Integral Equations for the Transition Matrices in the Static Meson Theory*

NOBUYUKI FUKUDA† AND JULIUS S. KOVACS‡
Department of Physics, Indiana University, Bloomington, Indiana
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The Chew-Low-Wick integral equation for the scattering matrix in the static meson theory has been generalized so as to make it possible to treat the various pion processes in which arbitrary numbers of pions are involved. Such a generalization is also necessitated if one wants to take into account the contribution of two- or more-meson configurations in pion-nucleon scattering. The outgoing or incoming wave functions corresponding to the many-meson initial states are defined in the same manner as by the above-mentioned authors in the one-meson problem, and are shown to be identical with those introduced by Lippmann and Schwinger. An approximate expression for the two-meson production matrix is obtained and some correction terms due to this production are derived for the Chew-Low one-meson equation.

1. INTRODUCTION

THE success of the Chew-Low[1] theory for the P-wave pion-nucleon interaction in the one-meson approximation[1–3] is well known in its application to the scattering[1] and photoproduction[2] of pions. With some accuracy the coupling constant[4] was determined and the value of the momentum cutoff was obtained so that the theory and low-energy experiments are in good agreement. In order that we can further verify this agreement, the contribution to these phenomena of the higher order configurations ought to be investigated. It is the purpose of this paper to generalize the integral equations for the transition matrices and, in so doing, to obtain correction terms to the Chew-Low one-meson equation due to the two-meson configuration.

In Sec. 2 we obtain the outgoing and incoming wave functions corresponding to the one-nucleon many-meson initial states and show these to be identical with those introduced by Lippmann and Schwinger.[5] In Sec. 3 the transition matrices for the many-meson processes are obtained and the integral equations that they satisfy are derived. In Sec. 4 the generalized Tamm-Dancoff method is introduced which does not violate the unitarity condition and the crossing theorem. The integral equations are specified in the two-meson approximation, neglecting all higher order contributions, and the reasonable expression for the two-meson production matrix is obtained. Making use of this expression, the corrections to the Chew-Low one-meson approximation are derived in Sec. 5. The numerical evaluation of these corrections is left for a later paper.

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‡ Now at Michigan State University, East Lansing, Michigan.

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[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).

[3] G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).

[4] See, however, M. Cini and S. Fubini, Nuovo cimento **3**, 764 (1956) and W. Thirring in *Proceedings of the Sixth Annual Rochester Conference on High Energy Physics, 1956* [Interscience Publishers, Inc., New York (to be published)]. M. Cini and S. Fubini, Phys. Rev. **102**, 1687 (1956).

[5] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

2. OUTGOING AND INCOMING WAVE SOLUTIONS FOR THE MANY-MESON STATES

Let us assume that n pions with momenta and isotopic spins $k_1, k_2, \cdots, k_n$ are incident on a fixed nucleon. The stationary outgoing or incoming solutions $\Psi^{(\pm)}(k_1, \cdots k_n)$ (abbreviated as $\Psi_n^{(\pm)}$) corresponding to this initial condition are expected to satisfy, in the static model, the Schrödinger equation

$$H\Psi_n^{(\pm)}=(\omega_1+\omega_2+\cdots+\omega_n)\Psi_n^{(\pm)}, \tag{2.1}$$

where H is the total Hamiltonian for the pion-nucleon system,

$$\begin{gathered} H=H_0+H_1-\Delta E, \\ H_0=\textstyle\sum_k\omega_k a^*(k)a(k), \quad H_1=\sum_k[V_k a(k)+V_k^\dagger a^*(k)], \\ V_k=if^{(0)}\dfrac{(\boldsymbol{\sigma}\cdot\mathbf{k})}{(2\omega_k)^{\frac{1}{2}}}\tau_k v(k), \end{gathered} \tag{2.2}$$

defined as in reference 1,[6] and ΔE is the self-energy of the nucleon. We assume that $\Psi_n^{(\pm)}$ will have the form

$$\Psi_n^{(\pm)}=\frac{1}{(n!)^{\frac{1}{2}}}a^*(k_1)\cdots a^*(k_n)\Psi_0+\chi_n^{(\pm)}, \tag{2.3}$$

where Ψ_0 represents the physical nucleon state. Inserting Eq. (2.3) into Eq. (2.1) and noting that

$$H\Psi_0=0, \tag{2.4}$$

we obtain

$$\begin{gathered} (H-E_n)\chi_n^{(\pm)} \\ =-\frac{1}{(n!)^{\frac{1}{2}}}\sum_{i=1}^{n}V_{k_i}a^*(k_1)\cdots{}^{(\prime)}\cdots a^*(k_n)\Psi_0, \\ E_n=\sum_{i=1}^{n}\omega_i, \end{gathered} \tag{2.5}$$

where the symbol ($'$) means that $a^*(k_i)$ is omitted in the product. It is natural, therefore, to define the

[6] The units $\hbar=c=\mu$ (pion mass) $=1$ are used throughout this work.

singularity in $\Psi_n^{(\pm)}$ as follows:

$$\chi_n^{(\pm)}=-\frac{1}{H-E_n\mp i\epsilon}\frac{1}{(n!)^{\frac{1}{2}}}\times\sum_{i=1}^{n}V_{k_i}a^*(k_1)\cdots{}^{(\prime)}\cdots a^*(k_n)\Psi_0, \quad (2.6)$$

where ϵ is a positive infinitesimal.

Next, we shall show that the solution (2.3) is identical with that defined by Møller[7] and Lippmann-Schwinger.[5] These authors defined $\Psi_n^{(\pm)}$ by

$$\Psi_n^{(\pm)}=\Omega^{(\pm)}\Phi_n. \quad (2.7)$$

$\Omega^{(\pm)}$ are conventional wave matrices and Φ_n is given by

$$\Phi_n=\frac{1}{(n!)^{\frac{1}{2}}}a^*(k_1)\cdots a^*(k_n)\Phi_0, \quad (2.8)$$

where Φ_0 is the bare-nucleon state. The $\Omega^{(\pm)}$ are given by the transformation matrix in the interaction representation as follows:

$$\Omega^{(+)}=U(0,-\infty),\quad \Omega^{(-)}=U(0,\infty),$$
$$i\partial U(t,t_0)/\partial t=H_1(t)U(t,t_0),\quad U(t_0,t_0)=1, \quad (2.9)$$
$$H_1(t)=e^{iH_0t}H_1e^{-iH_0t}=\sum_k[V_ka(k)e^{-i\omega_kt}+V_k^\dagger a^*(k)e^{i\omega_kt}].$$

Making use of the well-known power series expansion of the solution of Eq. (2.9) and noting $\Psi_0=\Omega^{(\pm)}\Phi_0$, we have

$$\Omega^{(+)}\Phi_n=\frac{1}{(n!)^{\frac{1}{2}}}a^*(k_1)\cdots a^*(k_n)\Psi_0+\frac{1}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}dt_1\cdots dt_m\times[P\{H_1(t_1)\cdots H_1(t_m)\},a^*(k_1)\cdots a^*(k_n)]\Psi_0. \quad (2.10)$$

It is easily shown that

$$\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}dt_1\cdots dt_m\times[P\{H_1(t_1)\cdots H_1(t_m)\},a^*(k_1)\cdots a^*(k_n)]$$
$$=(-i)\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}dt_1\cdots dt_mdtP\{H_1(t_1)\cdots\times H_1(t_m)[\sum_{i=1}^{n}V_{k_i}e^{-i\omega_it}a^*(k_1)\cdots{}^{(\prime)}\cdots a^*(k_n)]\}$$
$$=(-i)\int_{-\infty}^{0}dtU(0,t)\sum_{i=1}^{n}V_{k_i}e^{-i\omega_it}\times a^*(k_1)\cdots{}^{(\prime)}\cdots a^*(k_n)U(t,-\infty). \quad (2.11)$$

[7] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945).

We may write

$$U(t,-\infty)=U(t,0)U(0,-\infty), \quad (2.12)$$

and put

$$U(t,0)=e^{iH_0t}e^{-iHt},\quad U(0,t)=e^{iHt}e^{-iH_0t}. \quad (2.13)$$

Substituting Eq. (2.11) into (2.10), noting Eq. (2.4) and

$$e^{-iH_0t}a^*(k)e^{iH_0t}=a^*(k)e^{-i\omega_kt}, \quad (2.14)$$

we finally obtain

$$\Omega^{(+)}\Phi_n=\frac{1}{(n!)^{\frac{1}{2}}}a^*(k_1)\cdots a^*(k_n)\Psi_0-\frac{1}{H-E_n-i\epsilon}\frac{1}{(n!)^{\frac{1}{2}}}\times\sum_i V_{k_i}a^*(k_1)\cdots{}^{(\prime)}\cdots a^*(k_n)\Psi_0. \quad (2.15)$$

The same is true for $\Omega^{(-)}\Phi_n$, except that $-i\epsilon$ is replaced by $+i\epsilon$. Thus it is shown that the outgoing solution defined by Eqs. (2.3) and (2.6) is identical with that defined by Møller and Lippmann-Schwinger,[8] from which it follows that these solutions will form a complete set if there are no bound states of the pion-nucleon system.

3. TRANSITION MATRICES INVOLVING MANY MESONS AND THE INTEGRAL EQUATIONS FOR THESE MATRICES

Having shown that the $\Psi_n^{(\pm)}$ are actually the outgoing and incoming wave solutions, we can define as the transition matrix $T_{k_1\cdots k_n}(l_1\cdots l_m)$ [abbreviated as $T_n(m)$] for going from the initial state $(k_1,k_2,\cdots,k_n)$ to the final state $(l_1,l_2,\cdots,l_m)$, the quantity

$$T_n(m)=\frac{1}{(n!)^{\frac{1}{2}}}(\Psi_m^{(-)},\sum_{i=1}^{n}V_{k_i}a^*(k_1)\cdots{}^{(\prime)}\cdots a^*(k_n)\Psi_0). \quad (3.1)$$

That this transition matrix is equal to the conventional one on the energy shell is shown as follows.

The S matrix is defined by

$$\langle m|S|n\rangle=(\Phi_m,\Omega^{(-)\dagger}\Omega^{(+)}\Phi_n)=(\Psi_n^{(-)},\Psi_n^{(+)})$$
$$=\left(\Psi_m^{(-)},\left\{\Psi_m^{(-)}+\left[\frac{1}{H-E_n+i\epsilon}-\frac{1}{H-E_n-i\epsilon}\right]\times\frac{1}{(n!)^{\frac{1}{2}}}\sum_{i=1}^{n}V_{k_i}a^*(k_1)\cdots{}^{(\prime)}\cdots a^*(k_n)\Psi_0\right\}\right) \quad (3.2)$$
$$=\delta_{mn}-2\pi i\delta(E_m-E_n)T_n(m).$$

[8] This demonstration is in marked contrast with that given by Wick[3] in the one-meson solution, where he considers the wave packet as the incident meson wave. It seems much too complicated, however, to generalize his way of approach to the many-meson case.

This shows that, for $E_n = E_m$,

$$T_n(m) = (\Psi_m^{(-)}, H_1\Phi_n), \tag{3.3}$$

where the right-hand side is that defined by Lippmann and Schwinger.[5] From the relation (3.2), it follows that the unitarity condition $SS^\dagger = 1$ for the S matrix is equivalent to the following statement:

$$T_n^\dagger(m) - T_m(n) = 2\pi i \textstyle\sum_r \delta(E_r - E_n) T_n^\dagger(r) T_m(r), \tag{3.4}$$

where $E_n = E_m = E_r$. Here the symbol $\dagger$ means the Hermitean conjugate.

We are now in a position to set up the integral equations to connect various transition matrices. For this, however, we have to introduce the following auxiliary matrices:

$$S_n(m) = \frac{1}{(n!)^{\frac{1}{2}}} (\Psi_m^{(-)}, \sum_{i=1}^{n} V_{k_i}{}^\dagger a(k_1) \cdots {}^{(\prime)} \cdots a(k_i) \Psi_0). \tag{3.5}$$

Substituting Eq. (2.3) for $\Psi_m^{(-)}$ into Eq. (3.5) and making use of the relation

$$a(l_1) \cdots a(l_m) \Psi_0 = -\frac{1}{H + E_m} \sum_{i=1}^{m} V_{l_j}{}^\dagger a(l_1) \cdots {}^{(\prime)} \cdots a(l_m) \Psi_0, \tag{3.6}$$

(which is easily verified by operating with H on the left-hand side), we get

$$\begin{aligned} S_n(m) = {} & \frac{1}{(m!n!)^{\frac{1}{2}}} (\Psi_0, a(l_1) \cdots a(l_m) \sum_{i=1}^{n} V_{k_i}{}^\dagger a(k_1) \cdots {}^{(\prime)} \cdots a(k_n) \Psi_0) \\ & - \frac{1}{(m!n!)^{\frac{1}{2}}} \left(\Psi_0, \sum_{j=1}^{m} V_{l_j}{}^\dagger a(l_1) \cdots {}^{(\prime)} \cdots a(l_m) \frac{1}{H - E_m - i\epsilon} \sum_{i=1}^{n} V_{k_i}{}^\dagger a(k_1) \cdots {}^{(\prime)} \cdots a(k_n) \Psi_0 \right) \\ & = -\sum_r \frac{T_n^\dagger(r) S_m(r)}{E_r + E_m} - \sum_r \frac{T_m^\dagger(r) S_n(r)}{E_r - E_m - i\epsilon}. \end{aligned} \tag{3.7}$$

Here, $T_n(m)$ and $S_n(m)$ are taken to be operators in the spin and isotopic spin space of the nucleon.

In the same way, we obtain

$$\begin{aligned} T_n(m) = {} & \frac{1}{(m!n!)^{\frac{1}{2}}} (\Psi_0, a(l_1) \cdots a(l_m) \sum_{i=1}^{n} V_{k_i} a^*(k_1) \cdots {}^{(\prime)} \cdots a^*(k_n) \Psi_0) \\ & - \frac{1}{(m!n!)^{\frac{1}{2}}} \left(\Psi_0, \sum_{j=1}^{m} V_{l_j}{}^\dagger a(l_1) \cdots {}^{(\prime)} \cdots a(l_m) \frac{1}{H - E_m - i\epsilon} \sum_{i=1}^{n} V_{k_i} a^*(k_1) \cdots {}^{(\prime)} \cdots a^*(k_n) \Psi_0 \right) \\ = {} & -\sum_r \frac{S_n^\dagger(r) S_m(r)}{E_r + E_m} - \sum_r \frac{T_m^\dagger(r) T_n(r)}{E_r - E_m - i\epsilon} + \frac{1}{(n!m!)^{\frac{1}{2}}} (\Psi_0, \sum_{i=1}^{n} V_{k_i} [a(l_1) \cdots a(l_m), a^*(k_1) \cdots {}^{(\prime)} \cdots a^*(k_n)] \Psi_0). \end{aligned} \tag{3.8}$$

The last term of this equation can be reduced further and is a function of $S_p(r)$ $(p < n, p < m)$. For instance,

$$\begin{aligned} & \frac{1}{(m!n!)^{\frac{1}{2}}} (\Psi_0, \sum_{i=1}^{n} V_{k_i} [a(l_1) \cdots a(l_m), a^*(k_1) \cdots {}^{(\prime)} \cdots a^*(k_n)] \Psi_0) \\ & = -\frac{1}{(mn)^{\frac{1}{2}}} \sum_{q=1}^{n} \sum_{p=1}^{m} \delta(l_p, k_q) \sum_r \frac{S_{n-1}^\dagger(r) S_{m-1}(r)}{E_r + E_{m-1}} + \frac{1}{(m!n!)^{\frac{1}{2}}} \sum_{q=1}^{n} \sum_{p=1}^{m} \delta(l_p, k_q) \\ & \qquad \times (\Psi_0, \sum_{i \neq q} V_{k_i} [a(l_1) \cdots {}^{(\prime)} \cdots a(l_m), a^*(k_1) \cdots {}^{(\prime\prime)} \cdots a^*(k_n)] \Psi_0). \end{aligned} \tag{3.9}$$

Here the initial meson k_q is omitted in $S_{n-1}^\dagger(r)$, and the final meson l_p in $S_{m-1}(r)$. The symbol $('')$ in the last term means that k_i and k_q are omitted in the product. The complete reduction of this expression will be given in Appendix A. The set of integral Eqs. (3.7) and (3.8) are just enough to determine, in principle, the various transition matrices. It should be noted that Eq. (3.9) gives zero when the initial state is the one-meson state, or when any state of the final mesons is different from that of the initial ones. Thus the last term in Eq. (3.8) gives no contribution to the cross section; it will play, however, a predominant role in some cases through virtual states, as is seen in the following section.

Since

$$V_k^\dagger = -V_k, \tag{3.10}$$

it is obvious from the definition that

$$S_1(r) = -T_1(r). \tag{3.11}$$

For the special case $m, n=1$, using the foregoing relation, the ordinary Chew-Low-Wick equation follows from Eq. (3.7) or (3.8). Note that

$$T_n(0) = S_n{}^\dagger(0) = -\frac{1}{n^{\frac{1}{2}}} \sum_{i=1}^{n} \frac{S_{n-1}{}^\dagger(r) T_{k_i}(r)}{E_r + E_{n-1}}. \tag{3.12}$$

Here the initial meson k_i is omitted in $S_{n-1}{}^\dagger(r)$.

The physical meaning of $S_n(m)$ is not quite clear from the definition (3.5), though its introduction seems to be absolutely necessary for setting up the complete integral equations to determine the transition matrices.[9] There may be a close correspondence, however, between these quantities as indicated, for instance, by Eqs. (3.11) and (3.12). If we define the analytic function $s_{nm}(z)$ which is derived from Eq. (3.7) by replacing $E_m + i\epsilon$ with z, it satisfies the crossing theorem

$$s_{nm}(z) = s_{mn}(-z). \tag{3.13}$$

On the other hand, the transition matrix $T_n(m)$ which satisfies Eq. (3.8) automatically satisfies the unitarity condition (3.4), since it is shown in Appendix A that the last term of Eq. (3.8) turns out to be Hermitian on the energy shell.

4. GENERALIZED TAMM-DANCOFF APPROXIMATION

In order to replace an infinite set of the integral Eqs. (3.7) and (3.8) by a finite set, it seems natural to introduce the following approximation methods, generalizing the conventional Tamm-Dancoff method. That is, let us assume that

$$S_n(m) = 0, \quad T_n(m) = 0, \quad \text{for} \quad m > p, \quad n > p, \tag{4.1}$$

where p is a given positive integer. The Chew-Low one-meson approximation is a special case for $p=1$, so this approximation may be called the p-meson approximation.

The generalized Tamm-Dancoff method has the following advantages over the conventional one:

(1) The renormalization has been automatically performed in this scheme compared to the ordinary treatment in the Fock space; the unrenormalized coupling constant disappears completely in the result.

(2) The crossing theorem for $S_n(m)$ and the unitarity condition for $T_n(m)$ are always exactly satisfied, as is easily seen from Eqs. (3.7) and (3.8) cut off at $r=p$. In the conventional Tamm-Dancoff method the unitarity condition is satisfied, but the crossing theorem is usually violated even in the scattering process.

(3) The more the number of mesons involved in the process, the higher are the powers of the renormalized coupling constant, which is believed much smaller than the unrenormalized one, in the transition matrix. In addition, the higher the energy of the incident meson becomes, the more effective is the cut-off function acting at each vertex of the incident or outgoing mesons. Therefore, the generalized Tamm-Dancoff method seems to be very powerful in the static meson theory.

Now, especially in the two-meson approximation, we shall have seven coupled integral equations for $T_2(0)$, $T_1(1)$, $T_1(2)$, $T_2(1)$, $S_2(1)$, $T_2(2)$, and $S_2(2)$ from Eqs. (3.7) and (3.8). We shall write down among them the following four for later discussion:

$$T_q(p) = \frac{1}{\omega_p}[V_p{}^\dagger, V_q] - \sum_{r=1,2}\left[\frac{T_q{}^\dagger(r)T_p(r)}{E_r+\omega_p} + \frac{T_p{}^\dagger(r)T_q(r)}{E_r-\omega_p-i\epsilon}\right], \tag{4.2}$$

$$T_p(k_1k_2) = \frac{1}{\omega_1+\omega_2}[T_{k_1k_2}{}^\dagger(0), V_p] - \sum_{r=1,2}\left[\frac{S_p{}^\dagger(r)S_{k_1k_2}(r)}{E_r+\omega_1+\omega_2} + \frac{T_{k_1k_2}{}^\dagger(r)T_p(r)}{E_r-\omega_1-\omega_2-i\epsilon}\right], \tag{4.3}$$

$$T_{k_1k_2}(k) = \frac{1}{\sqrt{2}}[\delta(k_1k)V_{k_2} + \delta(k_2k)V_{k_1}] + \frac{1}{\omega_k}[T_{k_1k_2}(0), V_p] - \sum_{r=1,2}\left[\frac{S_{k_1k_2}{}^\dagger(r)S_k(r)}{E_r+\omega_k} + \frac{T_k{}^\dagger(r)T_{k_1k_2}(r)}{E_r-\omega_k-i\epsilon}\right], \tag{4.4}$$

$$T_{k_1k_2}(0) = \frac{1}{\sqrt{2}}\left(\frac{1}{\omega_2}V_{k_2}{}^\dagger V_{k_1} + \frac{1}{\omega_1}V_{k_1}{}^\dagger V_{k_2}\right) + \frac{1}{\sqrt{2}}\sum_{r=1,2}\left[\frac{T_{k_2}{}^\dagger(r)T_{k_1}(r)}{E_r+\omega_2} + \frac{T_{k_1}{}^\dagger(r)T_{k_2}(r)}{E_r+\omega_1}\right]. \tag{4.5}$$

Here the renormalized coupling constant f is used in the V's. We will make the following assumptions for the sake of mathematical simplicity. First, we may neglect the S terms in Eqs. (4.3) and (4.4) since the $S_n(m)$ is of at least $(n+m)$ order[10] in the renormalized coupling constant and the energy denominator is very large. This neglect does not violate the unitary condition, since the S terms in Eq. (3.8) have in general nothing to do with this condition. However, the crossing theorem no longer holds except for the scattering. The number of the coupled integral equations is then reduced from seven to five. Second, we will neglect the $T_2(2)$ terms, because those terms in Eqs. (4.3) and (4.4) are of at least fifth order in the coupling constant. Then, the unitarity condition, in addition to the crossing theorem, is violated in the sense that

$$T_{2'}{}^\dagger(2) - T_2(2') \neq \textstyle\sum_1 2\pi i\delta(E_2-E_1)T_{2'}{}^\dagger(1)T_2(1).$$

But the other unitarity conditions, Eq. (3.4) for $m=1$ or $n=1$, are still valid. The number of equations is now

[9] Drell, Friedman, and Zachariasen [Phys. Rev. 104, 236 (1956)] have defined a quantity in connection with S-state pion-nucleon scattering which is similar to $S_n(m)$.

[10] This fact is very easy to show using Eqs. (3.5), (3.6), and (2.15).

reduced to four. Third, we will assume further that $T_{k_1k_2}(k)$ in Eq. (4.3) may be approximated by the first term on the right hand side of Eq. (4.4) which is of the lowest order. The unitarity condition and the crossing theorem are now completely violated except for the scattering matrix. For this we have

$$\begin{aligned} t_{pq}(z)&=t_{qp}(-z), \quad [t_{pq}(z)=-s_{pq}(z)], \\ T_p{}^\dagger(q)-T_q(p)&=2\pi i\sum_{r=1,2}\delta(\omega_p-E_r)T_p{}^\dagger(r)T_q(r). \end{aligned} \tag{4.6}$$

Then it follows that

$$T_p(k_1k_2)=\frac{1}{\omega_1+\omega_2}T_{k_1k_2}{}^\dagger(0)V_p-\frac{1}{\omega_1+\omega_2}V_pT_{k_1k_2}{}^\dagger(0) +\frac{1}{\sqrt{2}}\left(\frac{V_{k_2}{}^\dagger T_p(k_1)}{\omega_2}+\frac{V_{k_1}{}^\dagger T_p(k_2)}{\omega_1}\right). \tag{4.7}$$

In spite of many assumptions made above, this expression has the following clear-cut physical meaning; for instance, the first term corresponds to the process in which the physical nucleon first absorbs the meson p and emits two mesons k_1, k_2 successively, and in the last term the nucleon scatters the incoming meson p into k_1 or k_2 and then emits k_2 or k_1. The effect of resonance scattering in the (3.3) state will come into play only in the last term of Eq. (4.7), but not apparently in the first or second term. It is shown in Appendix B that the lowest order terms of Eq. (4.7) give the Born approximation correctly with the unrenormalized coupling constant $f^{(0)}$ replaced by the renormalized f.

It should be noted here that $T_{k_1k_2}(0)$ may be expressed in the following alternative forms according as we make use of Eq. (3.6) or Eq. (2.15):

$$\begin{aligned} T_{k_1k_2}(0) &=\frac{1}{\sqrt{2}}\left(\Psi_0,\, V_{k_2}{}^\dagger\frac{1}{H+\omega_2}V_{k_1}\Psi_0\right)+(k_1\rightleftarrows k_2) && (4.8\text{a}) \\ &=-\frac{1}{\sqrt{2}}T_{k_1}{}^\dagger(k_2)-\frac{1}{\sqrt{2}}\left(\Psi_0,\, V_{k_1}{}^\dagger\frac{1}{H-\omega_2+i\epsilon}V_{k_2}\Psi_0\right)+(k_1\rightleftarrows k_2). && (4.8\text{b}) \end{aligned}$$

We prefer to take the first form and its expansion, Eq. (4.5), because here the energy denominators become larger and larger as we proceed to the higher configurations, while in Eq. (4.8b) there appears a pole at the energy equal to ω_1 or ω_2 of the intermediate states which seems to make the convergence of the expansion a little worse. Neglecting the two-meson contribution in Eq. (4.6), we obtain

$$T_{k_1k_2}(0)=\frac{1}{\sqrt{2}}\frac{1}{\omega_2}V_{k_2}{}^\dagger V_{k_1}+\frac{1}{\sqrt{2}}\sum_k\frac{T_{k_2}{}^\dagger(k)T_{k_1}(k)}{\omega_k+\omega_2}+(k_1\rightleftarrows k_2), \tag{4.9}$$

which is to be used in Eq. (4.7).

5. CORRECTIONS TO THE CHEW-LOW ONE-MESON APPROXIMATION

Following Chew and Low,[2] the scattering matrix $T_q(p)$ can conveniently be expanded in the eigenstates of angular momentum and isotopic spin as follows:

$$T_q(p)=-v(p)v(q)\frac{4\pi}{(4\omega_p\omega_q)^{\frac{1}{2}}}\sum_{\alpha=1}^{3}P_\alpha(p,q)h_\alpha(\omega_p), \tag{5.1}$$

where

$$\begin{aligned} P_1(p,q)&=\tfrac{1}{3}\tau_p\tau_q(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{q}) \\ P_2(p,q)&=\delta_{pq}(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{q})+\tfrac{1}{3}\tau_p\tau_q[3(\mathbf{p}\cdot\mathbf{q})-2(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{q})] \\ P_3(p,q)&=[\delta_{pq}-\tfrac{1}{3}\tau_p\tau_q][3(\mathbf{p}\cdot\mathbf{q})-(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{q})] \end{aligned} \tag{5.2}$$

are related to the projection operators for the four eigenstates of total angular momentum and isotopic spin (that is, $P_1=P_{11}$, $P_2=P_{13}+P_{31}$, $P_3=P_{33}$).[11] It is then easy to show[12] that

$$\begin{aligned} \mathrm{Re}h_\alpha(\omega)&=\frac{\lambda\alpha}{\omega}+\frac{1}{\pi}\int_1^\infty d\omega'\times\left\{\frac{\mathrm{Im}h_\alpha(\omega')}{\omega'-\omega}+\sum_\beta A_{\alpha\beta}\frac{\mathrm{Im}h_\beta(\omega')}{\omega'+\omega}\right\}, \\ \mathrm{Im}h_\alpha(\omega)&=\frac{1}{12\pi kv^2(k)}\sigma_\alpha(\omega), \end{aligned} \tag{5.3}$$

where

$$\lambda_\alpha=\frac{2}{3}\left(\frac{f^2}{4\pi}\right)\begin{bmatrix}-4\\-1\\2\end{bmatrix},\quad (A_{\alpha\beta})=\frac{1}{9}\begin{bmatrix}1&-8&16\\-2&7&4\\4&4&1\end{bmatrix}, \tag{5.4}$$

and $\sigma_\alpha(\omega)$ is the total cross section for the eigenstate (α).

It is to be noted that Eq. (4.9), with the above expansion, may be written as follows:

$$T_{k_1k_2}(0)=\frac{4\pi v(k_1)v(k_2)}{\sqrt{2}(4\omega_{k_1}\omega_{k_2})^{\frac{1}{2}}}\sum_{\alpha=1}^{3}P_\alpha(k_2,k_1)R_\alpha(\omega_2,\omega_1), \tag{5.5}$$

where

$$R_\alpha(\omega_2,\omega_1)=\frac{3}{\omega_2}\left(\frac{f^2}{4\pi}\right)\delta_{\alpha1}+\frac{3}{\omega_1}\left(\frac{f^2}{4\pi}\right)A_{\alpha1} +\frac{1}{\pi}\int d\omega k^3v(k)\left\{\frac{|h_\alpha(\omega)|^2}{\omega+\omega_2}+\sum_\beta A_{\alpha\beta}\frac{|h_\beta(\omega)|^2}{\omega+\omega_1}\right\}. \tag{5.6}$$

In the two-meson approximation, σ_α is separated into two terms

$$\sigma_\alpha(\omega)=\sigma_\alpha{}^{(1)}(\omega)+\sigma_\alpha{}^{(2)}(\omega). \tag{5.7}$$

Here $\sigma_\alpha{}^{(1)}$ is the cross section for the scattering given by

$$\sigma_\alpha{}^{(1)}(\omega)=12\pi k^4v^4(k)|h_\alpha(\omega)|^2, \tag{5.8}$$

and $\sigma_\alpha{}^{(2)}$ is that for the two-meson production, and is

[11] The subscripts 11, 13, 31, 33 label the eigenstates of the total angular momentum, J, and the isotopic spin, I, by $ij=(2J,2I)$.

[12] H. Miyazawa, Phys. Rev. **101**, 1564 (1956).

shown, after a lengthy calculation, to be given by

$$\sigma_\alpha^{(2)}(\omega)=18k^2v^2(k)\left(\frac{f^2}{4\pi}\right)\int_1^{\omega-1}d\omega_1k_1^3k_2^3v^2 \times(k_1)v^2(k_2)\Lambda_\alpha(\omega_1,\omega_2), \quad (\omega_2=\omega-\omega_1,\ \omega\geq 2), \tag{5.9}$$

where

$$\Lambda_\alpha(\omega_1,\omega_2)=|h_\alpha(\omega_1)|^2 +|h_\alpha(\omega_2)|^2+2A_{\alpha 1}\,\mathrm{Re}(h_\alpha^*(\omega_1)h_\alpha(\omega_2)) +\sum_\beta(\delta_{\alpha 1}\delta_{\beta 1}-A_{\alpha\beta}A_{\beta 1})\frac{2\,\mathrm{Re}(h_\alpha(\omega_2)R_\beta(\omega_2,\omega_1))}{\omega_1+\omega_2} +\sum_\beta(\delta_{\alpha 1}A_{1\beta}-\sum_\gamma A_{\alpha\gamma}A_{\gamma 1}A_{\gamma\beta})\frac{2\,\mathrm{Re}(h_\alpha(\omega_1)R_\beta(\omega_2,\omega_1))}{\omega_1+\omega_2} -\sum_{\gamma,\delta}\frac{8R_\gamma R_\delta Q_{\alpha,\gamma\delta}}{3^6(\omega_1+\omega_2)^2}, \tag{5.10}$$

and

$$Q_1=8\begin{bmatrix}-13 & -1 & 14\\ -1 & -25 & 26\\ 14 & 26 & -40\end{bmatrix},\quad Q_2=\begin{bmatrix}-11 & -5 & 16\\ -5 & -17 & 22\\ 16 & 22 & -38\end{bmatrix},\quad Q_3=8\begin{bmatrix}-1 & 2 & -1\\ 2 & -4 & 2\\ -1 & 2 & -1\end{bmatrix}. \tag{5.11}$$

If we insert Eqs. (5.8) and (5.9) into Eq. (5.3), correction terms due to the two-meson configuration are obtained. Making use of Salzman's solution for $h_\alpha(\omega)$ in the one-meson approximation to evaluate the two-meson production cross section, we will be able to obtain an improved value for $h_\alpha(\omega)$ from Eq. (5.3), which may in turn be used as the trial function. Convergence of this procedure in successive approximations will give a test for the one-meson approximation and make it possible to get an improved solution.

6. DISCUSSION

In Eqs. (3.7) and (3.8) we have obtained a set of integral equations which can, in principle, determine various transition matrices in the static meson theory. As an application of this generalization to the Chew-Low-Wick integral equation, we have found correction terms to the Chew-Low one-meson equation due to the two-meson configuration. It will be of interest to see in what way these correction terms, Eq. (5.9), affect the numerical solution of the one-meson approximation. In particular, it is hoped that these might remove some of the peculiarities in the high-energy behavior of $h_\alpha(\omega)$ reported by Salzman.[13]

As a further application which is still within the limits of validity of the static theory, these general equations can be applied to determine the transition matrix for the inelastic scattering process in which an extra pion is created.[14]

It should be noted, however, that some ambiguities always occur in applying the approximation method to the generalized Chew-Low-Wick equation, as was already pointed out in Sec. 4. Chew has suggested to us that such a difficulty may be due to the situation that the transition matrices for $n\geq 2$ defined by Eq. (3.1) always contain reducible processes corresponding to the last term in Eq. (3.8). In order to overcome this difficulty, it seems to be necessary to modify the present definition of the transition matrix such that all reducible processes are subtracted. To this end, the covariant generalization of the present formulation may be helpful.

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APPENDIX A. HOMOGENEOUS TERMS IN THE TRANSITION MATRICES

The complete reduction of the last term of Eq. (3.7) is given as follows:

$$\frac{1}{(m!n!)^{\frac{1}{2}}}(\Psi_0,\sum_{i=1}^n V_{k_i} \times[a(l_1)\cdots a(l_m),\,a^*(k_1)\cdots^{(i)}\cdots a^*(k_n)]\Psi_0)$$

$$=-\sum_{s=1}^{n-1}\left(\frac{(m-s)!(n-s)!}{m!n!}\right)^{\frac{1}{2}}\sum_{(i_1\cdots i_s)}\sum_{(j_1\cdots j_s)}\delta_{i_1j_1}\cdots\delta_{i_sj_s} \times\sum_r\frac{S_{n-s}^\dagger(r)S_{m-s}(r)}{E_r+E_{m-s}} \quad \text{for} \quad n\leq m, \tag{A1}$$

$$=-\sum_{s=1}^{m-1}(\text{same as above}) +\left(\frac{(n-m)!}{m!n!}\right)^{\frac{1}{2}}\sum_{(i_1\cdots i_m)}\delta_{i_11}\cdots\delta_{i_mm}S_{n-m}^\dagger(0) \quad \text{for} \quad n>m. \tag{A2}$$

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[13] G. Salzman, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* [Interscience Publishers, Inc., New York, (to be published)].

[14] Such a calculation has independently been done by J. Franklin [thesis, 1956, University of Illinois (unpublished)]. We are indebted to Professor G. F. Chew for bringing this work to our attention before publication. However, this work seems to be unsatisfactory in that the two-meson production matrix derived there does not agree with the Born approximation in the weak-coupling limit, and the last term of Eq. (4.7) expressing the resonance effect is omitted.

Here i and j stand for momenta k_i and l_j, and the sum over $(i_1,\cdots,i_s)$ and $(j_1,\cdots,j_s)$ runs over various s-combinations, disregarding the difference of order, of $(k_1,\cdots,k_n)$ and $(l_1,\cdots,l_m)$, respectively. In $S_{n-s}{}^\dagger(r)$ momenta $(i_1,\cdots,i_s)$ of the initial state, and in $S_{m-n}(r)$ momenta $(j_1,\cdots,j_s)$ of the final state, are omitted. It is evident from the above expression that the last term of Eq. (3.7) is Hermitian on the energy shell, so it does not contribute to the unitarity condition Eq. (3.4).

Each term of Eq. (A1) or (A2) corresponding to a specified s combination expresses the reducible process where the incident s mesons $(i_1,\cdots,i_s)$ go over, without interacting with the nucleon, into the outgoing s mesons $(j_1,\cdots,j_s)$, the rest of the mesons interacting with the nucleon. It should be noted here that $S_n(m)$ is of at least the $(n+m)$th order in the renormalized coupling constant and does not contain any reducible processes. When the initial state is that of the one-meson configuration $(n=1)$, homogeneous terms do not appear at all resulting in the absence of the term in Eq. (A1) corresponding to the last term of Eq. (A2).

APPENDIX B. BORN APPROXIMATION FOR THE TWO-MESON PRODUCTION

If we approximate $T_{k_1k_2}{}^\dagger(0)$ and $T_p(k)$ in Eq. (4.7) by their lowest order terms in the renormalized coupling constant, i.e., by the first term in Eq. (4.9) and in Eq. (4.2), respectively, we obtain, noting Eq. (3.10)

$$\begin{aligned}T_p(k_1k_2)=&\frac{1}{\sqrt{2}(\omega_1+\omega_2)}V_p\left\{\frac{1}{\omega_2}V_{k_1}{}^\dagger V_{k_2}{}^\dagger+\frac{1}{\omega_1}V_{k_2}{}^\dagger V_{k_1}{}^\dagger\right\}\\&-\frac{1}{\sqrt{2}(\omega_1+\omega_2)}\left\{\frac{1}{\omega_2}V_{k_1}{}^\dagger V_{k_2}{}^\dagger+\frac{1}{\omega_1}V_{k_2}{}^\dagger V_{k_1}{}^\dagger\right\}V_p\\&+\frac{1}{\sqrt{2}\omega_1\omega_2}\{V_1{}^\dagger V_2{}^\dagger+V_2{}^\dagger V_1{}^\dagger\}V_p\\&-\frac{1}{\sqrt{2}\omega_1\omega_2}\{V_2{}^\dagger V_pV_1{}^\dagger+V_1{}^\dagger V_pV_2{}^\dagger\}.\end{aligned}\tag{B1}$$

The second and third terms can be combined to give

$$\frac{1}{\sqrt{2}(\omega_1+\omega_2)}\left\{\frac{1}{\omega_1}V_{k_1}{}^\dagger V_{k_2}{}^\dagger+\frac{1}{\omega_2}V_{k_2}{}^\dagger V_{k_1}{}^\dagger\right\}V_p.\tag{B2}$$

On the energy shell $(\omega_r=\omega_1+\omega_2)$, the first term of Eq. (B1) and the expression (B2) are identical with the conventional Born approximation for the process in which the two mesons are emitted successively, the incoming meson being absorbed at the end or at the beginning. In the last term of Eq. (B1), the incoming meson is absorbed between the two outgoing mesons.

This result seems to show that it may be difficult to separate the first Born term from the expression for the general transition matrix in such a way as is done in pion-nucleon scattering or in photopion production.