussion, we immediately see that our problem is in general excessively underdetermined. We have eight independent states and, as each state requires two parameters (the real and imaginary part of the phase shift), there are sixteen theoretical parameters to evaluate. However, as we shall explicitly show later, there are only five independent experimental data for the case under consideration. The main purpose of this paper is to investigate reasonable ways of reducing the sixteen theoretical quantities to about five. The most drastic assumption is to take the various orbital states to be linked together by the use of a specific model, such as the refractive and absorptive homogeneous sphere of interaction, or equivalently a square-well complex potential. But these highly specialized models yield only three theoretical parameters to account for the five experimental data and can be only accidentally an adequate description. We can assume, however, different shapes for the potential and in principle we should be able to fit the data in this way. An interesting possibility of this type is to take a square-well potential with a central core: a four-parameter family. A less restrictive supposition is to take the various orbital states as independent, not connected through a common potential. Then we get eight theoretical parameters, which become reduced to six, only one in excess, for the 1-Bev data, because of a fortunate circumstance related to what we shall call the minimum theorem. Our most definite conclusions concern this case. We shall also look into the modification of the above situation when instead of $^3P_0=^3P_1=^3P_2$ we allow $^3P_0$ with $^3P_1=^3P_2=0$ and no change in the other admitted states. Other more involved j dependences, patterned on spin-orbit or tensor coupling, are worthy of study but will not be discussed in this paper.

I. Classical Black Sphere Diffraction

At 1 Bev, it was found[3] that the expression given by Fernbach, Serber, and Taylor[4] for the elastic scattering of neutrons by a black nucleus, when adapted to the p-p case, gave unusually good agreement. We have

$$\sigma(\theta) \sim J_1^2(kR\sin\theta)/\sin^2\theta, \tag{1a}$$

$$\sigma_t = 2\pi R^2; \quad k = (ME_L/2\hbar^2)^{\frac{1}{2}}, \tag{1b}$$

where E_L is the laboratory energy of the incident proton. The wave number k is calculated for one of the protons relative to the center of mass. The formula for k written in this way is valid for all energies. The total cross section σ_t was taken as 48 mb from the data of Shapiro, Leavitt, and Chen.[2] We get directly that $kR \sim 3.1$. We have a single adjustable constant in this theory. The fit, as measured by $\mathfrak{R} \equiv (1/n)(\sum|\Delta\sigma/\sigma|^2)^{\frac{1}{2}}$ at the six experimental points, is 1.0%.

However, the small value of $kR \sim 3$ throws doubt on the validity of Eq. (1a) which follows from a partial wave analysis if we assume that large l values contribute the major part of the scattering. The low value of kR suggests that no higher states than D should occur.

* Supported in part by the U. S. Atomic Energy Commission.
† Visiting professor at Case Institute of Technology, Cleveland, Ohio during the year 1955–1956.
[1] Smith, McReynolds, and Snow, Phys. Rev. **97**, 1186 (1955).
[2] Shapiro, Leavitt, and Chen, Phys. Rev. **95**, 663 (1954).
[3] R. Serber and W. Rarita, Phys. Rev. **99**, 629(A) (1955).
[4] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).

II. Special Models

We saw in the previous section that the angular distribution for 1 Bev was satisfied extremely well for the classical black-sphere model. The next approximation would be to use the ray optical model or W.K.B. method as discussed by Fernbach, Serber, and Taylor,[4] or alternatively to use the complex square-well potential to fit the p-p data. These direct attacks led to very poor results. The nature of these difficulties did not become clear until we made a phase shift analysis, as we shall discuss below in Sec. VI.

III. The Minimum Theorem

We proceed with a partial wave analysis. An important result, from both the experimental and theoretical aspects, is an inequality relating the forward angle scattering $\sigma(0°)$ and the total cross section σ_t. For p-p scattering, assuming no j dependence (or, in the language of potentials, limiting ourselves to a central potential), we can write

$$\sigma(\theta)=|f_1(\theta)|^2+3|f_3(\theta)|^2, \tag{2a}$$

$$\sigma_t=(2\pi/k)\,\mathrm{Im}[f_1(0°)+3f_3(0°)]. \tag{2b}$$

To get the total elastic cross section σ_e, we have to integrate $\sigma(\theta)$ over the forward hemisphere or 2π steradians. For later reference, we include here the explicit expressions for $f_1(\theta)$ and $f_3(\theta)$:

$$f_1(\theta)=\sum_{l=0}\xi_l \quad \text{and} \quad f_3(\theta)=\sum_{l=1}\xi_l, \tag{3a}$$

where

$$\xi_l=\frac{-(2l+1)(1-e^{2i\delta l})}{2ik}P_l(\cos\theta). \tag{3b}$$

The summation for the singlet scattering phase amplitude $f_1(\theta)$ is over even l and the triplet $f_3(\theta)$ is over odd l values. It is instructive to consider the "black" S state i.e., $e^{2i\delta_0}\to 0$. Then

$$\sigma(\theta)=1/4k^2 \quad \text{and} \quad \sigma_e=2\pi/4k^2,$$
$$\sigma_t=(2\pi/k)\cdot(1/2k)=2\pi/2k^2,$$
$$\sigma_a=\sigma_t-\sigma_e=2\pi/4k^2=\sigma_e,$$

as we expect.

Returning to the proof of the minimum theorem, we have

$$\sigma(0°)\geq[\mathrm{Im}f_1(0°)]^2+3[\mathrm{Im}f_3(0°)]^2. \tag{4}$$

The minimum of the right-hand side of Eq. (4) under the condition of Eq. (2b) is attained when $\mathrm{Im}f_1(0°)=\mathrm{Im}f_3(0°)$. As a consequence, we get

$$\sigma(0°)\geq 4[\mathrm{Im}f_1(0°)]^2=(k\sigma_t/4\pi)^2. \tag{5}$$

This inequality, which we shall name the minimum theorem, was obtained independently by Karplus and Ruderman.[5]

[5] R. Karplus and M. Ruderman (private communication).

IV. Experimental Determination of the Total Elastic Cross Section

Initially, we limit ourselves to states up to F; then

$$4k^2\sigma(\theta)=AP_0+BP_2+CP_4+DP_6. \tag{6}$$

That no Legendre polynomial of odd order enters, follows most readily from the indistinguishability of the two interacting protons, which entails forward and backward symmetry in the center-of-mass system. The coefficients A, B, C, and D are four parameters which are determined by a least-squares fit of the experimental data $\sigma(\theta)$. The measure of fit $\mathfrak{R}$ defined in Sec. I is 0.66%. The total cross section σ_t is the fifth experimental datum. For 1 Bev, we find that the inequality of Eq. (5) is not satisfied; i.e., $\sigma(0°)$ is less than the minimum required by σ_t. It was decided that the most suitable solution of this dilemma would be to use σ_t to give an additional point in the forward direction for $\sigma(\theta)$; i.e., we assume that $\sigma(0°)=(k\sigma_t/4\pi)^2$. The least-squares fit is done again for this new situation to redetermine the four parameters of Eq. (6). The $\mathfrak{R}$ value with the imposition of the minimum theorem is now 1.63%. From this analysis, the total elastic cross section σ_e is 22.3 mb. Without the use of the minimum theorem, the value of σ_e is 19.8 mb and agrees with that given by Smith, McReynolds, and Snow.[1] But we feel that our estimate is more reliable. Also, $\sigma_a=25.7$ mb and $\sigma_a/\sigma_e=1.15$.

V. Phase-Shift Analysis

Below 400 Mev, the scattering and polarization experiments of protons by protons[6] can be interpreted in terms of the components of the triplet ${}^3P_{0,1,2}$ having different phase shifts with the ${}^3P_0$ dominating the other two ${}^3P_{1,2}$ states. The components of the triplet ${}^3F_{2,3,4}$ are required to have small phase shifts. We shall see that we can make a complete analysis of the 1-Bev data by making the simple assumption that the triplet states are independent of j and that no states higher than F occur.

In Eq. (3b), let

$$(2l+1)(1-e^{2i\delta l})\equiv L_l+iM_l\equiv Q_l. \tag{7}$$

In the Bev range, as we are dealing with large absorptive processes, δ_l is complex and L_l and M_l can be treated as independent quantities, which in turn determine the real and imaginary parts α_l and κ_l or δ_l. This method is especially applicable in our case, but even for purely elastic scattering some advantage is gained by proceeding in a similar way. The device[7] of L_l and M_l is of course not limited to p-p scattering and has been applied to π-p scattering.[8]

From the experimental side, we have the five data

[6] R. Thaler and J. Bengston, Phys. Rev. **94**, 679 (1954).
[7] W. Rarita, Phys. Rev. **100**, 1241(A) (1955).
[8] W. Rarita, Phys. Rev. **102**, 486 (1956).

TABLE I. Phase-shift analysis for 1 Bev. $M_2=-M_0$; $M_3=-M_1$.

$\|M_0\|$	L_0	L_2	L_1	L_3	$\|M_1\|$
0	0.3920	4.211	3.053	1.550	1.467
0.5	0.5073	4.095	3.030	1.573	1.443
1.0	0.9824	3.620	2.935	1.668	1.332

A, B, C, and D of $\sigma(\theta)$, and σ_t. The available theoretical parameters (L_l,M_l) are eight with our assumption of no j dependence. Thus even with what look like highly restrictive conditions of maximal value of l equal to three and no j dependence, we still have an excess of three free parameters. But the minimum theorem again enters the picture. We saw in Sec. IV that the requirement $\sigma(0°)=(k\sigma_t/4\pi)^2$ was the best way to resolve the experimental situation, but we shall see that this condition will impose two additional independent relations on our parameters. In all there will be but a single free parameter. In this way we will find no unique set of phase shifts, but in fact continuous sets of permissible solutions expressible in terms of a single parameter.

At the energies of our study we shall ignore Coulomb effects, or for sufficiently small angles we assume that correction has been made for the Coulomb interaction, so that δ_l or (L_l,M_l) can be considered as purely nuclear in nature.

In terms of L_l and M_l, we have

$$\sum\equiv k^2\sigma_t/\pi=(L_0+L_2)+3(L_1+L_3). \tag{8}$$

Equation (8) is Eq. (2b) rewritten in our new notation.

Further, Eq. (2a) becomes

$$4k^2\sigma(\theta)=(L_0P_0+L_2P_2)^2+(M_0P_0+M_2P_2)^2 +3(L_1P_1+L_3P_3)^2+3(M_1P_1+M_3P_3)^2. \tag{9}$$

Comparing Eq. (6) and Eq. (9), we get

$$A=X_0+\tfrac{1}{5}X_2+X_1+(3/7)X_3, \tag{10a}$$

$$B=2Y_S+2X_1+(18/7)Y_T+(2/7)X_2+(4/7)X_3, \tag{10b}$$

$$C=(24/7)Y_T+(18/35)X_2+(54/77)X_3, \tag{10c}$$

$$D=(100/77)X_3, \tag{10d}$$

where

$$X_l=L_l^2+M_l^2;\quad l=0,1,2,3, \tag{11a}$$

$$Y_S=L_0L_2+M_0M_2, \tag{11b}$$

$$Y_T=L_1L_3+M_1M_3. \tag{11c}$$

In order that $\sigma(0°)$ be fitted at the minimum, we saw in Sec. III that $\mathrm{Im}f_1(0°)=\mathrm{Im}f_3(0°)$. Also a glance at Eq. (9) shows that the terms involving M_l must be set equal to zero for $\theta=0°$. Summarizing the equations for a minimum in the forward direction, we have

$$L_0+L_2=L_1+L_3=k^2\sigma_t/4\pi, \tag{12a}$$

$$M_0+M_2=M_1+M_3=0. \tag{12b}$$

TABLE II. Phase shifts for 1 Bev.

$\|M_0\|$	α_0	κ_0	α_2	κ_2	α_1	κ_1	α_3	κ_3
0.0	0	0.2488	0	0.9229	0.8034	0.3573	−0.1315	0.1070
0.5	0.3964	0.1769	−0.2524	0.7881	0.7957	0.3660	−0.1299	0.1102
1.0	0.7766	0	−0.3136	0.5382	0.7609	0.4055	−0.1224	0.1210

It appears that we have eight equations: four in Eqs. (12) and four in Eqs. (10) to determine the eight quantities L_l and M_l. But A, B, C, and D are not independent. They are restricted by $A+B+C+D=4(L_0+L_2)^2$.

The parameter M_0 occurs in our equations only in the form M_0^2. Thus we find it convenient to use $|M_0|$ as our free parameter. Further, $-1\leq M_0\leq 1$. We have made calculations at 1 Bev for three values of $|M_0|$. Our results are entered in Table I. We used $A=17.124$; $B=40.543$; $C=21.150$, and $D=5.915$.

For a given M_0, we observe that the L_l and M_2 are unique but that M_1 and M_3 have some sign freedom. However, M_1 and M_3 must have opposite signs from Eq. (12b).

From Eq. (7) we can now calculate δ_l or α_l and κ_l. The absorptive coefficients κ_l will be unique but the refractive coefficients α_l have various possibilities. We give the simplest set of phase shifts in Table II for M_0 and M_3 positive.

If we remove the restriction of no j dependence, we have up to D states for $\sigma(\theta)$.

$$4k^2\sigma(\theta)=|Q_0P_0+Q_2P_2|^2 +3\{|Q_{01}|^2+3|Q_{11}|^2+5|Q_{12}|^2\}P_1^2 -\tfrac{1}{2}\{4|Q_{01}-Q_{21}|^2+9|Q_{11}-Q_{21}|^2\}P_2, \tag{13}$$

where $Q_{j1}=(1-e^{2i\delta_{j1}})$ and $Q_l=(2l+1)(1-e^{2i\delta_l})$ as in Eq. (7). Also we have, defining $L_{j1}+iM_{j1}=Q_{j1}$, that

$$\Sigma\equiv k^2\sigma_t/\pi=(L_0+L_2)+L_{01}+3L_{11}+5L_{21}, \tag{14}$$

$$4k^2\sigma(0°)\geq(L_0+L_2)^2+(L_{01}+2L_{21})^2 +(9/2)(L_{11}+L_{21})^2. \tag{15}$$

The minimum of the right-hand side of Eq. (15) with the restriction of Eq. (14) requires

$$L_0+L_2=L_{01}+2L_{21}=\tfrac{3}{2}(L_{11}+L_{21}), \tag{16}$$

or

$$L_{01}=\tfrac{1}{2}(3L_{11}-L_{21}).$$

We see that j independence or $L_{01}=L_{11}=L_{21}$ is a special case that gives a minimum $\sigma(0°)$ for the P-state phase shifts.

Let us consider in detail the instance when ${}^3P_0\neq 0$ and ${}^3P_1={}^3P_2=0$ and there is a single phase shift for the ${}^3F$ states, or ${}^3F_2={}^3F_3={}^3F_4\equiv{}^3F$. We get

$$4k^2\sigma(\theta)=|Q_0P_0+Q_2P_2|^2+|Q_{01}P_1+Q_3P_3|^2 +|Q_{01}|^2(1-P_1^2)+2|Q_3|^2P_3^2, \tag{17}$$

and

$$k^2\sigma_t/\pi=(L_0+L_2)+L_{01}+3L_3. \tag{18}$$

The minimum for $\sigma(0°)$ is obtained when

$$L_0+L_2=L_{01}+L_3=L_3, \quad \text{or} \quad L_{01}=0, \tag{19}$$

and is the same as in Eq. (5). For 1 Bev, we find easily that no solution at all then exists.

VI. Conclusions

A. Ray Optical Model

We are now in a position to understand the difficulties encountered with the ray optical model (Sec. II). The principal results of this model are that

$$\frac{\delta_0}{\delta_2}=\frac{\alpha_0}{\alpha_2}=\frac{\kappa_0}{\kappa_2}=\frac{[(kR)^2-(1/2)^2]^{\frac{1}{2}}}{[(kR)^2-(5/2)^2]^{\frac{1}{2}}}\equiv\frac{s_0}{s_2}>1, \tag{20a}$$

$$\frac{\delta_1}{\delta_3}=\frac{\alpha_1}{\alpha_3}=\frac{\kappa_1}{\kappa_3}=\frac{[(kR)^2-(3/2)^2]^{\frac{1}{2}}}{[(kR)^2-(7/2)^2]^{\frac{1}{2}}}\equiv\frac{s_1}{s_3}>1. \tag{20b}$$

The phase shifts δ_l are proportional to their effective path lengths s_l through the refractive and absorptive sphere of interaction. To test these requirements we concentrate on the κ_l, as the α_l are indeterminate and can be adjusted by adding factors of π. We note that the triplet states fulfill Eq. (20b) but that the singlet states are in gross disagreement with Eq. (20a). Thus this form of the optical theory of the interaction between two protons is bound to fail. Using the triplet absorptive coefficients, we get $kR\sim3.6$ to be compared to 3.1 from Eq. (1b). At the beginning of this study, a direct attack using the optical ray model led to very poor agreement. Our phase-shift analysis enables us to understand the failure.

B. Complex Square-Well Potential

Even if we assume that the singlet states ^1S and ^1D are connected through a potential and that the triplet states ^3P and ^3F are connected by a different potential, we conclude that in general no such potentials exist. The singlet states have four phase quantities α_0, κ_0, α_2, κ_2 to be determined by means of the potential which has three adjustable parameters: one is R, its range of interaction, and the other two are the real and imaginary parts of the potential. Of course, we can take general shapes for the potential and get the extra freedom we need. In this conjunction, a square well with a central core, suggested by the singlet data, is an especially attractive possibility. Further, it has the right number (four) of available parameters. The square-well complex potential (S.W.C.P.) and the homogeneous refractive and absorptive sphere of interaction (H.R.A.S.) both have three parameters but the potential seems to have a greater diversity of solutions. For instance, the ray model (H.R.A.S.) requires $\kappa_2<\kappa_0$ but the potential (S.W.C.P.) can have $\kappa_2>\kappa_0$. In fact we shall discuss in the next paragraph the condition for $\kappa_2\to\infty$, for the potential case.

In our study of the complex square-well potential, we investigated the conditions for which a given state becomes opaque, or $e^{2i\delta_l}\to0$. In the notation adopted by Kessler and Lederman,[9] we find with $x\equiv kR$ for the S, P, and D states that

$$S: z_0=1+ix, \tag{21a}$$

$$P: z_1=2-x^2/(1+ix), \tag{21b}$$

$$D: z_2=3+x^2(1+ix)/(x^2-3-3ix). \tag{21c}$$

The complex functions z are related to the wave function inside the complex potential well.

The task of obtaining a complex square-well potential to fit the given ^1S and ^1D phase shifts δ_0 and δ_2 can be easily formulated, but such a solution can exist only by accident or in a best-fit sense.

In summary, we have outlined some general methods of attacking the p-p scattering problem in the Bev range and have given detailed calculations for 1 Bev.

ACKNOWLEDGMENTS

We thank Professor R. Serber, who collaborated on the initial phases of this work, and also Professor L. Foldy, who gave many useful suggestions.

[9] J. Kessler and L. Lederman, Phys. Rev. 94, 689 (1954).