nucleon pairs. The matrix elements we want are simply

$$\langle \bar{p}|\Gamma_i|n\rangle = -\langle p|\gamma_5\Gamma_i|n\rangle.$$

Referring to (4.5), we find that the results for g''/g for P and A are given by the brackets g'/g appropriate to S and V.

6. CONCLUSIONS

The *a priori* attractive suggestion that Fermi interactions have some simple form with respect to bare nucleons has been examined. It is found very difficult to test experimentally. With regard to a universal Fermi interaction, no direct verification seems likely. The suggestion does not lead to immediate simplifications, although it is indicated that the universal couplings that should be considered are different from those which have received most attention recently. With regard to forbidden β-decay processes, we can in principle observe the effects of this suggestion. But these effects depend theoretically on models for both a recoiling nucleon and for the nucleus.

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Interaction of *P*- and *S*-Wave Pions with Fixed Nucleons

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A method is given of replacing pion scattering parts in a Feynman diagram by experimentally observed quantities. In this paper momenta of nucleons are neglected, although nucleon pair creation is not. It is assumed that only P- and S-wave pions interact with nucleons. No interaction is assumed between pions. Two examples are given: (1) The anomalous magnetic moment of the proton is rigorously expressed in terms of pion-nucleon scattering amplitudes, or, alternatively, in terms of the renormalized coupling constant and the total cross sections for pions. (2) The internucleon potential is also expressed by means of scattering quantities. In this case the number of virtual pions exchanged between the two nucleons is limited to two, although the number of pions emitted and absorbed by the same nucleon is not limited.

I. INTRODUCTION

THE static model of the pion-nucleon interaction has proved to be quite powerful in correlating certain experiments. As far as the low-energy scattering of P-wave pions by a nucleon is concerned, this theory is very successful. Experiments show that δ_{33}, the phase shift for the state with $J=\frac{3}{2}$ and $I=\frac{3}{2}$ is very large, while the other three phase shifts are small. This comes from the simple fact that the pion-nucleon interaction for the $\frac{3}{2}$–$\frac{3}{2}$ state is attractive while for all other states it is repulsive. Thus almost every method, the Tamm-Dancoff approximation,[1] the Tomonaga intermediate coupling approximation,[2] or the Chew-Low method,[3] gives satisfactory agreement with experiment if the field reaction is taken into account.

Granted that this scattering problem has been solved, how can other quantities like the anomalous magnetic moment or nuclear forces be calculated with similar accuracy? The purpose of this paper is to describe a method to express these quantities in terms of scattering quantities.

As an example, suppose that one wants to calculate the anomalous magnetic moment of the proton. One draws a Feynman diagram as in Fig. 1. The shaded area contains a number of virtual pions emitted and absorbed by the nucleon. The sum over the virtual interactions represented by this shaded area is identical to the graph which appears also in pion-nucleon scattering. Let us call this contribution a scattering part, which means the sum of all Feynman graphs with two external free nucleon lines and two external (free or virtual) pion lines. This scattering part is equal to the S-matrix element if the two pion lines are free. The difference here is that the pions are virtual and do not satisfy the energy relation $k_0^2=k^2+\mu^2$ as real pions do. This difficulty is overcome, however, in the static approximation.

We make the following assumptions:

(1) The static approximation is applicable; that is, the momenta of the nucleons and antinucleons (if any) can be neglected.

(2) There is no interaction between pions.

(3) The pion-nucleon interaction does not contain higher derivatives of the field. For P waves it is sufficient to assume the usual pseudovector coupling, although we do not use its explicit form. The S-wave interaction is unknown. We assume only the regularity of the interaction (see Sec. V). D and higher waves are assumed to have no interaction with the nucleon.

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[1] G. F. Chew, Phys. Rev. **89**, 591 (1953); K. Sawada, Progr. Theoret. Phys. (Japan) **9**, 455 (1953).
[2] G. Takeda, Phys. Rev. **95**, 1078 (1954). Friedman, Lee, and Christian, Phys. Rev. **100**, 1494 (1956).
[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

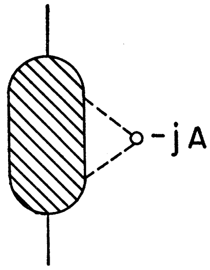


FIG. 1. Feynman graph for the nucleon magnetic moment.

Under these assumptions it is possible to bring the scattering part off the energy shell to the virtual case. For the moment we shall consider only the P wave, which is well known.

Suppose one changes the space momentum k of one of the pions of a scattering part without changing its fourth component k_0. The pion line does not interact with other pion lines but is absorbed by the nucleon. Since the momentum of the nucleon is neglected, the change of momentum of the pion gives no effect except at the time of absorption. This interaction is proportional to k since the P-wave pion is absorbed. The whole scattering part, therefore, depends linearly on the space momentum although it depends nontrivially on the fourth component. The general case of a virtual pion with momentum k', k_0 can be calculated from the value of the real scattering with momentum $k[=(k_0^2-\mu^2)^{\frac{1}{2}}]$, k_0 merely by multiplying with k'/k.[4] This is an important element of simplicity in the static approximation.

In the evaluation of the S matrix, the integral over k_0 extends from minus infinity to plus infinity. On the other hand, real scattering occurs only for $k_0>\mu$. This gap can be filled by the use of dispersion relations[5] or Low's equations.[6] These relations show that the scattering amplitudes can be expanded in a Mittag-Leffler series, with coefficients given by the total cross sections and the renormalized coupling constant. In this form the scattering amplitude is defined for all values of k_0 from minus infinity to plus infinity. Thus the scattering part is completely known.

Application of this method to the anomalous magnetic moment problem is given in Sec. III. As an example of the application to many-nucleon problems, the fourth order nuclear potential is calculated in Sec. IV. In Sec. V, the effect of S-wave interaction is investigated. The next Sec. II, is devoted to preliminaries for the following sections.

II. S MATRIX FOR FIXED NUCLEONS

In this section we summarize the S-matrix formulas for the case of fixed nucleons. Only P waves are considered in this and the following two sections. The rules for constructing the S matrix are:

(1) For each pion line, we use

$$\frac{i\delta_{ij}}{(2\pi)^4}\left(\frac{1}{k_0^2-\omega_k^2+i\epsilon}\right),\quad \omega_k^2=k^2+\mu^2.$$

(2) For each nucleon line, which merely gives time-ordering in this case, we use

$$\frac{i}{2\pi}\left(\frac{1}{k_0+i\epsilon}\right).$$

(3) If the pseudovector interaction is assumed, for each interaction,[7] we use

$$2\pi\delta(\textstyle\sum k_0)(f_0/\mu)\tau_i\boldsymbol{\sigma}\cdot\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}},$$

where f_0 is the unrenormalized coupling constant and $\mathbf{x}$ is the position of the nucleon.

(4) After these substitutions, we integrate over all momenta of internal lines.

The matrix element for the scattering part of an ith pion with momentum k to a jth pion with momentum q is given by

$$\langle j,q|S|i,k\rangle=-\int e^{ikx-iqy}\langle 1|P(O_j(y),O_i(x))|\rangle d^4x d^4y,$$

where $|\rangle$ and $|1\rangle$ are the initial and final nucleon states respectively and

$$O_i(x)=(\Box^2-\mu^2)\phi_i(x).$$

We express this quantity as

$$\begin{aligned}\langle j,q|S|i,k\rangle=2\pi i\delta(k_0-q_0)[A(k_0)\tau_i\tau_j\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{q}+B(k_0)\\ \times(\tau_i\tau_j\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}+\tau_j\tau_i\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{q})\\ +C(k_0)\tau_j\tau_i\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}]e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}}.\end{aligned}\quad (1)$$

The dispersion relations or the Low equations show that

$$\begin{aligned}A(k_0)&=\frac{f^2}{\mu^2}\left(\frac{1}{k_0-i\epsilon}\right)+\frac{1}{4\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_{33}(p)}{\omega_p-k_0-i\epsilon}\right)\\&\quad+\frac{1}{36\pi}\left(\frac{dp}{\omega_p}\right)\left(\frac{4\sigma_{11}+4\sigma_{13}+\sigma_{33}}{\omega_p+k_0-i\epsilon}\right),\\ B(k_0)&=\frac{1}{12\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_{33}+2\sigma_{13}}{\omega_p-k_0-i\epsilon}\right)\\&\quad+\frac{1}{12\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_{33}+2\sigma_{13}}{\omega_p+k_0-i\epsilon}\right),\qquad(2)\\ C(k_0)&=-\frac{f^2}{\mu^2}\left(\frac{1}{k_0+i\epsilon}\right)+\frac{1}{36\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{4\sigma_{11}+4\sigma_{13}+\sigma_{33}}{\omega_p-k_0-i\epsilon}\right)\\&\quad+\frac{1}{4\pi}\int_0^\infty\frac{dp}{\omega_p}\left(\frac{\sigma_{33}}{\omega_p+k_0-i\epsilon}\right),\end{aligned}$$

[4] For the sake of simplicity, the cutoff factor is omitted throughout this paper. If an explicit cut-off factor $v(k)$ is included in the Hamiltonian, we must multiply by $k'v(k')/kv(k)$.

[5] Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955); R. Oehme, Phys. Rev. **102**, 1174 (1956).

[6] F. Low, Phys. Rev. **97**, 1392 (1955).

[7] This expression is the interaction multiplied by $-i$. The $-i$ comes from $(-i)^n$ in the nth-order term of the S matrix.

where f is the renormalized coupling constant and σ_{ij} is the total cross section of the pure $i/2$, $j/2$ state. A, B, and C satisfy the crossing relation

$$A(-k_0)=C(k_0),\quad B(-k_0)=B(k_0). \tag{3}$$

The scattering part given by Eq. (1) is linear in k and q. For the reason stated in the introduction, the relation (1) holds for all types of pions. That is, k and q do not necessarily satisfy the relation

$$k_0^2=k^2+\mu^2,\quad q_0^2=q^2+\mu^2,\quad \text{or}\quad k_0>\mu,\quad q_0>\mu,$$

as real pions do.

III. ANOMALOUS MAGNETIC MOMENT OF THE PROTON

We shall calculate only the anomalous magnetic moment that comes from the pion current. For this we calculate the S-matrix element for the diagram of Fig. 1 according to the rules of Sec. II. The interaction of the pion line with an external constant magnetic field $\mathbf{H}$ is given by[7]

$$\langle j,q|H|i,k\rangle=-\tfrac{1}{2}ie\mathbf{H}\cdot(2\pi)^4(\delta_{1i}\delta_{2j}-\delta_{2i}\delta_{1j})\times(\mathbf{k}+\mathbf{q})\times\nabla\delta^4(k-q).$$

Since the shaded area is a scattering part, we have

$$\sum_{ij}\int\int\langle j,q|S|i,k\rangle\langle j,-q|H|i,-k\rangle\times\frac{-1}{(2\pi)^8}\left(\frac{d^4k}{k_0^2-\omega_k^2}\right)\left(\frac{d^4q}{q_0^2-\omega_q^2}\right), \tag{4}$$

where ω's are supposed to have a small negative imaginary part. This is not the correct matrix element, however. The scattering part contains two types of graphs, one with crossed and the other with uncrossed external pion lines. Both types give identical graphs if the two external pion lines are joined to the same point as in Fig. 1. The correct matrix element is, therefore, one half of (4).

$$S=2\pi i\times\tfrac{1}{2}\int\int\delta(k_0-q_0)[A(k_0)\tau_i\tau_j\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{q}+\cdots]\times e^{i\mathbf{x}\cdot(\mathbf{k}-\mathbf{q})}\frac{-1}{(2\pi)^8}\left(\frac{d^4kd^4q}{(k_0^2-\omega_k^2)(q_0^2-\omega_q^2)}\right)\times\left(-i\frac{e\mathbf{H}}{2}\right)(2\pi)^4(\delta_{1i}\delta_{2j}-\delta_{2i}\delta_{1j})(\mathbf{k}+\mathbf{q})\times\nabla\delta^4(k-q).$$

After carrying out the integration over q and the angles of $\mathbf{k}$, this reduces to

$$S=\delta(0)\frac{e\tau_3\boldsymbol{\sigma}\cdot\mathbf{H}}{(2\pi)^2}\times\frac{4}{3}\int\frac{A(k_0)-2B+C}{(k_0^2-\omega_k^2)^2}k^4dkdk_0.$$

Substituting (2) for A, B, and C and performing the k_0 integration, we have

$$S=-2\pi i\delta(0)(-\mathbf{H})\cdot\boldsymbol{\sigma}\tau_3\left\{e\left(\frac{f^2}{4\pi}\right)\left(\frac{4}{3\pi}\right)\left(\frac{1}{\mu^2}\right)\int\frac{k^4}{\omega_k^4}dk+\frac{e}{54\pi^3}\int\int[\sigma_{11}(p)-2\sigma_{13}+\sigma_{33}]\times\left[\frac{1}{\omega_k}+\frac{1}{\omega_k+\omega_p}\right]\left(\frac{k^4dkdp}{\omega_k^2\omega_p(\omega_k+\omega_p)}\right)\right\}.$$

Here $2\pi\delta(0)$ represents the time T between past and future. The quantity multiplying $-2\pi i\delta(0)$ gives the change of energy, and the expression within the curly brackets represents the anomalous magnetic moment. Expressed in terms of nuclear magnetons, it is

$$\mu=\left(\frac{f^2}{4\pi}\right)\left(\frac{8}{3\pi}\right)\left(\frac{M}{\mu^2}\right)\int\frac{k^4}{\omega_k^4}dk+\frac{M}{27\pi^3}\times\int\int[\sigma_{11}(p)-2\sigma_{13}+\sigma_{33}]\left[\frac{1}{\omega_k}+\frac{1}{\omega_k+\omega_p}\right]\times\frac{k^4dkdp}{\omega_k^2\omega_p(\omega_k+\omega_p)}, \tag{5}$$

where M is the nucleon mass.

The use of the dispersion relation (2) is not essential. For the calculation of

$$\int\frac{A(k_0)}{(k_0^2-\omega_k^2)^2}dk_0=\left(\frac{1}{2\omega_k}\right)\frac{d}{d\omega_k}\int\frac{A(k_0)}{k_0^2-\omega_k^2}dk_0, \tag{6}$$

we split A as well as $1/(k_0^2-\omega_k^2)$ into real and imaginary parts. Observing that the answer must be purely imaginary (since no real process occurs in the intermediate states), we have

$$\int\frac{A(k_0)}{k_0^2-\omega_k^2}dk_0=-i\pi\frac{f^2}{\mu^2}\left(\frac{1}{\omega_k^2}\right)-\frac{i\pi}{2\omega_k}[\mathrm{Re}A(\omega_k)+\mathrm{Re}C(\omega_k)]+iP\int_\mu^\infty\frac{dP_0}{P_0^2-\omega_k^2}[\mathrm{Im}A(P_0)+\mathrm{Im}C(P_0)]. \tag{7}$$

For practical calculations, Eq. (5) is more convenient than (6) and (7), since the differentiation of the experimental curves is not desirable.

The same expression (5) can be obtained by the Wick-Chew-Low method.[8] The advantage of the present method is that in this form it is very easy to generalize

[8] H. Miyazawa, Phys. Rev. **101**, 1564 (1956).

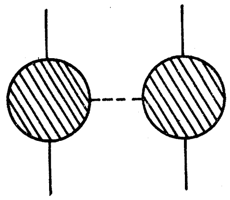


FIG. 2. Feynman graph for the second-order nuclear potential.

to the case of many-body problems and to S-wave problems.

IV. NUCLEAR FORCES

In applying this method to many-nucleon problems, we expand the S matrix in the following way. The first term is to contain all graphs in which no pion is exchanged between any pair of nucleons, although an arbitrary number of pions are emitted and absorbed on the same nucleon line. The second term will contain all graphs in which only one pion is exchanged between one pair of nucleons. In the next term, two pions are exchanged between one pair of nucleons, or one pion each is exchanged between two pairs of nucleons, and so on. We shall call these terms the zeroth-order term, second-order term, fourth-order term, and so on, respectively, although their meaning is different from that of perturbation theory. In this section we shall calculate the two-body nuclear force up to the fourth-order term.[9]

The second order term is represented by the graph in Fig. 2. The shaded areas are vertex parts. Since all nucleon lines are free, each vertex part is given by the renormalized pseudovector interaction, that is, the usual pseudovector interaction with the renormalized coupling constant f. The second order potential is, therefore, identical to the usual one except that it has the renormalized coupling constant instead.

$$V_2=\frac{f^2}{4\pi}(\tau^1\tau^2)(\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}^2\boldsymbol{\nabla})\frac{e^{-\mu x}}{x}.$$

The fourth-order potential is given by Fig. 3(a). Figure 3(b) does not give a different graph since the crossing of pion lines is already included in the shaded areas. Further, we see that in Fig. 3(a) we are counting the same graph twice, and we must divide by two for the correct S matrix. It is

$$S=-\tfrac{1}{2}\times\frac{1}{(2\pi)^8}\times\int\int\frac{\langle j,q|S^1|i,k\rangle\langle j,-q|S^2|i,-k\rangle}{(k_0{}^2-\omega_k{}^2)(q_0{}^2-\omega_q{}^2)}d^4kd^4q,$$

where the suffixes 1 and 2 on S label the two nucleons. In order to obtain the usual perturbation-theoretic fourth-order term, we simply replace the scattering matrices by their lowest-order expressions, which are, as is well known, very poor approximations.

After substitution of (1) for S_i, S consists of four terms:

$$\begin{aligned}S=-2\pi i\delta(0)\int\int[&(\tau^1\tau^2)^2\boldsymbol{\sigma}^1\cdot\mathbf{k}\boldsymbol{\sigma}^1\cdot\mathbf{q}\boldsymbol{\sigma}^2\cdot\mathbf{k}\boldsymbol{\sigma}^2\cdot\mathbf{q}X\\&+(\tau^1\tau^2)^2\boldsymbol{\sigma}^1\cdot\mathbf{k}\boldsymbol{\sigma}^1\cdot\mathbf{q}\boldsymbol{\sigma}^2\cdot\mathbf{q}\boldsymbol{\sigma}^2\cdot\mathbf{k}Y\\&+\tau_i^1\tau_j^1\tau_j^2\tau_i^2\boldsymbol{\sigma}^1\cdot\mathbf{k}\boldsymbol{\sigma}^1\cdot\mathbf{q}\boldsymbol{\sigma}^2\cdot\mathbf{k}\boldsymbol{\sigma}^2\cdot\mathbf{q}Y\\&+\tau_i^1\tau_j^1\tau_j^2\tau_i^2\boldsymbol{\sigma}^1\cdot\mathbf{k}\boldsymbol{\sigma}^1\cdot\mathbf{q}\boldsymbol{\sigma}^2\cdot\mathbf{q}\boldsymbol{\sigma}^2\cdot\mathbf{k}Z]e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}}d^3kd^3q,\end{aligned}$$

$$x=x_1-x_2.$$

X, Y, and Z are given, after using the symmetry property of (3), by

$$X=ac+bb,$$
$$Y=2ab,$$
$$Z=aa+bb,$$

where

$$ac=\frac{i}{(2\pi)^7}\int\frac{A(k_0)C(k_0)}{(k_0{}^2-\omega_k{}^2)(k_0{}^2-\omega_q{}^2)}dk_0,$$

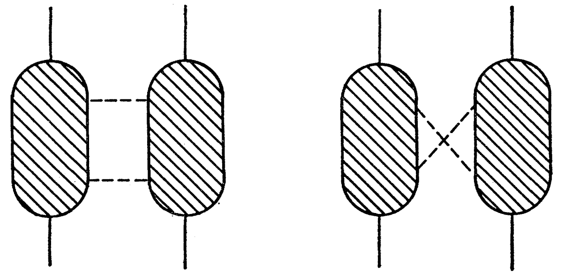
and similar expressions for bb, ab, and aa.

In the calculation of the first terms of ac, we meet with some trouble, since the contribution from the pole $k_0=0$ gives a term which is proportional to $1/\epsilon$. That is,

$$S=-2\pi i\delta(0)\frac{1}{2i\epsilon}\left[\frac{1}{(2\pi)^3}\left(\frac{f^2}{\mu^2}\right)(\tau^1\tau^2)\times\int\frac{\boldsymbol{\sigma}^1\cdot\mathbf{k}\boldsymbol{\sigma}^2\cdot\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega_k{}^2}dk\right]^2+\cdots.$$

We can easily see, however, that this term is the repetition of the second-order energy and does not contribute to the fourth-order energy. After this has been omitted, all the integrals are well defined. The quantity multiplying $-2\pi i\delta(0)$ in S gives the nuclear potential.

$$\begin{aligned}V_4=-[&(\tau^1\tau^2)^2\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla}_y\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla}_z\boldsymbol{\sigma}^2\cdot\boldsymbol{\nabla}_y\boldsymbol{\sigma}^2\cdot\boldsymbol{\nabla}_z\Xi(y,z)\\&+(\tau^1\tau^2)^2\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla}_y\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla}_z\boldsymbol{\sigma}^2\cdot\boldsymbol{\nabla}_z\boldsymbol{\sigma}^2\cdot\boldsymbol{\nabla}_yY(y,z)\\&+\tau_i^1\tau_j^1\tau_j^2\tau_i^2\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla}_y\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla}_z\boldsymbol{\sigma}^2\cdot\boldsymbol{\nabla}_y\boldsymbol{\sigma}^2\cdot\boldsymbol{\nabla}_zY(y,z)\\&+\tau_i^1\tau_j^1\tau_j^2\tau_i^2\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla}_y\boldsymbol{\sigma}^1\cdot\boldsymbol{\nabla}_z\boldsymbol{\sigma}^2\cdot\boldsymbol{\nabla}_z\boldsymbol{\sigma}^2\cdot\boldsymbol{\nabla}_yZ(y,z)]_{y=z=x},\end{aligned}$$



(a) (b)

FIG. 3. Feynman graphs for the fourth-order nuclear potential. Figure 3(b) is not needed for the calculation (see text).

[9] The Wick-Chew-Low method was applied to this problem by Sato [S. Sato (private communication)].

where Ξ, Y, and Z are functions involving the coupling constant and cross sections. Since their complete expressions are rather long, we shall write here the simplified expressions obtained by assuming that σ_{11} and σ_{13} are zero.

$$\Xi(y,z)=-\frac{f^4}{\mu^4}F_{00}(y,z)+\frac{f^2}{9\pi\mu^2}\times\int\frac{dp}{\omega_p}\sigma_{33}(p)[5F_{\omega_p 0}(y,z)+G_{\omega_p 0}(y,z)]+\frac{1}{162\pi^2}\int\int\frac{dpdq}{\omega_p\omega_q}\sigma_{33}(p)\sigma_{33}(q)\times[17F_{\omega_p\omega_q}(y,z)+25G_{\omega_p\omega_q}(y,z)],$$

$$Y(y,z)=\frac{f^2}{3\pi\mu^2}\int\frac{dp}{\omega_p}\sigma_{33}(p)[F_{\omega_p 0}(y,z)+G_{\omega_p 0}(y,z)]+\frac{5}{54\pi^2}\int\int\frac{dpdq}{\omega_p\omega_q}\sigma_{33}(p)\sigma_{33}(q)\times[F_{\omega_p\omega_q}(y,z)+G_{\omega_p\omega_q}(y,z)],$$

$$Z(y,z)=\frac{f^4}{\mu^4}F_{00}(y,z)+\frac{f^2}{9\pi\mu^2}\times\int\frac{dp}{\omega_p}\sigma_{33}(p)[5F_{\omega_p 0}(y,z)+9G_{\omega_p 0}(y,z)]+\frac{1}{162\pi^2}\int\int\frac{dpdq}{\omega_p\omega_q}\sigma_{33}(p)\sigma_{33}(q)\times[17F_{\omega_p\omega_q}(y,z)+9G_{\omega_p\omega_q}(y,z)],$$

where F and G are given by

$$F_{\lambda\mu}(y,z)=\frac{1}{2(2\pi)^3 yz}\int\frac{k\sin k(y+z)}{\omega_k(\omega_k+\lambda)(\omega_k+\mu)}dk,$$

$$G_{\lambda\mu}(y,z)=\frac{1}{2(2\pi)^3 yz(\lambda+\mu)}\int\frac{k\sin k(y+z)}{(\omega_k+\lambda)(\omega_k+\mu)}dk.$$

In these expressions, the terms proportional to f^4 are fourth-order terms of perturbation theory.

V. INTERACTION OF S-WAVE PIONS WITH NUCLEONS

Little is known about the mechanism of S-wave interaction, since elementary theory fails to explain the large splitting of the phase shifts δ_1 and δ_3. Perhaps the creation of nucleon-antinucleon pairs might play an important role. Still we assume the static approximation, implying that momenta of nucleons can be neglected. Also we assume that the interaction is sufficiently regular. In this case the matrix element for the scattering part depends trivially on the pion's momentum: it does not depend on the momentum, since an S wave is emitted or absorbed.

The dispersion relation for S-wave pions, within the framework of the static model, was derived by Oehme.[5] We write the scattering matrix of S waves as

$$\langle j,q|S_s|i,k\rangle=2\pi i\delta(k_0-q_0)\times[D(k_0)\tau_i\tau_j+E(k_0)\tau_j\tau_i]e^{i\mathbf{x}\cdot(\mathbf{k}-\mathbf{q})},\quad(8)$$

where

$$D(k_0)=\frac{2\pi}{3}(a_1+2a_3)-\frac{2\pi k_0}{3\mu}(a_1-a_3)+\left(\frac{k_0^2-\mu^2}{2\pi}\right)\times\int_0^\infty\frac{dp}{\omega_p}\left[\frac{\sigma_3(p)}{\omega_p-k_0-i\epsilon}+\frac{\sigma_3+2\sigma_1}{3(\omega_p+k_0-i\epsilon)}\right],$$
$$E(k_0)=\frac{2\pi}{3}(a_1+2a_3)-\frac{2\pi k_0}{3\mu}(a_1-a_3)+\left(\frac{k_0^2-\mu^2}{2\pi}\right)\times\int\frac{dp}{\omega_p}\left[\frac{\sigma_3+2\sigma_1}{3(\omega_p-k_0-i\epsilon)}+\frac{\sigma_3}{\omega_p+k_0-i\epsilon}\right].\quad(9)$$

Here σ_i is the total cross section of the pure state for $I=i/2$ and a_i is the scattering length in the $i/2$ state. Equation (8) is to be taken with the right-hand side of (1), and the calculation proceeds as before. In Eq. (9), terms involving cross sections are small compared with other terms. The scattering lengths a_i are proportional to δ_i, the S-phase shifts, whereas the σ_i are proportional to δ_i^2; and δ_i are small. Thus for practical purposes, D and E can be written as

$$D(k_0)=\frac{2\pi}{3}(a_1+2a_3)-\frac{2\pi k_0}{3\mu}(a_1-a_3),$$
$$E(k_0)=\frac{2\pi}{3}(a_1+2a_3)+\frac{2\pi k_0}{3\mu}(a_1-a_3).\quad(10)$$

In fact, Eq. (10) gives a fairly good fit with the experimental points calculated from Orear's phase shifts.

From (10), we have a fourth-order nuclear potential due to S waves:

$$V_{ss}=-\mu^3\frac{(a_1+2a_3)^2}{18\pi}\left(\frac{k_1(2\mu x)}{(\mu x)^2}\right)+(\tau^1\tau^2)\mu^3\frac{(a_1-a_3)^2}{36\pi}\left[\frac{k_0(2\mu x)}{(\mu x)^3}+\frac{k_1(2\mu x)}{(\mu x)^4}\right].\quad(11)$$

This potential corresponds to the two-pair term of the usual fourth-order nuclear potential. Two points are to be noted. One is the damping of the pair term. The potential (11) is very much smaller than the usual one due to the repulsive force of S-wave pion-nucleon interaction. The other point is that (11) is dependent on τ spin. This is due to the mysterious splitting of δ_1 and δ_3. The potential corresponding to the one-pair term is

$$V_{sp}=8\pi(a_1+2a_3)\left[\frac{f^2}{\mu^2}F_0(x)+\frac{4}{9\pi}\int\frac{dp}{\omega_p}\sigma_{33}(p)F_{\omega_p}(x)\right]$$
$$+(\tau^1\tau^2)\frac{8\pi}{3\mu}(a_1-a_3)$$
$$\times\left[\frac{f^2}{\mu^2}G_0(x)+\frac{5}{18\pi}\int\frac{dp}{\omega_p}\sigma_{33}(p)G_{\omega_p}(x)\right],$$

$$F_\lambda(x)=\nabla_y\cdot\nabla_z\frac{1}{2(2\pi)^3yz}\int\frac{k\sin k(y+z)}{\omega_k(\omega_k+\lambda)}dk\bigg|_{y=z=x},$$

$$G_\lambda(x)=\nabla_y\cdot\nabla_z\frac{1}{2(2\pi)^3yz}\int\frac{k\sin k(y+z)}{\omega_k+\lambda}\bigg|_{y=z=x}.$$

The splitting of δ_1 and δ_3 has the consequence that S-wave pions contribute to the electron-neutron interaction and to the charge distribution around the proton. These interactions are also proportional to a_1-a_3. These points are discussed in a separate paper.

VI. CONCLUSIONS AND DISCUSSIONS

The theory of pion-nucleon interaction is greatly simplified by assuming the static model. In this case, a matrix element depends trivially on the spatial momenta, and it is easy to calculate the value of the matrix element off the energy shell. Thus the expectation value in the one-proton state of any quantity bilinear in the pion field can be rigorously expressed if the total cross sections are known.

The approximation of static interaction is of course not valid if high-energy pions are involved. The anomalous magnetic moment of the proton, for instance, depends rather sensitively on the cutoff. Therefore, numerical values obtained by this method cannot be taken too seriously. The nuclear force problem, on the other hand, is an example to which the static model can be applied. If the internucleon distance is a pion Compton wavelength or larger, only low-energy matrix elements are important and the present method can be safely applied.

It is interesting to note that, so far as P waves are concerned, identical results can be obtained from a different approach, namely, by the Wick-Chew-Low method. The present approach, however, does not need the explicit interaction Hamiltonian, and can be generalized to the case of S waves for which the interaction is not known.

The technique of extending matrix elements to the case of virtual pions is not limited to the scattering part only. If the structure of an n vertex (sums of graphs with two external nucleon lines and n external pion lines) is known, either experimentally or theoretically, the calculation of the S matrix is very much simplified. For instance, the proton expectation value of a quantity quadrilinear in the pion field can be calculated from the 4 vertex.

Rules for constructing the S matrix are as follows. Draw as many points on a paper as the number of nucleons in question. Then draw as many pion lines linked to these points as he wishes. Twice the number of lines represents the order of the graph. Replace each pion line by the propagation function. Replace each nucleon point by the n vertex if n-pion lines start from that point. Divide by an appropriate number[10] in order not to count the same Feynman graphs many times. Finally integrate over all virtual pion momenta to obtain the S matrix. This way of computing the S matrix gives far better convergence than the usual perturbation expansion if the internucleon distance is not small.

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[10] This number can be obtained as follows. Two or more pion lines are called equivalent if they start from the same point and end on the same point. In a graph, lines are classified according to equivalence. If there are m nonequivalent classes, containing $n_1, n_2, \cdots, n_m$ equivalent lines, this graph must be divided by $n_1!n_2!\cdots n_m!$. In the examples of Secs. III and IV, $m=1$ and $n_1=2$.