Normalization of Bethe-Salpeter Wave Functions

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The scalar product between any two bound states is expressed covariantly in terms of the Bethe-Salpeter component of the one, and the adjoint Bethe-Salpeter component of the other.

1. INTRODUCTION

VARIOUS authors[1] have suggested that the covariant integral equation which Bethe and Salpeter[2] introduced for the treatment of bound states must be supplemented by a normalization condition which, as for the nonrelativistic Schrödinger equation, will serve to distinguish the physical solutions by their normalizability. Moreover, a knowledge of normalization integrals is always essential in finding the physical probabilities predicted by the theory. Once the normalization problem is solved, the construction of correctly normalized transition amplitudes and expectation values is formally very simple, as has been shown by Mandelstam.[3]

In the present work, a general covariant expression for the scalar product of two bound states is derived by studying the connection between the Bethe-Salpeter wave functions and the appropriate propagation function.

2. AMPLITUDES AND COMPONENTS

The method will be illustrated by a bound state of two fermions. The extension to other bound systems is quite trivial, apart from the necessity of generalizing the definitions (1)–(3) in such a way that Eq. (4) and Eqs. (11)–(15) are still applicable.

This section will be largely devoted to the establishment of a formal framework within which the problem can be held.

A two-fermion bound state $|a\rangle$ is characterized in a relativistic theory by a Bethe-Salpeter *component* $\chi_a(x_1,x_2)$,

$$\chi_a(x_1,x_2)=\langle 0|\mathrm{T}\psi(x_2)\psi(x_1)|a\rangle. \qquad (1)$$

The symbol $\langle 0|$ stands here for the true vacuum state, $\psi(x)$ is a renormalized Heisenberg operator, and T denotes the usual chronological ordering with Fermi sign. The state may also be characterized by the adjoint Bethe-Salpeter component $\bar{\chi}_a(x_1,x_2)$,

$$\bar{\chi}_a(x_1,x_2)=\langle a|\mathrm{T}\bar{\psi}(x_1)\bar{\psi}(x_2)|0\rangle. \qquad (2)$$

Information on many of the properties of the two-fermion system is contained within the appropriate propagator G,

$$G(x_1,x_2;y_1,y_2)=\langle 0|\mathrm{T}\psi(x_2)\psi(x_1)\bar{\psi}(y_1)\bar{\psi}(y_2)|0\rangle. \qquad (3)$$

The containing interaction responsible for bound states is taken properly into account by noting that the exact G obeys an integral equation of the form,[4]

$$G(x_1,x_2;y_1,y_2)=G_0(x_1,x_2;y_1,y_2)+\int G_0(x_1,x_2;y_1',y_2')d^4y_1'd^4y_2'K(y_1',y_2';x_1',x_2')\times d^4x_1'd^4x_2'G(x_1',x_2';y_1,y_2). \qquad (4)$$

We may abbreviate (4) as follows:

$$G=G_0+G_0KG. \qquad (4a)$$

We also have

$$G=G_0+GKG_0. \qquad (4b)$$

In practice suitable approximations must be made for G_0 and K; and then the *same* approximations should presumably be understood to hold throughout the following, irrespective of where G_0 and K appear.

It was recognized by Nishijima[5] that the rich content of the covariant theory of propagators can only be uncovered with the help of another kind of wave function, which we shall call an *amplitude* and denote by the letter f. State vectors are to be built up by applying various operators to the true vacuum with the complex amplitudes f as weighting factors.[6] This aspect of the formalism can be greatly simplified by bringing in integrations over large but finite time intervals τ in addition to the usual space integrations of the Tamm-Dancoff theory.[7,8] Thus we shall build up our two-fermion bound state $|a\rangle$ by using the following time-

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[1] J. Goldstein, Phys. Rev. **91**, 1516 (1953); F. L. Scarf, Phys. Rev. **100**, 912 (1955); G. C. Wick, Phys. Rev. **96**, 1124 (1954).

[2] H. A. Bethe and E. E. Salpeter, Phys. Rev. **84**, 1232 (1951).

[3] S. Mandelstam, Proc. Roy. Soc. (London) **A233**, 248 (1955).

[4] As is shown in references 2 and 6.

[5] K. Nishijima, Progr. Theoret. Phys. (Japan) **10**, 549 (1953); **12**, 279 (1954).

[6] This seldom used distinction between the words "amplitude" and "component" is suggested in lieu of Nishijima's terminology of "contravariant components" and "covariant components" respectively. In general, the amplitudes are closely related to the familiar expansion of a state vector in terms of probability amplitudes and basis vectors, while the components are related to the scalar products between basis vectors and the state vector. In relativistic formulations with interaction the basis vectors are not orthogonal, so that the distinction between amplitudes and components is very necessary.

[7] Time averaging was introduced by R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950), and was used in the treatment of bound states by Gell-Mann and Low.[8]

[8] M. Gell-Mann and F. E. Low, Phys. Rev. **84**, 350 (1951).

averaged operators, with weighting amplitude $f_a(y_1,y_2)$,

$$|a\rangle=\frac{1}{2\tau_a}\int_{t_a-\tau_a}^{t_a+\tau_a} d^4y_1d^4y_2\{\mathrm{T}\bar{\psi}(y_1)\bar{\psi}(y_2)\}f_a(y_1,y_2)\ |0\rangle. \quad (5)$$

The amplitude $f_a(y_1,y_2)$ must satisfy certain simple conditions in order that (5) shall represent a bound state. These conditions are connected with the translational invariance of the theory, and may therefore be expressed succinctly in terms of center coordinates Y^μ and relative coordinates y^μ. For the present problem we define

$$Y^\mu=(\alpha_1y_1^\mu+\alpha_2y_2^\mu)/(\alpha_1+\alpha_2), \quad (6)$$

$$y^\mu=y_1^\mu-y_2^\mu, \quad (7)$$

so that Y^μ carries all the translational dependence. Here α_1 and α_2 are real numbers such that $\alpha_1+\alpha_2\neq 0$. Their choice is a matter of convenience. We require, first and foremost, that $f_a(y_1,y_2)$ is a positive energy solution of the following Klein-Gordon equation,

$$(\partial^2/\partial Y^\mu\partial Y_\mu+M_a^2)f_a(y_1,y_2)=0, \quad (8)$$

where M_a is the rest mass eigenvalue of the bound state $|a\rangle$. The time averaging in Eq. (5) will then automatically ensure that only states with the correct rest mass appear in (5), and that (5) is independent of the time t_a about which the integrations are carried out. It is of course necessary that τ_a be sufficiently long to reject to a high accuracy all contributions from neighboring states. This is simply a manifestation of the uncertainty relation between energy and time.

Another important requirement is set by the existence of rotation, inversion, and other invariance groups. These are liable to lead to degenerate mass levels in which case the correct degenerate state must be picked out by a suitable choice for the dependence of $f_a(y_1,y_2)$ on the internal coordinates y. Accidental degeneracy will entail similar conditions.

Apart from the aforementioned requirements, the dependence of $f_a(y_1,y_2)$ on y does not matter. The precise form taken by the time averaging is also irrelevant, provided the time used is sufficiently long. Thus it will be seen that the amplitudes f are far too arbitrary to be taken as a suitable basis for the final formulas of the theory. They must ultimately be eliminated in favor of the components χ. In the initial formulation of the problem, however, the introduction of amplitudes f is very desirable, since one cannot easily discuss normalization integrals without an explicit representation of the state vectors concerned. The present way of using the f is particularly well adapted to their eventual elimination from the theory.

3. EXPRESSION OF SCALAR PRODUCTS AND COMPONENTS IN TERMS OF AMPLITUDES

If we take the Hermitian conjugate of Eq. (5), the chronological product changes into an antichronological product. This rather awkward feature may be overcome here by stipulating that $f_a(y_1,y_2)$ shall be zero for time-like relative coordinates.

$$f_a(y_1,y_2)=0 \quad \text{when} \quad (y_1^\mu-y_2^\mu)(y_{1\mu}-y_{2\mu})\geq 0. \quad (9)$$

With this proviso (which in no way curtails the general suitability of the formulation), conjugation of Eq. (5) gives

$$\langle a|=\frac{1}{2\tau_a}\int_{t_a-\tau_a}^{t_a+\tau_a} d^4y_1d^4y_2f_a(y_1,y_2)^* \times\beta_{(1)}\beta_{(2)}\langle 0|\mathrm{T}\psi(y_2)\psi(y_1). \quad (10)$$

The scalar product $\langle b|a\rangle$ is now easily expressed in terms of G, f_b^*, and f_a. Taking $t_b-\tau_b>t_a+\tau_a$, we have, using Eqs. (3), (5), and (10),

$$\langle b|a\rangle=\frac{1}{2\tau_b}\int_{t_b-\tau_b}^{t_b+\tau_b} d^4x_1d^4x_2\frac{1}{2\tau_a}\int_{t_a-\tau_a}^{t_a+\tau_a} d^4y_1d^4y_2f_b(x_1,x_2)^* \times\beta_{(1)}\beta_{(2)}G(x_1,x_2;y_1,y_2)f_a(y_1,y_2). \quad (11)$$

In a similar manner, we have

$$\chi_a(x_1,x_2)=\lim_{t_a\to-\infty}\frac{1}{2\tau_a}\int_{t_a-\tau_a}^{t_a+\tau_a} G(x_1,x_2;y_1,y_2) \times d^4y_1d^4y_2f_a(y_1,y_2), \quad (12)$$

and

$$\bar{\chi}_b(y_1,y_2)=\lim_{t_b\to+\infty}\frac{1}{2\tau_b}\int_{t_b-\tau_b}^{t_b+\tau_b} f_b(x_1,x_2)^* \times\beta_{(1)}\beta_{(2)}d^4x_1d^4x_2G(x_1,x_2;y_1,y_2). \quad (13)$$

The limits $t_a\to-\infty$, $t_b\to+\infty$ are taken so that Eqs. (12) and (13) will be valid for *all* values of x_1, x_2, and y_1, y_2, respectively. Otherwise they would be valid only for x_1^0, $x_2^0>t_a+\tau_a$, and y_1^0, $y_2^0<t_b-\tau_b$.

If we insert for G in Eq. (12) the right hand side of Eq. (4a) and take note that G_0 has no asymptotic behavior characteristic of a rest mass M_a, we obtain

$$\chi_a=G_0K\chi_a, \quad (14)$$

and similarly,

$$\bar{\chi}_a=\bar{\chi}_aKG_0. \quad (15)$$

All four-fold integrations in Eqs. (14) and (15) extend over the whole of space time.

The above proof of the Bethe-Salpeter equations rests on essentially the same ideas as those used by Gell-Mann and Low,[8] although in appearance it is rather different. It is to be noted that we are interested only in the positive energy solutions of (14) and the negative energy solutions of (15).

With these preliminaries settled, the nature of our task becomes clear. We must use the information contained in Eqs. (4), (12), (13), (14), and (15) to eliminate f_b^* and f_a from the normalization integral (11).

At this stage it must be admitted that it is rather unsatisfactory that no explicit expression of the relationship between $\bar{\chi}_b$ and χ_b has yet been discovered. For practical applications, however, we may take note that $\bar{\chi}_b(x_1,x_2)$ is that solution of Eq. (15) which reduces, for spacelike x_1-x_2, to $\chi_b(x_1,x_2)^*\beta_{(1)}\beta_{(2)}$. This prescription is expected to lead to a unique relation between $\bar{\chi}$ and χ, although we cannot be quite sure until a more explicit treatment is available.[9] This temporary inadequacy of the formulation will not however hamper our further investigation of $\langle b|a\rangle$; we shall just take $|b\rangle$ to be characterized by $\bar{\chi}_b$ rather than by χ_b.

Since $\langle b|a\rangle$ is obviously zero for $M_b\neq M_a$, we shall in the following take $M_b=M_a$, $=M$ say.

4. ASYMPTOTIC BEHAVIOR OF THE PROPAGATOR

Referring to Eq. (8), we note that the time-averaging over X and Y in the scalar product (11) will pick out from G only those positive-energy Fourier components in the very immediate neighborhood of the mass eigenvalue M. This suggests that we Fourier-analyze the dependence of G on the center coordinates X and Y and investigate this neighborhood closely. Thus, with

$$G(x_1,x_2;y_1,y_2)=(2\pi)^{-4}\int d^4p G(p,x,y)\exp[-ip(X-Y)], \quad (16)$$

and with similar Fourier expansions for $G_0(x_1,x_2;y_1,y_2)$ and $K(x_1,x_2;y_1,y_2)$, Eq. (4) becomes,

$$G(p,x,y)=G_0(p,x,y)+\int G_0(p,x,y')d^4y'K(p,y',x')d^4x'G(p,x',y), \quad (17)$$

which we write for short as

$$G(p)=G_0(p)+G_0(p)K(p)G(p). \quad (17a)$$

We also have

$$G(p)=G_0(p)+G(p)K(p)G_0(p). \quad (17b)$$

The solutions of the Bethe-Salpeter equations (14) and (15) may be similarly analyzed. Thus if P^μ is any positive-energy vector such that

$$P^\mu P_\mu=M^2, \quad (18)$$

then the functions $u_\alpha(P,x)\exp(-iPX)$ and $\bar{v}_\beta(P,y)\times\exp(iPY)$ will be, respectively, solutions of Eqs. (14) and (15) provided that u and $\bar{v}$ satisfy the following equations,

$$u_\alpha(P,x)=\int G_0(P,x,y)d^4yK(P,y,x')d^4x'u_\alpha(P,x'), \quad (19)$$

$$\bar{v}_\beta(P,y)=\int \bar{v}_\beta(P,y')d^4y'K(P,y',x)d^4xG_0(P,x,y). \quad (20)$$

[9] The recipe given by Mandelstam[3] is not sufficient to determine $\bar{\chi}$ from χ uniquely.

We must now determine the behavior of $G(p,x,y)$ in the immediate neighborhood of points $p=P$. Since G will have a pole for $p=P$, we shall carry out the analysis in terms of a new function $g(p,x,y)$ which is regular across the pole,

$$g(p,x,y)\equiv(p^2-M^2)G(p,x,y). \quad (21)$$

Evidently the normalization of states will be determined entirely by $g(P,x,y)$.

Now for $g(p,x,y)$ we have the equation

$$g(p)=(p^2-M^2)G_0(p)+G_0(p)K(p)g(p). \quad (22)$$

(Here and in much of the following, the internal coordinates, and the four-dimensional integrations concerned with them, are suppressed for the sake of brevity. To obtain the full formulas, the arguments $x\cdots$, volume elements $d^4x\cdots$, and integration signs should be inserted.)

When $p=P$, Eq. (22) becomes homogeneous. Thus if $u_\alpha(P,x)$, $\alpha=1, 2, \cdots, n$ and $\bar{v}_\beta(P,y)$, $\beta=1, 1, \cdots, n$ are respectively complete sets of linearly independent solutions of (19) and (20) (n denotes the degeneracy of the the states of mass M and is usually determined from consideration of the invariance groups), we shall have

$$g(P,x,y)=\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}u_\alpha(P,x)\lambda_{\alpha\beta}(P)\bar{v}_\beta(P,y), \quad (23)$$

where the numbers $\lambda_{\alpha\beta}(P)$ are not, for the moment, determined. We now try to solve Eq. (22) in the neighborhood of the pole, i.e., for points $p^\mu=P^\mu(1+\epsilon)$ with ϵ small. Inserting the ansatz

$$g((P(1+\epsilon),x,y)=\sum_\alpha\sum_b u_\alpha(P,x)\lambda_{\alpha\beta}(P)\bar{v}_\beta(P,y)+\epsilon h(P,x,y)+O(\epsilon^2), \quad (24)$$

and taking only the terms linear in ϵ, we find for $h(P,x,y)$ the following equation,

$$h(P)=2M^2G_0(P)+G_0(P)K(P)h(P)+\{G_0'(P)K(P)+G_0(P)K'(P)\}\times\sum_\alpha\sum_\beta u_\alpha(P)\lambda_{\alpha\beta}(P)\bar{v}_\beta(P), \quad (25)$$

where

$$G_0'(P,x,y)=\lim_{\epsilon\to 0}\epsilon^{-1}\{G_0(P(1+\epsilon),x,y)-G_0(P,x,y)\}, \text{ etc.} \quad (26)$$

Equation (25) is analogous to a nonhomogeneous linear equation with zero determinant, and therefore serves to give information not only on $h(P)$ but also on the inhomogeneous term [i.e., on the numbers $\lambda_{\alpha\beta}(P)$]. In fact, if we multiply on the left by $\int\bar{v}_\gamma(P,x')d^4x'K\times(P,x',x)$ and integrate over x, the terms involving h cancel out altogether, by virtue of Eq. (20), and we are left with

$$\sum_\alpha\sum_\beta[\bar{v}_\gamma(P)\{K'(P)+K(P)G_0'(P)K(P)\}u_\alpha(P)]\times\lambda_{\alpha\beta}(P)\bar{v}_\beta(P)=-2M^2\bar{v}_\gamma(P). \quad (27)$$

Equation (27) is simply an $n\times n$ matrix equation for the $n\times n$ matrix $\lambda_{\alpha\beta}(P)$, since the end factors $\bar{v}(P)$ appear on both sides. [The necessary and sufficient condition that Eq. (25) should have a solution $h(P)$ is evidently that (27) should be obeyed with γ (and hence also α and β) running over complete sets of solutions of the homogeneous equations; i.e., the degeneracy must be fully explored.]

The asymptotic behavior of the propagator $G\times(x, X-Y, y)$ as X is made very much later than Y depends only on the immediate neighborhood of the singularities in its Fourier transform $G(p,x,y)$, the integrations over d^4p canceling out elsewhere by interference. The singularities in $G(p,x,y)$ must be circumnavigated in Feynman's well-known manner. Thus, when X is very far to the future of Y the part of $G(x, X-Y, y)$ descriptive of the bound states is given by the following expression [using Eqs. (16), (21), and (23)],

$$G(x, X-Y, y)\sim\frac{-2\pi i}{(2\pi)^4}\int\frac{d^3P}{2P^0}\sum_\alpha\sum_\beta u_\alpha(P,x)\lambda_{\alpha\beta}\times(P)\bar{v}_\beta(P,y)e^{-iP(X-Y)}, \quad (28)$$

with the $\lambda_{\alpha\beta}(P)$ completely determined by Eq. (27).

5. EXPRESSION OF SCALAR PRODUCTS IN TERMS OF COMPONENTS

We Fourier-analyze $\chi_a(x_1,x_2)$ and $\bar{\chi}_b(y_1,y_2)$ as follows:

$$\chi_a(x_1,x_2)=\frac{1}{(2\pi)^3}\int\frac{d^3P}{2P^0}\chi_a(P,x)e^{-iPX}, \quad (29)$$

$$\bar{\chi}_b(y_1,y_2)=\frac{1}{(2\pi)^3}\int\frac{d^3P}{2P^0}\bar{\chi}_b(P,y)e^{iPY}. \quad (30)$$

The scalar product will be best expressed in terms of $\chi_a(P,x)$ and $\bar{\chi}_b(P,y)$. These functions must be linear combinations of the basic solutions $u_\alpha(P,x)$ and $\bar{v}_\beta(P,y)$, thus,

$$\chi_a(P,x)\equiv\textstyle\sum_\alpha a_\alpha(P)u_\alpha(P,x), \quad (31)$$

$$\bar{\chi}_b(P,y)\equiv\textstyle\sum_\alpha \bar{b}_\alpha(P)\bar{v}_\alpha(P,y). \quad (32)$$

The coefficients a_α and $\bar{b}_\alpha$ can be explicitly determined in terms of the amplitudes $f_a(x_1,x_2)$ and $f_b(y_1,y_2)^*$ by inserting the bound state part of the asymptotic formula for G [Eq. (28)] into Eqs. (12) and (13). The use of the asymptotic formula instead of the exact propagator is to be justified by noting that the integrations in Eqs. (12) and (13) are taken in the far past and far future, respectively. In this way we obtain

$$a_\alpha(P)=\sum_\beta\lambda_{\alpha\beta}(P)\times\left\{\frac{-i}{2\tau_a}\int_{t_a-\tau_a}^{t_a+\tau_a}\bar{v}_\beta(P,y)e^{iPY}f_a(y_1,y_2)d^4y_1d^4y_2\right\}, \quad (33)$$

and

$$\bar{b}_\alpha(P)=\sum_\beta\left\{\frac{-i}{2\tau_b}\int_{t_b-\tau_b}^{t_b+\tau_b}f_b(x_1,x_2)^*\beta_{(1)}\beta_{(2)}e^{-iPX}u_\beta(P,x)d^4x_1d^4x_2\right\}\times\lambda_{\beta\alpha}(P). \quad (34)$$

In these formulas t_a and t_b are once again arbitrary, as in Eqs. (5) and (10).

We now proceed to evaluate the scalar product $\langle b|a\rangle$, given by Eq. (11). Here again, the use of the asymptotic propagator is justified since t_b may be taken into the far future, and t_a into the far past. In this way, from Eqs. (11), (28), (33), and (34), we find

$$\langle b|a\rangle=\sum_{\alpha,\beta}\frac{i}{(2\pi)^3}\int\frac{d^3P}{2P^0}\bar{b}_\alpha(P)[\lambda(P)^{-1}]_{\alpha\beta}a_\beta(P), \quad (35)$$

where $[\lambda(P)^{-1}]_{\alpha\beta}$ is the $n\times n$ matrix reciprocal to $\lambda_{\alpha\beta}(P)$; that is, from Eq. (27),

$$[\lambda(P)^{-1}]_{\alpha\beta}=-(2M^2)^{-1}\bar{v}_\alpha(P)\times\{K'(P)+K(P)G_0'(P)K(P)\}u_\beta(P). \quad (36)$$

Inserting (36) into (35) and using (31) and (32), we obtain the desired formula,

$$\langle b|a\rangle=\frac{1}{2M^2}\frac{-i}{(2\pi)^3}\int\frac{d^3P}{2P^0}\bar{\chi}_\beta(P)\times\{K'(P)+K(P)G_0'(P)K(P)\}\chi_a(P). \quad (37)$$

In (37), integrations over the relative space time coordinates, of which there will be two sets in the first term and four in the second, is understood. The primes on K and G_0 denote a differentiation across the mass hyperboloid, as specified by Eq. (26), and M is the rest mass of the bound state.

The expression (37) is applicable to any kind of bound state, since its derivation depends only on translational and Lorentz invariance.