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Construction of the Adiabatic Nuclear Potential: Formalism

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A new formalism is presented for the construction of the two-nucleon potential, whose salient characteristic is that it involves an expansion only in the number of mesons exchanged, the self-mesonic field of each nucleon being treated, in principle, exactly. Access to the potential is achieved through the intermediary of the scattering matrix. With the assumption that nonlinear meson propagation may be neglected, alternative versions of this matrix are derived, only one of which is exploited in this paper. The connection between the scattering matrix and the potential is discussed, and it is emphasized again that the transition between the two requires a knowledge of non-energy-conserving matrix elements of the potential, which can be obtained only if the underlying Schrödinger equation is known. The potential involving the exchange of at most two P-wave mesons is computed and shown to depend on the renormalized coupling constant, the single-nucleon source function, and the total cross sections for pion-nucleon interaction. The numerical evaluation of these formulas is not here attempted.

I. INTRODUCTION

THE concept of a two-nucleon potential together with the definition of its proper domain of applicability has been clarified considerably in recent years by quantum field theory.[1] In describing the interaction between two nucleons by a Schrödinger-like equation, that term corresponding to the "potential" has, in general, little structural resemblance to its classical counterpart.[2] This "potential" or, more properly, kernel is characteristically both nonlocal and energy-dependent. Another feature common to all investigations has been the expansion of the interaction as a sequence of terms, each term describing the exchange by the nucleons of successively larger numbers of mesons. Confronted with this unbounded quagmire, impenetrable from a technical point of view, we are led to formulate a very modest question: Can one find a local, energy-independent potential, which for energies below a prescribed maximum energy, $E_{\max}$, and for internucleon separations beyond a certain minimum distance, $r_{\min}$, is an accurate representation of the full kernel?

As stated above, the question has an obviously affirmative answer. What leads us to anticipate these twin limitations upon the full interaction function? Such nonlocal and energy dependent deviations of the full kernel from an energy-independent local interaction should be nominal for nucleon relative kinetic energies, p^2/M, not larger than μc^2, the pion rest energy. Considering p as the relative momentum in an actual collision provides the aforementioned energy criterion which furthermore is the condition that we are below the threshold for meson production; considering p as the nucleon recoil momentum in an intermediate state we obtain, from the uncertainty relation, the validity of the local approximation to the nonlocal behavior for nucleon-nucleon separations:

$$r > 0.55\times 10^{-13} \text{ cm}.$$

A second aspect of our answer concerns the possible convergence of the series in the exchange of successively larger numbers of virtual mesons. The qualitative discussion can be based upon range considerations which inform us that the exchange of n mesons is associated with a potential function $V^{(2n)}$ of extent $\hbar/n\mu c$. This argument alone guarantees that for a sufficiently large internucleon separation the potential will be dominated by the second-order part of range, $\hbar/\mu c$, for a somewhat smaller distance by the second- plus fourth-order part, and so on. The argument may be continued until one reaches a separation within which the local approximation itself ceases to be valid. It is reasonable *a priori* as well as confirmed by detailed calculation[3] that sixth and higher order potentials first enter appreciably at distances of the order of 0.6×10^{-13} cm, i.e., that separation at which the local approximation itself becomes questionable. Furthermore, the interaction arising from the exchange of K mesons begins to be felt within this same neighborhood. On these four grounds then, a separation of the order of $\frac{1}{3}(\hbar/\mu c)$ represents the critical distance $r_{\min}$ inside which one has no basis for extending the concept of point interaction of pi-mesonic origin, and outside which can reasonably expect the interaction to be well approximated by the local potential arising from the *exchange* of at most two mesons.

Before undertaking the description of the actual program to be expounded in this and succeeding papers, it

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[1] The pioneers of this effort were Taketani, Nakamura, and Sasaki, Progr. Theoret. Phys. (Japan) **6**, 581 (1951); M. M. Lévy, Phys. Rev. **88**, 725 (1952).

[2] See, for instance, M. M. Lévy, Phys. Rev. **88**, 72 (1952); A. Klein, Phys. Rev. **90**, 1101 (1953).

[3] A. Klein, Phys. Rev. **91**, 740 (1953); **92**, 1017 (1953); K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953); E. M. Henley and M. A. Ruderman, Phys. Rev. **92**, 1036 (1953); S. Machida and K. Senba, Progr. Theoret. Phys. (Japan) **13**, 389 (1955).

may be well to consider what has been achieved by the above considerations. For nucleon-nucleon interactions below a few hundred Mev, ($E_{\max}\sim\mu c^2$), it is reasonable to expect that the S wave alone will be strongly sensitive to the form of the interaction in the inside region. On the other hand, P and higher wave interactions should be determined largely by the form of the interaction in the outside region. Accordingly, p-p scattering at intermediate energies may well represent the most sensitive test of this "long range" part of the potential. However, the best method to adjust the phenomenological interaction in the inner and unknown domain, and to join to the known potential outside is not the proper task of this immediate paper. Such prescriptions, as in the selection of appropriate core radii, are presumably to be read from the data itself.

Our immediate goal, then, is to compute the adiabatic approximation to the nuclear potential through the fourth order as accurately as possible, and ultimately to compare the result with experiment. A question of basic interest here is whether the nuclear forces can be understood completely in terms of quantities supplied by other experiments, such as meson-nucleon scattering, or whether new parameters will enter essentially, for instance, the meson-meson interaction. In the conservative approach taken in this paper, we shall make two major assumptions: first, that we may neglect possible nonlinear propagation of mesons, and secondly, that the nuclear potential through fourth order is determined largely by P-wave mesons. Of the two assumptions there is some theoretical justification for neglecting S-wave interactions.[4] On the other hand, both restrictions will be reexamined in subsequent publications. With these assumptions, our major result is as follows: through the fourth order the adiabatic potential can be expressed completely in terms of the renormalized coupling constant for P-wave mesons, the form factor or source function for individual nucleons, and the total cross sections for pion-nucleon interactions.

Several points require special emphasis. One should note that only with the neglect of meson-meson interaction can one properly speak of the exchange of a distinct number of virtual mesonic quanta. The analysis presented then clearly separates *exchange* mesons and the quanta of the nucleon self-fields; it is upon the former alone that the expansion of the nucleon-nucleon interaction given here depends. That part of the potential arising from the exchange of at most one mesonic quantum is, except for small corrections, the second order potential of the perturbation theory expressed in terms of the renormalized P-wave coupling constant. This result, including fully all self-interaction, will be recognized as tantamount to a low-energy theorem.[5] Further, those contributions to the potential from the simultaneous existence in the field of two virtual mesons are correlated directly with the observed meson-nucleon scattering. The method to be described has the virtue that we can relate the off-the-energy-shell meson-nucleon scattering required directly and unambiguously to the observable on-the-energy-shell scattering by the same technique as has been recently employed for meson-nucleon scattering,[6] and therefore can express our results directly in terms of the total cross sections for pion-nucleon interaction.

The fourth-order potential has been extensively studied in recent years,[1–3,7,8] including several attempts to correctly evaluate the contribution of pion-nucleon scattering. The most schematic of these attempts was that of Sugawara *et al.*[8] who utilized the model of a discrete isobar state for the nucleon of angular momentum and isotopic spin $\frac{3}{2}$. Two principal efforts have been made to compute directly the effect of the virtual scattering. The considerations of Brueckner and Watson[3] represent a qualitative attempt to compare the corrections to the perturbation result and to justify the neglect of the former compared to the latter. Our own considerations are most closely akin to those of Henley and Ruderman[3] although they will differ both in the choice of the perturbation result, as well as in the form of the scattering corrections.

In common with the latter authors, we find it convenient to arrive at the potential through the intermediary of scattering considerations. In Sec. II, the scattering amplitude for two nucleons is obtained as a power series in the number of mesons exchanged. Alternative versions are exhibited, only one of which is exploited in this paper. The definition of the potential and of its adiabatic limit is dealt with in Sec. III. Whereas the adiabatic limit of the second order and of the nonperturbative corrections to the fourth-order term are found to be unambiguously determined from the scattering amplitude, this is not the case for the perturbative part of the fourth-order potential. In the recent literature, this same ambiguity manifests itself in competing versions of the perturbation result: that of Taketani *et al.*, and of other authors,[1–3,7] and the alternative of Brueckner and Watson.[3] The relationship between the two forms has been previously analyzed by Fukuda *et al.*,[9] by one of the authors,[10] and others.[7] The analysis given here traces this ambiguity back to a corresponding uncertainty in the relationship between the on- and off-the-energy-shell scattering of *two*

[4] A. Klein, Phys. Rev. **95**, 1061 (1954).

[5] The theorem stated here has the same origin as the P-wave theorem for meson-nucleon scattering discussed by A. Klein, Phys. Rev. **99**, 998 (1955) and by G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[6] G. F. Chew and F. E. Low, reference 5.

[7] Taketani, Machida, and Onuma, Progr. Theoret. Phys. (Japan) **7**, 45 (1952), I. Sato, Progr. Theoret. Phys. (Japan) **10**, 323 (1953); J. Iwadare, Progr. Theoret. Phys. (Japan) **14**, 16 (1955); D. Feldman, Phys. Rev. **98**, 1456 (1955).

[8] Matsumoto, Hamada, and Sugawara, Progr. Theoret. Phys. (Japan) **10**, 199 (1953).

[9] Fukuda, Sawada, and Taketani, Progr. Theoret. Phys. (Japan) **12**, 156 (1954).

[10] A. Klein, Phys. Rev. **94**, 195 (1954).

nucleons, and in this form is well known.[11] We actually require the nondiagonal matrix elements of the second-order potential, and these can be stated only in conjunction with the underlying Schrödinger equation. With appropriate selection of these nondiagonal elements of $V^{(2)}$, the fourth-order perturbation result for the potential agrees in essence with that suggested by Brueckner and Watson.

The evaluation of this perturbation result is described in Sec. IV, whereas the concluding section of the body of the paper analyzes the nonperturbative corrections to the fourth-order interaction and exhibits our principal new results. Appendix A contains a derivation of the formula fundamental to the separation of the potential into its various orders.

II. TWO-NUCLEON SCATTERING MATRIX

We wish to demonstrate in this section that the S matrix for nucleon-nucleon scattering can be formally exhibited as a sum of contributions, each term, $S^{(2m)}$, having its origin in the exchange of m-virtual mesons between the two coupled nucleons. The prescription to be given here avoids detailed consideration of radiative corrections, vertex diagrams, and similar correlatives of a microanalysis of the emission and absorption mechanism for those mesons ultimately exchanged between the two nucleons.

The two-nucleon S matrix may be conveniently constructed with the aid of the familiar "in" and "out" operator formalism of scattering theory.[12] If $p_1,\lambda_1;p_2,\lambda_2$ are the initial momenta and spin states of nucleons one and two, respectively, and $p_1',\lambda_1';p_2',\lambda_2'$ similar quantities for the final states, then we may write the two-nucleon S matrix as

$$(p_1',\lambda_1';p_2',\lambda_2'|S|p_1,\lambda_1;p_2,\lambda_2) = (\Psi_0,\chi(\lambda_1'p_1')^{(\text{out})}\chi(\lambda_2'p_2')^{(\text{out})} \times\bar{\chi}(\lambda_2p_2)^{(\text{in})}\bar{\chi}(\lambda_1p_1)^{(\text{in})}\Psi_0). \quad (1)$$

Those components of the "in" and "out" fields referring to nucleons (as opposed to anti-nucleons) are given by the following surface integrals:

$$\bar{\chi}(\lambda p)^{(\text{in})}=\lim(\sigma_2\to-\infty)Z_2^{-\frac{1}{2}}\int d\sigma_2\bar{\psi}(x)\gamma_0\psi_{\lambda p}(x), \quad (2)$$

$$\chi(\lambda p)^{(\text{out})}=\lim(\sigma_1\to+\infty)Z_2^{-\frac{1}{2}}\int d\sigma_1\bar{\psi}_{\lambda p}(x)\gamma_0\psi(x), \quad (3)$$

with

$$\psi_{\lambda p}(x)=C_p u(\lambda p)e^{ipx}, \quad C_p=[M/(2\pi)^3E(p)]^{\frac{1}{2}}. \quad (4)$$

Here $\psi(x)$, $\bar{\psi}(x)$ are, as usual, the Heisenberg variables of the Dirac field, and $u(\lambda p)$ is a free-particle Dirac spinor. The parameter Z_2 will be identified later as the nucleon field variable renormalization constant of Dyson. Within this formalism then, we have

$$(p_1',\lambda_1';p_2',\lambda_2'|S|p_1,\lambda_1;p_2,\lambda_2) =\lim(\sigma_2\to-\infty;\sigma_1\to+\infty) \times\int d\sigma_1 d\sigma_1' d\sigma_2 d\sigma_2'(\bar{\psi}_{\lambda_1'p_1'}(x_1')\gamma_0^{(1')}) \times(\bar{\psi}_{\lambda_2'p_2'}(x_2')\gamma_0^{(2')})(-)G_{12}(x_1',x_2';x_1,x_2) \times(\gamma_0^{(2)}\psi_{\lambda_2p_2}(x_2))(\gamma_0^{(1)}\psi_{\lambda_1p_1}(x_1)), \quad (5)$$

where

$$G_{12}(1',2';1,2)=-Z_2^{-2}\epsilon\langle(\psi(1')\psi(2')\bar{\psi}(2)\bar{\psi}(1))_+\rangle. \quad (6)$$

The factor ϵ indicates a sign change for an odd permutation of the operators from the standard order indicated.

In order to proceed further, we must explicitly construct the two-nucleon Green's function G_{12}. Recent developments by Schwinger and others[13] show that the entire structure of the theory of coupled fields is most readily exhibited in terms of an hypothesized dependence of the single-particle propagators upon fictitious external sources. A suitably restricted form of the general framework, in which we neglect the possibility of nonlinear meson propagation, permits us to exhibit G_{12} in terms of the functional dependence of the single nucleon Green's function $G(x,x')=G[\phi]$ upon a classical meson field ϕ. The following formula, as derived in Appendix A, gives

$$G_{12}(x_1',x_2';x_1,x_2) =Z_2^{-2}G(x_1',x_1)G(x_2',x_2) +Z_2^{-2}\int(dx_1'')(dx_2'')(dx_1''')(dx_2''')G(x_1',x_1'') \times G(x_2',x_2'')I(x_1'',x_2'';x_1''',x_2''')G(x_1''',x_1) \times G(x_2''',x_2)-(x_1'\leftrightarrow x_2'), \quad (7)$$

with

$$I(x_1',x_2';x_1,x_2) =\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int(d1)\cdots(dn') \times\left(x_1'\left|G^{-1}\frac{\delta^n G}{\delta\phi(1)\cdots\delta\phi(n)}G^{-1}\right|x_1\right)\Bigg|_{\phi=0} \times\Delta''(1,1')\cdots\Delta''(n,n') \times\left(x_2'\left|G^{-1}\frac{\delta^n G}{\delta\phi(1')\cdots\delta\phi(n')}G^{-1}\right|x_2\right)\Bigg|_{\phi=0}. \quad (8)$$

[11] For example, see Y. Nambu, Progr. Theoret. Phys. (Japan) 5, 614 (1950).
[12] C. N. Yang and D. Feldman, Phys. Rev. 79, 972 (1950); G. Källén, Arkiv Physik 2, 371 (1950).
[13] J. Schwinger, Proc. Natl. Acad. Sci. 37, 452, 455 (1951) and unpublished lectures; K. Symanzik, Z. Naturforsch. 9, 809 (1954); H. Umezawa and A. Visconti, Nuovo cimento 1, 1079 (1955).

Here $\Delta''(\xi,\eta)$ is the renormalized meson propagator of the causal type.

Making use of such renormalization statements as

$$\lim(\sigma_1\to+\infty)\int d\sigma\bar{\psi}_{\lambda p}(x')\gamma_0 G(x',x)$$
$$=iZ_2\bar{\psi}_{\lambda p}(x),\quad (x_0<x_0'\epsilon\sigma_1),\quad (9)$$

and the orthogonality condition

$$\int d\sigma\bar{\psi}_{\lambda' p'}(x)\gamma_0\psi_{\lambda p}(x)=\delta_{\lambda\lambda'}\delta(p'-p),\qquad (10)$$

the S matrix developed from Eqs. (7) and (8) assumes the form[14]

$$(p_1'\lambda_1';p_2'\lambda_2'|S|p_1\lambda_1;p_2\lambda_2)$$
$$=\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\delta(p_1'-p_1)\delta(p_2'-p_2)$$
$$-C(p_1')C(p_2')C(p_1)C(p_2)\int(dx_1')(dx_2')(dx_1)(dx_2)$$
$$\times\exp(-ip_1'x_1'-ip_2'x_2')\bar{u}(\lambda_1'p_1')\bar{u}(\lambda_2'p_2')$$
$$\times I(x_1',x_2';x_1,x_2)u(\lambda_1,p_1)u(\lambda_2,p_2)$$
$$\times\exp(ip_1x_1+ip_2x_2).\qquad (11)$$

Actually Eq. (11) was the first form of the scattering matrix used by the authors[15] to obtain information about nuclear forces from the meson-nucleon scattering experiments. A detailed account of this work will be postponed, however. The developments of this paper will be based on an alternative version which we now describe.

We suppose the interaction terms in the Lagrangian density to be

$$\mathcal{L}'(x)=j(x)\phi(x)+J(x)Z_3^{-\frac{1}{2}}\phi(x),\qquad (12)$$

where $\phi(x)$ is here the dynamical meson field, $j(x)$ the nucleon source density, and $J(x)$ an external source density. It is the essence of the linear meson propagation approximation that

$$(-i)^n\frac{\delta^n}{\delta J(\xi_1)\cdots\delta J(\xi_n)}Z_3^{-\frac{1}{2}}\langle\phi(\xi)\rangle\bigg|_{J=0}=\begin{cases}\Delta'(\xi_1,\xi), & n=1\\ 0\quad, & n>1\end{cases},\qquad (13)$$

where, by Eq. (12), $\Delta'(\xi,\xi')$ is the renormalized meson propagator

$$\Delta'(\xi,\xi')=iZ_3^{-1}\langle(\phi(\xi)\phi(\xi'))_+\rangle|_{J=0}$$

(explicitly defining the renormalization constant Z_3). It is therefore possible within the above approximation to assert

$$Z_3^{-\frac{1}{2}}\langle\phi(\xi)\rangle=\int(d\xi')\Delta'(\xi,\xi')J(\xi'),\qquad (14)$$

where we will henceforth refer to the renormalized vacuum expectation value $Z_3^{-\frac{1}{2}}\langle\phi(\xi)\rangle$ as the quantity ϕ_e'. If we now conveniently introduce a renormalized current operator

$$j_R(\xi)=Z_3^{-\frac{1}{2}}\Delta'^{-1}\phi(\xi),$$

we can replace Eq. (12) for $\mathcal{L}'(x)$ by the equivalent expression

$$\mathcal{L}'(x)=j(x)\phi(x)+j_R(x)\phi_e'(x).\qquad (15)$$

If we employ the techniques of Schwinger,[13] it then follows that if $\langle O\rangle$ is the vacuum matrix element of the operator O, in his sense, then

$$\delta/\delta\phi_e'(\xi)\langle O\rangle=i[\langle(j_R(\xi)O)_+\rangle-\langle j_R(\xi)\rangle\langle O\rangle]$$
$$\equiv i\langle(j_R(\xi)O)_+'\rangle.\qquad (16)$$

The physical significance of the primed matrix element is that if $\langle O\rangle$ describes only connected processes, then $\langle(j_R(\xi)O)_+'\rangle$ retains this characteristic feature. This remarkable property follows from the definition of $\langle O\rangle$ as the ratio of a transition amplitude to the vacuum-to-vacuum amplitude.

The formulation contained in Eqs. (14)–(16) has almost immediate application to a rewording of G_{12} as expressed by Eqs. (7) and (8). We merely must replace the classical field $\phi_e=\langle\phi(\xi)\rangle$ referred to there with $\phi_e'(\xi)$, and concurrently substitute for $\Delta''(\xi,\xi')$, the renormalized meson propagator $\Delta'(\xi,\xi')$ for those mesons which are *exchanged*.[16] The consistent neglect of nonlinear meson propagation permits, accordingly, a transparent renormalization of the nucleon source density operator, and of the exchange meson propagation. With $x_0>\xi_{10}\cdots\xi_{n0}>x_0'$, we then have (dropping the subscript R):

$$(-i)^n[\delta^n/\delta\phi_e'(\xi_1)\cdots\delta\phi_e'(\xi_n)]G(x',x;\phi_e)|_{\phi_e=0}$$
$$=i\langle\psi(x)(j(\xi_1)\cdots j(\xi_n))_+'\bar{\psi}(x')\rangle.\qquad (17)$$

By means of the consequent expression for G_{12}, remembering Eqs. (2) and (3) and the definition of the real one-nucleon states,

$$\bar{\chi}(\lambda p)^{(\mathrm{in})}\Psi_0=\bar{\chi}(\lambda p)^{(\mathrm{out})}\Psi_0=|p\lambda),\qquad (18)$$

[14] Here and in what follows, we shall not record the explicit antisymmetrization.

[15] B. H. McCormick and A. Klein, Phys. Rev. **99**, 618 (1955).

[16] In the actual calculations of the text we shall replace $\Delta'(\xi,\xi')$ by the free-meson propagator appropriate to mesons of the observed mass μ. The renormalization theory tells us that we thereby ignore the contribution of virtual pair formation to the renormalized meson Green's function. These excited states of mass $\geqslant 3\mu$ contribute to the adiabatic potential terms of range comparable to those from three meson exchanges, and may be therefore legitimately dropped. A similar approximation in the case of virtual mesons, because of the unrestricted energy and momentum which they bear, would have been completely unjustified.

we can record our alternative form of the S matrix,

$$(p_1',\lambda_1';p_2',\lambda_2'|S|p_1,\lambda_1;p_2,\lambda_2)$$
$$=\delta(\lambda_1'\lambda_1)\delta(\lambda_2'\lambda_2)\delta(p_1'-p_1)\delta(p_2'-p_2)+\sum_{n=1}^{\infty}\frac{(i)^n}{n!}$$
$$\times\int(d1)\cdots(dn')(p_1'\lambda_1'|(j(1)\cdots j(n))_+'|p_1\lambda_1)$$
$$\times\Delta'(1,1')\cdots\Delta'(n,n')$$
$$\times(p_2'\lambda_2'|(j(1')\cdots j(n'))_+'|p_2\lambda_2)$$
$$\equiv\delta(\lambda_1'\lambda_1)\delta(\lambda_2'\lambda_2)\delta(p_1'-p_1)\delta(p_2'-p_2)$$
$$+\sum_{n=1}^{\infty}(p_1',p_2'|T^{(2n)}|p_1,p_2).\quad(19)$$

The remainder of this paper will be concerned with the evaluation of the contributions of $T^{(2)}$ and $T^{(4)}$ to the nuclear potential.

III. DEFINITION OF THE POTENTIAL

The quantity to be termed the potential will be related below to the reduced T matrix, $t(\mathbf{p}',\mathbf{p};P_0)$ defined as follows:

$$(p_1',p_2'|T|p_1,p_2)$$
$$=-2\pi i\delta(P_0'-P_0)\delta(\mathbf{P}-\mathbf{P}')T(\mathbf{p}',\mathbf{p};\mathbf{P},P_0),\quad(20)$$

where relative momentum coordinates, $\mathbf{p}$, $\mathbf{p}'$ are introduced according to

$$p_1=\tfrac{1}{2}P+p,\quad p_1'=\tfrac{1}{2}P'+p',$$
$$p_2=\tfrac{1}{2}P-p,\quad p_2'=\tfrac{1}{2}P'-p',\quad(21)$$

and

$$P_0=E(p_1)+E(p_2),\quad P_0'=E(p_1')+E(p_2').\quad(22)$$

The reduced T matrix, $t(\mathbf{p}',\mathbf{p};P_0)$, defined in the center-of-mass frame, $\mathbf{P}=0$, then is given by

$$t(\mathbf{p}',\mathbf{p};P_0)=(\mathbf{p}'|t(P_0)|\mathbf{p})$$
$$=T(\mathbf{p}',\mathbf{p};\mathbf{P}=0,P_0).\quad(23)$$

We shall suppose that $t(P_0)$ is the transition matrix which results from the solution of a Schrödinger-like equation with the nonlocal kernel $(\mathbf{p}'|v(P_0)|\mathbf{p})=v(\mathbf{p}',\mathbf{p};P_0)$. The two quantities $t(P_0)$ and $v(P_0)$ are accordingly related by the familiar Lippmann-Schwinger integral equation, which in the center-of-mass frame of the two nucleons, reads

$$v(\mathbf{p}',\mathbf{p};P_0)=t(\mathbf{p}',\mathbf{p};P_0)$$
$$-\int d^3\mathbf{p}''\frac{v(\mathbf{p}',\mathbf{p}'',P_0)t(\mathbf{p}'',\mathbf{p};P_0)}{P_0+i\epsilon-2E(p'')}.\quad(24)$$

We shall use Eq. (24) to *define* $v(\mathbf{p}',\mathbf{p};P_0)$. In particular $v^{(2)}$ and $v^{(4)}$, the second- and fourth-order contributions, are given as

$$v^{(2)}(\mathbf{p}',\mathbf{p};P_0)=t^{(2)}(\mathbf{p}',\mathbf{p};P_0),\quad(25a)$$
$$v^{(4)}(\mathbf{p}',\mathbf{p};P_0)=t^{(4)}(\mathbf{p}',\mathbf{p};P_0)$$
$$-\int d^3\mathbf{p}''\frac{v^{(2)}(\mathbf{p}',\mathbf{p}'';P_0)v^{(2)}(\mathbf{p}'',\mathbf{p};P_0)}{P_0+i\epsilon-2E(p'')}.\quad(25b)$$

Our ultimate interest is in the adiabatic potential, a local function in coordinate space. We have, first of all,

$$v(\mathbf{r},\mathbf{r}';P_0)=(2\pi)^{-3}\int d^3\mathbf{p}d^3\mathbf{p}'e^{i\mathbf{p}'\cdot\mathbf{r}'}v(\mathbf{p}',\mathbf{p};P_0)e^{-i\mathbf{p}\cdot\mathbf{r}}$$
$$=(2\pi)^{-3}\int d^3(\mathbf{p}'-\mathbf{p})d^3\tfrac{1}{2}(\mathbf{p}'+\mathbf{p})$$
$$\times V(\mathbf{p}'-\mathbf{p};\tfrac{1}{2}(\mathbf{p}'+\mathbf{p});P_0)$$
$$\times e^{i(\mathbf{p}'-\mathbf{p})\cdot\frac{1}{2}(\mathbf{r}+\mathbf{r}')}e^{i\frac{1}{2}(\mathbf{p}'+\mathbf{p})\cdot(\mathbf{r}'-\mathbf{r})},\quad(26)$$

with

$$v(\mathbf{p}',\mathbf{p};P_0)\equiv V(\mathbf{p}'-\mathbf{p};\tfrac{1}{2}(\mathbf{p}'+\mathbf{p});P_0).\quad(27)$$

We shall expand $V(\mathbf{p}'-\mathbf{p};\frac{1}{2}(\mathbf{p}'+\mathbf{p});P_0)$ in powers of the variable $\frac{1}{2}(\mathbf{p}'+\mathbf{p})$ and of the difference P_0-2M, ignoring nucleon kinetic energies compared to the nucleon rest mass, and identify the leading term, to be called $v(\mathbf{p}'-\mathbf{p})$, with the adiabatic nuclear potential. The justification for this procedure is that by Eq. (26),

$$v(\mathbf{p}'-\mathbf{p})\to v_{\text{adiabatic}}(\mathbf{r}',\mathbf{r})=V(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'),\quad(28)$$

with

$$V(\mathbf{r})=\int d^3\mathbf{k}v(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}).\quad(29)$$

There is just one impediment to effectuating the program outlined above with the information at hand. We require, according to Eq. (25b), $v^{(2)}(\mathbf{p}',\mathbf{p};P_0)$ off-the-energy shell to determine the fourth-order potential unambiguously, whereas Eq. (20) ultimately determines only the diagonal elements of that quantity. To circumvent this difficulty, we must actually derive the Schrödinger equation with the correct nonlocal kernel to lowest order in the coupling constant. Similarly, if we were interested in the adiabatic potential through order $2n$, we would require knowledge of the full kernel through order $2(n-1)$. Fortunately, the procedure for obtaining the desired result to a sufficient approximation is known.[17] It consists first in defining a covariant two-body equation whose solution is our scattering matrix and subsequently in reducing the equation to a single-time Schrödinger form. To lowest order, all that is here required, a knowledge of previous work should convince the reader that $v^{(2)}(\mathbf{p}',\mathbf{p};P_0)$ will be given by Eq. (37) below.

[17] This procedure has been described in great detail by A. Klein, Phys. Rev. **94**, 1052 (1954) and reference 2; W. Macke, Z. Naturforsch. **8a**, 599 (1953).

IV. PERTURBATION RESULTS

We turn to the detailed evaluation of the transition matrix. Consider first

$$(p_1',p_2'|T^{(2)}|p_1,p_2)=i\int(d\xi)(d\eta)(\mathbf{p}_1'|j(\xi)|\mathbf{p}_1)\times(\mathbf{p}_2'|j(\eta)|\mathbf{p}_2)\Delta(\xi-\eta). \quad (30)$$

It is convenient to translate the nucleon density operators to the space-time origin and thus to write

$$(\mathbf{p}'|j_i(\xi)|\mathbf{p})=e^{-i(p'-p)\xi}(\mathbf{p}'|j_i(0)|\mathbf{p}). \quad (31)$$

We define

$$(\mathbf{p}'|j(0)|\mathbf{p})=(2\pi)^{-3}(\mathbf{p}'|J(\mathbf{p}-\mathbf{p}')|0\rangle \cong(2\pi)^{-3}\langle 0|J(\mathbf{p}-\mathbf{p}')|0\rangle, \quad (32)$$

where the nonrelativistic form of the matrix element of the reduced current operator

$$\langle 0|J_i(\mathbf{p}-\mathbf{p}')|0\rangle=(f/\mu)[i\sigma\cdot(\mathbf{p}-\mathbf{p}')\tau_i]\rho(|\mathbf{p}-\mathbf{p}'|) \quad (33)$$

follows from the usual symmetry considerations. Here f is the renormalized P-wave coupling constant and $\rho(|\mathbf{p}-\mathbf{p}'|)$, the nucleon form factor, is normalized to $\rho(0)=1$. By means of Eq. (33) and either of the standard representations for $\Delta(\xi)$,

$$\Delta(\xi)=(2\pi)^{-4}\int(dk)e^{ik\xi}[k^2+\mu^2-i\epsilon]^{-1} =i(2\pi)^{-3}\int d^3k(2\omega)^{-1}\exp(i\mathbf{k}\cdot\xi-i\omega|\xi_0|) \quad (34)$$

we then easily determine that

$$(p_1',p_2'|T^{(2)}|p_1,p_2)=(2\pi i)\delta(P_0'-P_0)\delta(\mathbf{P}'-\mathbf{P})\times(2\pi)^{-3}(\mathbf{p}'|J_i^{(1)}(\mathbf{p}-\mathbf{p}')|0\rangle(-\mathbf{p}'|J_i^{(2)}(-(\mathbf{p}-\mathbf{p}'))|0\rangle\times\{\omega(\mathbf{p}'-\mathbf{p})^2-[E(p)-E(p')]^2-i\epsilon\}. \quad (35)$$

The comparison with Eq. (20) and our previous remarks allows us to conclude merely that

$$v^{(2)}(\mathbf{p}',\mathbf{p};P_0)=t^{(2)}(\mathbf{p}',\mathbf{p};P_0) =-(2\pi)^{-3}(\mathbf{p}'|J_i^{(1)}(\mathbf{p}-\mathbf{p}')|0\rangle\times(-\mathbf{p}'|J_i^{(2)}(-(\mathbf{p}-\mathbf{p}'))|0\rangle[\omega(\mathbf{p}'-\mathbf{p})]^{-2} +[P_0-E(p)-E(p')]\delta t^{(2)}(\mathbf{p}',\mathbf{p};P_0), \quad (36)$$

where $\delta t^{(2)}(\mathbf{p}',\mathbf{p};P_0)$ is itself finite on the energy shell. Thus we argue that in fact only the adiabatic part of $v^{(2)}(\mathbf{p}',\mathbf{p};P_0)$ as given by the first term of Eq. (36) is unambiguously determined by the scattering considerations. This term is basically the familiar second-order static potential. The procedure described at the end of the previous section leads one to regard the correct form of $v^{(2)}(\mathbf{p}',\mathbf{p};P_0)$ to be that represented by the expression

$$v^{(2)}(\mathbf{p}',\mathbf{p};P_0)=-(2\pi)^{-3}\times\frac{(\mathbf{p}'|J_i^{(1)}(\mathbf{p}-\mathbf{p}')|0\rangle(-p'|J_i^{(2)}(\mathbf{p}'-\mathbf{p})|0\rangle}{\omega(\mathbf{p}'-\mathbf{p})[\omega(\mathbf{p}'-\mathbf{p})+E(p)+E(p')-P_0]} \quad (37)$$

since this is the form which results from the reduction of a suitable covariant two-body equation to a single-time Schrödinger-like equation. Equation (37) will be employed below in the evaluation of $v^{(4)}$.

We turn then to the reduction of

$$(p_1',p_2'|T^{(4)}|p_1,p_2)=\tfrac{1}{2}(i)^2\int(d\xi_1)(d\xi_2)(d\eta_1)(d\eta_2)\times(\mathbf{p}_1'|(j(\xi_1)j(\xi_2))_+|\mathbf{p}_1)(\mathbf{p}_2'|(j(\eta_1)j(\eta_2))_+|\mathbf{p}_2)\times\Delta(\xi_1-\eta_1)\Delta(\xi_2-\eta_2). \quad (38)$$

Removing the center-of-mass motion by translation of the current operators in line with our previous treatment of $T^{(2)}$, and carrying out the time integrations we find

$$\begin{aligned}t^{(4)}(\mathbf{p}',\mathbf{p})=&\frac{i}{4\pi}\int d^3\mathbf{p}''d^3\xi d^3\boldsymbol{\eta}\int dk_0\\&\times\exp[-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\tfrac{1}{2}(\xi-\boldsymbol{\eta})]\{(k_0^2-\omega_1^2+i\epsilon)\\&\times[(k_0+E(p)-E(p'))^2-\omega_2^2+i\epsilon)\}^{-1}\\&\times\sum_n\left\{\frac{(\mathbf{p}'|j(\tfrac{1}{2}\xi)|n)(n|j(-\tfrac{1}{2}\xi)|\mathbf{p})}{[E_n-i\epsilon-E(p)-k_0]}+\frac{(p'|j(-\tfrac{1}{2}\xi)|n)(n|j(\tfrac{1}{2}\xi)|p)}{[E_n-i\epsilon-E(p')+k_0]}\right\}\\&\times\sum_{n'}\left\{\frac{(-\mathbf{p}'|j(\tfrac{1}{2}\boldsymbol{\eta})|n')(n'|j(-\tfrac{1}{2}\boldsymbol{\eta})|-\mathbf{p})}{[E_{n'}-i\epsilon-E(p')+k_0]}+\frac{(-\mathbf{p}'|j(-\tfrac{1}{2}\boldsymbol{\eta})|n')(n'|j(\tfrac{1}{2}\boldsymbol{\eta})|-\mathbf{p})}{[E_{n'}-i\epsilon-E(p)-k_0]}\right\},\end{aligned} \quad (39)$$

where $\mathbf{k}_1=\mathbf{p}'-\mathbf{p}''$, $\mathbf{k}_2=\mathbf{p}''-\mathbf{p}$. Here the symbolic summations, $\sum_n$, $\sum_{n'}$, extend over some complete set of scattering (plus bound) states of the coupled meson-nucleon fields, the vector $|n)=|E_n,\mathbf{P},\gamma)$ being specified as an eigenvector of energy E_n of the total Hamiltonian, of total momentum $\mathbf{P}$, and some completing set of commuting operators γ.

The ξ, $\boldsymbol{\eta}$ coordinate integrations can be expressed in terms of the Fourier transform $j(\mathbf{k})$ of the current operator, where

$$j(\mathbf{k})=\int d^3\xi\exp(-i\mathbf{k}\cdot\xi)j(\xi). \quad (40)$$

However it proves more convenient to reintroduce with renewed generality the concept of a reduced current operator $J(\mathbf{k})$ already profitably employed in the treatment of $T^{(2)}$ [Eq. (32)]. Let $J(\mathbf{k})$ be defined by the expression

$$(\mathbf{p}|j(\mathbf{k})|n)=\delta(\mathbf{p}+\mathbf{k}-\mathbf{P}_n)(\mathbf{p}|J(\mathbf{k})|m\rangle, \tag{41}$$

where $|m\rangle$ designates just those mesons and nucleon pairs explicitly contained in $|n)$. That is, on the momentum shell we have explicitly

$$(\mathbf{p}|J(\mathbf{k})|m\rangle=(2\pi)^3(\mathbf{p}|j(\xi=0)|n). \tag{42}$$

With these preliminaries, we deduce

$$\begin{aligned} t^{(4)}(\mathbf{p}',\mathbf{p})=&\tfrac{1}{2}i(2\pi)^{-7}\int dk_0d^3\mathbf{p}''\{(k_0^2-\omega_1^2+i\epsilon)\\ &\times[(k_0+E(p)-E(p'))^2-\omega_2^2+i\epsilon)\}^{-1}\\ &\times\sum_m\left\{\frac{(\mathbf{p}'|J_i^{(1)}(-\mathbf{k}_1)|m\rangle\langle m|J_j^{(1)}(-\mathbf{k}_2)|\mathbf{p})}{[E_m-i\epsilon-E(p)-k_0]}\right.\\ &\left.+\frac{(\mathbf{p}'|J_j^{(1)}(-\mathbf{k}_2)|m\rangle\langle m|J_i^{(1)}(-\mathbf{k}_1)|\mathbf{p})}{[E_m-i\epsilon-E(p')+k_0]}\right\}\\ &\times\sum_{m'}\left\{\frac{(-\mathbf{p}'|J_i^{(2)}(k_1)|m'\rangle\langle m'|J_j^{(2)}(\mathbf{k}_2)|\mathbf{p})}{[E_{m'}-i\epsilon-E(p')+k_0]}\right.\\ &\left.+\frac{(-\mathbf{p}'|J_j^{(2)}(\mathbf{k}_2)|m'\rangle\langle m'|J_i^{(2)}(\mathbf{k}_1)|-\mathbf{p})}{[E_{m'}-i\epsilon-E(p)-k_0]}\right\}. \end{aligned} \tag{43}$$

It is our object in this section to study exclusively those contributions to Eq. (43) arising from the single nucleon states, typically $|\mathbf{p}''\lambda)$. This portion of $T^{(4)}$, which we shall call $T_{\text{pert}}^{(4)}$ can be readily written down.

We obtain, on the energy shell [i.e., we set $E(p')=E(p)$], this specialization not affecting the ultimate value of the adiabatic potential,

$$\begin{aligned} &t_{\text{pert}}^{(4)}(p',p;P_0=2E(p)=2E(p'))\\ &=\tfrac{1}{2}i(2\pi)^{-7}\int dk_0d^3\mathbf{p}''(k_0^2-\omega_1^2+i\epsilon)^{-1}(k_0^2-\omega_2^2+i\epsilon)^{-1}\\ &\times\left\{\frac{(\mathbf{p}'|J_i^{(1)}(-\mathbf{k}_1)|0\rangle\langle 0|J_j^{(1)}(-\mathbf{k}_2)|\mathbf{p})}{[E(\mathbf{p}+\mathbf{k}_2)-E(p)-k_0-i\epsilon]}\right.\\ &\left.\quad+\frac{(\mathbf{p}'|J_j^{(1)}(-\mathbf{k}_2)|0\rangle\langle 0|J_i^{(1)}(-\mathbf{k}_1)|\mathbf{p})}{[E(\mathbf{p}+\mathbf{k}_1)-E(p')-k_0-i\epsilon]}\right\}\\ &\times\left\{\frac{(-\mathbf{p}'|J_i^{(2)}(\mathbf{k}_1)|0\rangle\langle 0|J_j^{(2)}(\mathbf{k}_2)|-\mathbf{p})}{[E(\mathbf{p}+\mathbf{k}_2)-E(p')+k_0-i\epsilon]}\right.\\ &\left.\quad+\frac{(-\mathbf{p}'|J_j^{(2)}(\mathbf{k}_2)|0\rangle\langle 0|J_i^{(2)}(\mathbf{k}_1)|-\mathbf{p})}{[E(\mathbf{p}+\mathbf{k}_1)-E(p)-k_0-i\epsilon]}\right\}. \end{aligned} \tag{44}$$

The next step involves the performance of the k_0 interaction. Upon suitable rearrangement the result can be exhibited in a familiar form, namely, as the sum of contributions from all possible time-ordered diagrams describing the exchange of two mesons in which the nucleons are always in positive energy states, the so-called no-pair diagrams.[2] We may then pass immediately to the adiabatic limit except for those diagrams which contain an intermediate state with only two nucleons. The contribution from these latter potentially singular terms is found to be

$$\frac{1}{(2\pi)^6}\int d^3\mathbf{p}''\frac{\begin{array}{c}(\mathbf{p}'|J_i^{(1)}(-\mathbf{k}_1)|0\rangle\langle 0|J_j^{(1)}(-\mathbf{k}_2)|\mathbf{p})\\ \times(-\mathbf{p}'|J_i^{(2)}(\mathbf{k}_1)|0\rangle\langle 0|J_j^{(2)}(\mathbf{k}_2)|-\mathbf{p})\end{array}}{\begin{array}{c}2\omega_1\omega_2[E(p)-E(p'')-\omega_1]\\ \times[E(p)-E(p'')-\omega_2][E(p)-E(p'')]\end{array}}. \tag{45}$$

Turning momentarily to the computation of the potential $v_{\text{pert}}^{(4)}(\mathbf{p}',\mathbf{p};P_0=2E(p)=2E(p'))$, Eq. (45) is seen to be canceled precisely by the contribution to Eq. (25b) arising from the iteration of $v^{(2)}(\mathbf{p}',\mathbf{p};P_0)$. The remaining part of $t_{\text{pert}}^{(4)}(\mathbf{p}',\mathbf{p};P_0)$ may be identified immediately with $v_{\text{pert}}^{(4)}(\mathbf{p}'-\mathbf{p})$ in the adiabatic limit and is readily shown to be

$$\begin{aligned} &v_{\text{pert}}^{(4)}(\mathbf{p}'-\mathbf{p})\\ &=-(2\pi)^{-6}\int\frac{d^3\mathbf{p}''}{2\omega_1\omega_2}(\mathbf{p}'|J_i^{(1)}(-\mathbf{k}_1)|0\rangle\langle 0|J_j^{(1)}(-\mathbf{k}_2)|\mathbf{p})\\ &\times\left\{\frac{(-\mathbf{p}'|J_i^{(2)}(\mathbf{k}_1)|0\rangle\langle 0|J_j^{(2)}(\mathbf{k}_2)|-\mathbf{p})}{\omega_1\omega_2(\omega_1+\omega_2)}\right.\\ &\quad+\frac{(-\mathbf{p}'|J_j^{(2)}(\mathbf{k}_2)|0\rangle\langle 0|J_i^{(2)}(\mathbf{k}_1)|-\mathbf{p})}{(\omega_1+\omega_2)}\\ &\left.\quad\times\left(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}+\frac{1}{\omega_1\omega_2}\right)\right\}. \end{aligned} \tag{46}$$

After the introduction of the nonrelativistic forms of the J operators, Eq. (46) is in agreement with the form of the potential first proposed by Brueckner and Watson,[3] with one essential difference. The perturbation result which we have derived refers completely to renormalized vertices. Besides its proportionality, therefore, to the renormalized coupling constant, it also contains the nuclear form factor $\rho(k)$, if k is the momentum transfer at the vertex. As pointed out by Gartenhaus,[18] the assumption of a suitable form for this function—in the absence of a means of calculating it from first principles—may provide a cutoff at small distances of the otherwise singular potentials, without

[18] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).

having to introduce such cutoffs *ad hoc* in coordinate space. However we prefer to interpret the introduction of cores as reflecting the essential breakdown of the potential concept within separations $r_{\min}\sim\frac{1}{3}\mu$.

We may summarize the results of this section in the potentials

$$V^{(2)}(\mathbf{r})=-(2\pi)^{-3}(f/\mu)^2\int d^3\mathbf{k}\omega^{-2}|\rho(k)|^2 \times\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\boldsymbol{\sigma}^{(1)}\cdot\mathbf{k}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (47)$$

$$\begin{aligned}V_{\text{pert}}^{(4)}(\mathbf{r})=&-(2\pi)^{-6}(f/\mu)^4\\ &\times\int d^3\mathbf{k}_1d^3\mathbf{k}_2(2\omega_1\omega_2)^{-1}|\rho(k_1)|^2|\rho(k_2)|^2\\ &\times e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}\boldsymbol{\sigma}^{(1)}\cdot\mathbf{k}_2\boldsymbol{\sigma}^{(1)}\cdot\mathbf{k}_1\tau_j^{(1)}\tau_i^{(1)}\\ &\times\{\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k}_2\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k}_1\tau_j^{(2)}\tau_i^{(2)}[\omega_1\omega_2(\omega_1+\omega_2)]^{-1}\\ &+\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k}_1\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k}_2\tau_i^{(2)}\tau_j^{(2)}[(\omega_1^2(\omega_1+\omega_2))^{-1}\\ &\times(\omega_2^2(\omega_1+\omega_2))^{-1}+(\omega_1\omega_2(\omega_1+\omega_2))^{-1}]\}. \quad (48)\end{aligned}$$

V. SCATTERING CORRECTIONS

The remainder of our work will be concerned with an evaluation of that contribution to the potential from the nonperturbation part of $T^{(4)}$, which we shall continue to label by the same symbol. In this instance, the adiabatic limit of $t^{(4)}(\mathbf{p}',\mathbf{p};P_0)$ is unambiguous, finite and synonymous with that of $v^{(4)}(\mathbf{p}',\mathbf{p};P_0)$. In order to be able to evaluate these corrections, our considerations will perforce be more special than those of the previous sections. Here we shall find it expedient to consider the fixed extended source limit of the theory, in which pair intermediate states are excluded and to replace the Heisenberg operator $j_i(\xi)$ by its extended source analog. We shall assume further that the source interacts only with mesons in P states.

On examination of Eq. (43), we see that we require the fixed source limit of a matrix element such as $(\mathbf{p}|J(\mathbf{q})|m\rangle$, a quantity which can be nonvanishing off the momentum shell. It is therefore that quantity with a fixed source analog,

$$(\mathbf{p}|J(\mathbf{q})|m\rangle\cong\langle 0|J(\mathbf{q})|m\rangle, \quad (49)$$

where $J(\mathbf{q})$ is the Fourier transform of the fixed source nucleon density and the eigenstates are those of the fixed source theory. $J(\mathbf{q})$ is taken to be independent of meson variables and to have the one nucleon expectation value

$$\langle 0|J_i(\mathbf{q})|0\rangle=(f/\mu)(i\boldsymbol{\sigma}\cdot\mathbf{q}\tau_i\rho(q)), \quad (50)$$

where $\rho(q)$ may be presumed to be the same source function as occurs in the relativistic theory.

The passage to the adiabatic potential will be made rather more directly here than previously. Introducing the results of Eqs. (43) and (49) into Eq. (29), we obtain directly for the nonperturbative fourth-order potential:

$$\begin{aligned}V^{(4)}(\mathbf{r})=&\tfrac{1}{2}i(2\pi)^{-7}\int d^3\mathbf{k}_1d^3\mathbf{k}_2dk_0(k_0^2-\omega_1^2+i\epsilon)^{-1}\\ &\times(k_0^2-\omega_2^2+i\epsilon)^{-1}e^{-i\mathbf{k}_1\cdot\mathbf{r}}e^{i\mathbf{k}_2\cdot\mathbf{r}}\\ &\times\sum_m{}'\left[\frac{\langle 0|J_i^{(1)}(\mathbf{k}_1)|m\rangle\langle m|J_j^{(1)}(-\mathbf{k}_2)|0\rangle}{(E_m-k_0-i\epsilon)}+\frac{\langle 0|J_j^{(1)}(-\mathbf{k}_2)|m\rangle\langle m|J_i^{(1)}(\mathbf{k}_1)|0\rangle}{(E_m+k_0-i\epsilon)}\right]\\ &\times\sum_{m'}{}'\left[\frac{\langle 0|J_i^{(2)}(-\mathbf{k}_1)|m'\rangle\langle m'|J_j^{(2)}(\mathbf{k}_2)|0\rangle}{(E_{m'}+k_0-i\epsilon)}+\frac{\langle 0|J_j^{(2)}(\mathbf{k}_2)|m'\rangle\langle m'|J_i^{(2)}(-\mathbf{k}_1)|0\rangle}{(E_{m'}-k_0-i\epsilon)}\right]. \quad (51)\end{aligned}$$

The energies E_n are henceforth to be measured relative to that of the one-nucleon state. In the primed summations over m and m' all values occur except that both indices cannot simultaneously refer to the ground state since this contribution is already contained in $V_{\text{pert}}^{(4)}(\mathbf{r})$.

We shall convert Eq. (51) to an expression involving the total cross sections for pion-nucleon interaction. We introduce the total cross section $\sigma_{2I,2J}(E_n)$, in a state of definite isotopic spin I and angular momentum J. In particular, in the elastic approximation $\sigma_{2I,2J}$ reduces to the familiar form

$$\sigma_{2I,2J}(E_n)=(2\pi/k_n^2)(2J+1)\sin^2\delta_{2I,2J}(E_n). \quad (52)$$

Evaluation of the potential, Eq. (51), is then expedited by the following lemma (proven in Appendix B):

$$\begin{aligned}\sum_{m=1}G(E_m)\langle 0|J_i(\mathbf{k})|m\rangle\langle m|J_j(-\mathbf{k}')|0\rangle =&\sum_{I,J}(2J+1)^{-1}(i|P(2I)|j)(\hat{k}|F(2J)|\hat{k}')8kk'\rho(k)\rho(k')\\ &\times\int d\omega_n\frac{G(\omega_n)\sigma_{2I,2J}(\omega_n)}{k_n|\rho(k_n)|^2}, \quad (53)\end{aligned}$$

where $\omega_n^2=k_n^2+\mu^2$, and $G(\omega_n)$ is some reasonably behaved function of ω_n. The existence of the above relationship depends crucially upon the possibility of relating the meson-nucleon T-matrix values off and on the energy shell in the fixed source gradient coupling theory. The equations

$$\sum_{J_3}(\hat{k}_1|J,J_3)(J,J_3|\hat{k}_2)=(\hat{k}_1|F(2J)|\hat{k}_2), \quad (54)$$

$$\sum_{I_3}(i|I,I_3)(I,I_3|j)=(i|P(2I)|j), \quad (55)$$

define the matrix elements of the projection operators for angular momentum J and isotopic spin I, respectively, suitably compounded from the ingredient angular momenta and isotopic spins. For the P states we recall that

$$(\hat{k}_1|F(2J)|\hat{k}_2)=\frac{1}{4\pi}\left\{\binom{1}{2}\hat{k}_1\cdot\hat{k}_2+\binom{1}{-1}i\boldsymbol{\sigma}\cdot\hat{k}_1\times\hat{k}_2\right\} \quad (56)$$

for $J=\frac{1}{2}$, and $\frac{3}{2}$, respectively, and

$$(i|P(2I)|j)=\tfrac{1}{3}\left\{\binom{1}{2}\delta_{ij}+\binom{1}{-1}i\epsilon_{ijl}\tau_l\right\}. \quad (57)$$

We next state the result of inserting Eq. (53) into Eq. (51). Making use of the symmetry properties of the integral, we write

$$V^{(4)}(\mathbf{r})=V_{\mathrm{I}}^{(4)}(\mathbf{r})+V_{\mathrm{II}}^{(4)}(\mathbf{r}), \quad (58)$$

where $V_{\mathrm{I}}^{(4)}(\mathbf{r})$ is the term which contains scattering corrections for only one of the nucleons, chosen as nucleon two in the following unsymmetrical mode of writing:

$$\begin{aligned}V_{\mathrm{I}}^{(4)}(\mathbf{r})=&\,i(2\pi)^{-7}\int d^3\mathbf{k}_1d^3\mathbf{k}_2dk_0(k_0^2-\omega_1^2+i\epsilon)^{-1}\\&\times(k_0^2-\omega_2^2+i\epsilon)^{-1}e^{-i\mathbf{k}_1\cdot\mathbf{r}}e^{i\mathbf{k}_2\cdot\mathbf{r}}\\&\times 8(f/\mu)^2k_1^2k_2^2|\rho(k_1)|^2|\rho(k_2)|^2\\&\times\left\{\frac{\boldsymbol{\sigma}^{(1)}\cdot\hat{k}_1\boldsymbol{\sigma}^{(1)}\cdot\hat{k}_2\tau_i^{(1)}\tau_j^{(1)}}{-k_0-i\epsilon}+\frac{\boldsymbol{\sigma}^{(1)}\cdot\hat{k}_2\boldsymbol{\sigma}^{(1)}\cdot\hat{k}_1\tau_j^{(1)}\tau_i^{(1)}}{k_0-i\epsilon}\right\}\\&\times\sum_{I,J}(2J+1)^{-1}\int_\mu^\infty\frac{d\omega_p\sigma_{2I,2J}}{p|\rho(p)|^2}\\&\times\{(i|P^{(2)}(2I)|j)(\hat{k}_1|F^{(2)}(2J)|\hat{k}_2)\\&\times(\omega_p+k_0-i\epsilon)^{-1}+(j|P^{(2)}(2I)|i)\\&\times(\hat{k}_2|F^{(2)}(2J)|\hat{k}_1)(\omega_p-k_0-i\epsilon)^{-1}\}. \quad (59)\end{aligned}$$

It is left implicit that $V_{\mathrm{I}}^{(4)}$ is to be identified with that part of the above expression symmetric in $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$; $\boldsymbol{\tau}^{(1)}$, $\boldsymbol{\tau}^{(2)}$.

On the other hand, $V_{\mathrm{II}}^{(4)}(r)$ describes scattering corrections on both nucleons,

$$\begin{aligned}V_{\mathrm{II}}^{(4)}(\mathbf{r})=&\,i(2\pi)^{-7}\int d^3\mathbf{k}_1d^3\mathbf{k}_2dk_0\\&\times(k_0^2-\omega_1^2+i\epsilon)^{-1}(k_0^2-\omega_2^2+i\epsilon)^{-1}\\&\times e^{-i\mathbf{k}_1\cdot\mathbf{r}}e^{i\mathbf{k}_2\cdot\mathbf{r}}64k_1^2k_2^2|\rho(k_1)|^2|\rho(k_2)|^2\\&\times\sum_{I,J}\sum_{I',J'}(2J+1)^{-1}(2J'+1)^{-1}\\&\times\int_\mu^\infty\frac{d\omega_p\sigma_{2I,2J}(\omega_p)}{p|\rho(p)|^2}\int_\mu^\infty\frac{d\omega_t\sigma_{2I',2J'}(\omega_t)}{t|\rho(t)|^2}\\&\times\{(i|P^{(1)}(2I)|j)(\hat{k}_1|F^{(1)}(2J)|\hat{k}_2)\\&\times(\omega_p-k_0-i\epsilon)^{-1}+(j|P^{(1)}(2I)|i)\\&\times(\hat{k}_2|F^{(1)}(2J)|\hat{k}_1)(\omega_p+k_0-i\epsilon)^{-1}\}\\&\times\{(i|P^{(2)}(2I')|j)(\hat{k}_1|F(2J')|\hat{k}_2)\\&\times(\omega_t+k_0-i\epsilon)^{-1}\}. \quad (60)\end{aligned}$$

The remaining task for the present paper is to perform the well-defined k_0 integrals. For $V_{\mathrm{I}}^{(4)}(\mathbf{r})$, we obtain rather immediately

$$\begin{aligned}V_{\mathrm{I}}^{(4)}(\mathbf{r})=&\,(2\pi)^{-6}\int d^3\mathbf{k}_1d^3\mathbf{k}_2\\&\times e^{-i\mathbf{k}_1\cdot\mathbf{r}}e^{i\mathbf{k}_2\cdot\mathbf{r}}8(f/\mu)^2k_1^2k_2^2|\rho(k_1)|^2|\rho(k_2)|^2\\&\times\int_\mu^\infty\frac{d\omega_p}{p|\rho(p)|^2}\sum_{I,J}(2J+1)^{-1}\sigma_{2I,2J}(\omega_p)\\&\times(i|P^{(2)}(2I)|j)(\hat{k}_1|F^{(2)}(2J)|\hat{k}_2)\\&\times\left\{\boldsymbol{\sigma}^{(1)}\cdot\hat{k}_1\boldsymbol{\sigma}^{(2)}\cdot\hat{k}_2\tau_i^{(1)}\tau_j^{(1)}\left[-\frac{2}{\omega_p\omega_1^2\omega_2^2}\right.\right.\\&\left.-\frac{1}{(\omega_1^2-\omega_2^2)}\left(\frac{1}{\omega_1^2(\omega_1+\omega_p)}-(1\leftrightarrow 2)\right)\right]\\&+\boldsymbol{\sigma}^{(1)}\cdot\hat{k}_2\boldsymbol{\sigma}^{(1)}\cdot\hat{k}_1\tau_j^{(1)}\tau_i^{(1)}\\&\left.\times\left[\frac{1}{(\omega_1^2-\omega_2^2)}\left(\frac{1}{\omega_1^2(\omega_1+\omega_p)}-(1\leftrightarrow 2)\right]\right]\right\}. \quad (61)\end{aligned}$$

For $V_{\mathrm{II}}^{(4)}(r)$, after some algebraic manipulation, we find

$$
\begin{aligned}
V_{\mathrm{II}}^{(4)}(\mathbf{r}) = {} & (2\pi)^{-6}\int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \\
& \times e^{-i\mathbf{k}_1\cdot\mathbf{r}} e^{i\mathbf{k}_2\cdot\mathbf{r}} 64 k_1^2 k_2^2 |\rho(k_1)|^2 |\rho(k_2)|^2 \\
& \times \int_\mu^\infty \frac{d\omega_p}{p|\rho(p)|^2} \int_\mu^\infty \frac{d\omega_t}{t|\rho(t)|^2} \sum_{I,J}\sum_{I',J'} (2J+1)^{-1} \\
& \times (2J'+1)^{-1} \sigma_{2I,2J}(\omega_p) \sigma_{2I'2J'}(\omega_t) \\
& \times \{(i|P^{(2)}(2I')|j)(\hat{k}_1|F^{(2)}(2J')|\hat{k}_2)\} \\
& \times \Bigg\{ (i|P^{(1)}(2I)|j)(\hat{k}_1|F^{(1)}(2J)|\hat{k}_2) \\
& \times \left[\frac{1}{2}\frac{1}{(\omega_1^2-\omega_2^2)}\left(\frac{1}{\omega_1(\omega_1+\omega_t)(\omega_p+\omega_t)} \right.\right. \\
& \left.\left. + \frac{1}{\omega_1(\omega_1+\omega_p)(\omega_p+\omega_t)} - (1\leftrightarrow 2)\right)\right] \\
& + (j|P^{(1)}(2I)|i)(\hat{k}_2|F^{(1)}(2J)|\hat{k}_1)\left[\frac{1}{2}\frac{1}{(\omega_1^2-\omega_2^2)} \right. \\
& \left. \times \left(\frac{1}{\omega_1(\omega_1+\omega_p)(\omega_1+\omega_t)} - (1\leftrightarrow 2)\right)\right]. \qquad (62)
\end{aligned}
$$

The total proposed potential therefore consists of

$$V(\mathbf{r}) = V^{(2)}(\mathbf{r}) + V_{\mathrm{pert}}^{(4)}(\mathbf{r}) + V_{\mathrm{I}}^{(4)}(\mathbf{r}) + V_{\mathrm{II}}^{(4)}(\mathbf{r}), \qquad (63)$$

as given by Eqs. (47), (48), (61), and (62), respectively. The evaluation of these expressions and their comparison with experiment will be studied in subsequent publications. We shall then also consider alternative formulations and possible extensions.

APPENDIX A

We turn to the derivation of the expansion, Eqs. (7) and (8) of the text for the two-nucleon Green's function G_{12}. Schwinger[13] has shown that $G_{12}[\langle\phi\rangle]$ satisfies the equation

$$\mathfrak{F}_1\mathfrak{F}_2 G_{12}[\langle\phi\rangle] = 1, \qquad (A.1)$$

where

$$\mathfrak{F} = \gamma p + M - \bar{\gamma}(\langle\phi\rangle - i\delta/\delta J) \equiv G^{-1}[\langle\phi\rangle - i\delta/\delta J] \quad (A.2)$$

is the differentio-functional operator which is the inverse of the single nucleon propagator, $\bar{\gamma}$ characterizes the particular form of the coupling, and 1 is the unit antisymmetric matrix. Neglecting the antisymmetrization once again, we are interested in $G_{12} = G_{12}[0]$,

$$G_{12} = G_1[\langle\phi\rangle - i\delta/\delta J] G_2[\langle\phi\rangle - i\delta/\delta J]|_{\langle\phi\rangle=0}, \quad (A.3)$$

a structure which is actually more general than the explicit linear dependence on the meson field operator implied by (A.2).

The reworking of (A.3) makes repeated use of the operational form of the functional Taylor series. For example, we first invoke the relation

$$
\begin{aligned}
& G_2[\langle\phi\rangle - i\delta/\delta J]|_{\langle\phi\rangle=0} \\
& \quad = \exp[\langle\phi\rangle\delta/\delta\chi] G_2[\chi - i\delta/\delta J - i\Delta'\delta/\delta\chi]|_{\chi=\langle\phi\rangle=0}, \quad (A.4)
\end{aligned}
$$

where $\Delta' = \delta\langle\phi\rangle/\delta J$, and then the relation

$$
\begin{aligned}
& G_1[\langle\phi\rangle - i\delta/\delta J]\exp[\langle\phi\rangle\delta/\delta\chi]|_{\langle\phi\rangle=0} \\
& \qquad = G_1[\langle\phi\rangle - i\delta/\delta J - i\Delta'\delta/\delta\chi], \quad (A.5)
\end{aligned}
$$

to establish that without approximation

$$
\begin{aligned}
G_{12} &= G_1[\langle\phi\rangle - i\delta/\delta J - i\Delta'\delta/\delta\chi] \\
& \qquad \times G_2[\chi - i\delta/\delta J - i\Delta'\delta/\delta\chi]|_{\chi=\langle\phi\rangle=0} \\
&= G_1[\langle\phi\rangle - i\Delta'\delta/\delta\langle\phi\rangle - i\Delta'\delta/\delta\chi] \\
& \qquad \times G_2[\chi - i\Delta'\delta/\delta\langle\phi\rangle - i\Delta'\delta/\delta\chi]|_{\chi=\langle\phi\rangle=0}. \quad (A.6)
\end{aligned}
$$

Further progress will be considered here only with the restriction to the linear meson approximation defined by the condition

$$\delta^n\Delta'/\delta\langle\phi\rangle^n = 0, \quad n \geqslant 1. \qquad (A.7)$$

Under the assumption (A.7), it is then true that

$$
\begin{aligned}
G_{12} &= \exp[-i(\delta/\delta\langle\phi\rangle)\Delta'\delta/\delta\chi] \\
& \qquad \times G_1[\langle\phi\rangle - i\Delta'\delta/\delta\langle\phi\rangle] G_2[\chi - i\Delta'\delta/\delta\chi]|_{\chi=\langle\phi\rangle=0} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (\delta/\delta\langle\phi\rangle)^n G_1|_{\langle\phi\rangle=0} (\Delta'\delta/\delta\chi)^n G_2|_{\chi=0}, \quad (A.8)
\end{aligned}
$$

which was to be established.

APPENDIX B

We wish to convert a summation of the form

$$\textstyle\sum_m G(E_m)\langle 0|J_i(\mathbf{k})|m\rangle\langle m|J_j(-\mathbf{k}')|0\rangle$$

into an expression involving only the total cross-sections for pion-nucleon interaction. For this purpose we require the connection between the reduced nonrelativistic T matrix and matrix elements of the reduced current operator $J_i(\mathbf{k})$, mainly

$$\langle m|T|\mathbf{k},i\rangle = -(2\pi)^{-\frac{3}{2}}(2\omega)^{-\frac{1}{2}}\langle m|J^\dagger(\mathbf{k})|0\rangle. \qquad (B1)$$

We choose, however, to derive the more general relativistic formula

$$
\begin{aligned}
& (n|T|\mathbf{p},\lambda;\mathbf{k},i) \\
& \quad = -\delta(\mathbf{P}_n - \mathbf{k} - \mathbf{p})(2\pi)^{-\frac{3}{2}}(2\omega)^{-\frac{1}{2}}\langle m|J_i^\dagger(\mathbf{k})|\mathbf{p},\lambda), \quad (B2)
\end{aligned}
$$

where the relativistic matrix element of the current operator has been previously defined in Eq. (41). In the center-of-mass frame the nonrelativistic approximation, Eq. (B1), obviously follows from Eq. (B2) by Eq. (49) of the text. The renormalized S matrix to some arbitrary final state $|n)$ can be given as

$$(n|S|\mathbf{p},\mathbf{k})=\lim(\sigma_2\to-\infty)\int d\sigma_2 Z_3^{-\frac{1}{2}}(n|\phi(\xi)|\mathbf{p})\times i(\overrightarrow{\partial}_0-\overleftarrow{\partial}_0)\Phi_k(\xi), \quad \text{(B3)}$$

where

$$\Phi_k(\xi)=(2\pi)^{-\frac{3}{2}}(2\omega_k)^{-\frac{1}{2}}e^{ik\cdot\xi}.$$

The dynamical variable $\phi(\xi)$ can by Eq. (14) be replaced by

$$Z_3^{-\frac{1}{2}}\phi(\xi)=Z_3^{-\frac{1}{2}}\phi_0(\xi)+\int(d\xi')\Delta'(\xi,\xi')j_R(\xi'), \quad \text{(B4)}$$

where $\phi_0(\xi)$, the homogeneous solution, gives rise in Eq. (B3) to the unit matrix term of the S matrix. We have then

$$(n|T|\mathbf{p},\mathbf{k})=\lim(\sigma_2\to-\infty)\int d\sigma_2\int(d\xi')(n|j_R(\xi')|\mathbf{p})\times\Delta'(\xi',\xi)(\overrightarrow{\partial}_0-\overleftarrow{\partial}_0)\Phi_k(\xi). \quad \text{(B5)}$$

If we evoke the renormalization statement

$$i\int_{\sigma_2} d\sigma_2\Delta'(\xi',\xi)(\overrightarrow{\partial}_0-\overleftarrow{\partial}_0)\Phi_k(\xi)=i\Phi_k(\xi), \quad (\xi_0'>\xi_0), \quad \text{(B6)}$$

we readily deduce

$$(n|T|\mathbf{p},\mathbf{k})=i(2\pi)^{-\frac{3}{2}}(2\omega)^{-\frac{1}{2}}\int(d\xi')(n|j_R(\xi')|\mathbf{p})e^{ik\xi}, \quad \text{(B7)}$$

or upon reduction, Eq. (B2).

The total cross section for an incident meson of isotopic index i is

$$\sigma_i(\omega_k)=(2\pi)^4(\omega_k/k)\textstyle\sum_n\delta(\omega_k-E_n)|(n|T|\mathbf{k},i)|^2, \quad \text{(B8)}$$

and if $G(\omega_k)$ is a function of ω_k, we have

$$\int G(\omega_k)\sigma_i(\omega_k)d\omega_k=\textstyle\sum_n(2\pi)^4(\omega_n/k_n)G(\omega_n)|(n|T|\mathbf{k},i)|^2, \quad \text{(B9)}$$

where $\mathbf{k}_n$ is a vector in the same direction as $\mathbf{k}$ but of length determined by the equation $\omega_n^2=k_n^2+\mu^2$.

In Eq. (53) we have a somewhat more general expression to consider, namely, one of the form

$$\sum_{m\neq 0}G(E_n)\langle\mathbf{k}_1,i|T^\dagger|m\rangle\langle m|T|\mathbf{k}_2,j\rangle. \quad \text{(B10)}$$

For the set n, we choose E_n, J, J_Z, I, I_3, γ which, in addition to the energy, comprise the total angular momentum and its z component, the total isotopic spin and its third component, and the completing set γ. We then have, for example,

$$(n|T|\mathbf{k}_2,j)=\sum_{J_3,I_3}(E_n,J,I,\gamma|T|\mathbf{k}_2,J,I)\times(J,J_3|\hat{k}_2)(I,I_3|j). \quad \text{(B11)}$$

where, on the energy shell, we shall introduce a quantity

$$T_{2I,2J,\gamma}(E_n)=(k_n\omega_n)^{\frac{1}{2}}(\omega_n,J,I|T|k_n,J,I). \quad \text{(B12)}$$

Finally, we shall define the total cross section, $\sigma_{2I,2J}(E_n)$, in a state of definite isotopic spin and angular momentum, by the equation

$$\sigma_{2I,2J}(E_n)=\textstyle\sum_\gamma(2\pi/k_n^2)(2J+1)\pi^2|T_{2I,2J,\gamma}(E_n)|^2. \quad \text{(B13)}$$

This formula goes over in the elastic approximation, where we agree to limit the sum over γ to the single meson states only, to the more usual form

$$\sigma_{2I,2J}(E_n)=(2\pi/k_n^2)(2J+1)\sin^2\delta_{2I,2J}(E_n) \quad \text{(B14)}$$

upon the recognition that, for $\gamma=0$,

$$T_{2I,2J}(E_n)=-(1/\pi)e^{i\delta_{2I,2J}}\sin\delta_{2I,2J}. \quad \text{(B15)}$$

The essential simplicity of the fixed source gradient coupling limit is that it becomes possible to relate the T-matrix values off and on the energy shell according to

$$\begin{aligned}(E_n,J,I|T|k_2,J,I)&=[\rho(k_2)/\rho(k_n)](k_2E_n^{\frac{1}{2}}/k_n\omega_2^{\frac{1}{2}})(E_n,J,I|T|k_n,J,I)\\&=[\rho(k_2)/\rho(k_n)](k_2/k_n^{\frac{3}{2}}\omega_2^{\frac{1}{2}})T_{2I,2J,\gamma}(E_n).\end{aligned} \quad \text{(B16)}$$

Collecting the results of Eqs. (B11)–(B13), (B16) and Eqs. (54), (55) of the text, inserting these into Eq. (B10), we obtain the requisite lemma, Eq. (53) of Sec. V.