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Quantum Calculation of Coulomb Excitation. *M*1 and *M*1-*E*2 Mixed Transitions and Classical Approximation*

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Formulas are derived for magnetic-dipole Coulomb excitation, including spin effects. Numerical results are presented graphically. The mixed *M*1-*E*2 transitions are similarly discussed, and the simple pure-mixed analysis is shown to apply except for anomalous cases of slight interest.

A classical approximation is discussed in detail, and applied to higher multipoles.

I. INTRODUCTION

ELECTRIC dipole and electric quadrupole Coulomb excitation have been discussed in detail in previous papers.[1] The present work concerns itself first with magnetic dipole and mixed magnetic dipole-electric quadrupole Coulomb excitation. Secondly, a classical approximation to the general Coulomb excitation function is presented and discussed. In terms of this approximation reliable results for the general multipole can be obtained; the *E*3, *E*4, *M*2, and *M*3 excitation functions are explicitly discussed and curves presented.

The determination of the multipolarity of an experimentally observed Coulomb excitation is of prime importance, and not always straightforward. The straightforward procedure, for example, would use the energy dependence of the total cross section. However, in some instances this type of measurement alone may not suffice, and, in fact, may not even differentiate Coulomb excitation from a nuclear reaction,[2] particularly where the energy resolution is not good or where thick targets must be employed. A comparison of proton to alpha-particle yields may likewise provide a test of the multipolarity,[3] but this is not always feasible. In such cases, and as corroborative evidence, the further measurement of the angular distribution of the γ rays serves to identify the process. To this end, the particle parameters must be known to better accuracy than is given by any classical approximation, and so must be calculated quantum-mechanically. In contrast, to this situation (see Sec. IV), a suitably defined classical approximation suffices for the calculation of the excitation function, and hence the total cross section, over the entire range of experimental interest.

For mixed *M*1-*E*2 transitions, such as generally obtain for odd-*A* target nuclei, measurement of the angular correlation affords a sensitive means of determining both the relative probabilities of the two modes of decay and the relative phase (plus or minus) of the nuclear matrix elements, just as in γ–γ cascades.[4] As the experiments of McGowan and Stelson[5] have shown, such a directional correlation, when combined with a polarization-direction correlation to eliminate ambiguities, can yield valuable nuclear data. These are precision experiments, however, and it is essential to note that whenever such a mixed transition exists, the correlation is in principle a *mixed-mixed* correlation. This is, of course, due to the fact that for Coulomb excitation both the excitation and decay usually involve the same nuclear levels. Although mixed-mixed correlations in general have been difficult to interpret, for Coulomb excitation considerable simplification results from the fact that the same mixture is involved, since the process is a *reaction* rather than a *cascade*.

Estimates of the relative probabilities of *M*1 *versus* *E*2 excitation using the classical approximation indicate that the mixed process is unlikely to be of general importance. This is hardly surprising since the Coulomb excitation is primarily an electric process. Nevertheless, it is not clear on these grounds that the mixed process is entirely without interest, since (1) the interference enters as the square root of the ratio of excitation probabilities, (2) the size of the mixture coefficient for the interference term, $b_2(1m,2e)$, has never been calculated, and (3) the mixture terms are usually very anisotropic. Despite these possibilities, the results presented below indicate that except for anomalous cases[6] the mixed excitation process need not be considered. This state of affairs greatly simplifies the analysis of Coulomb excitation experiments involving mixed transitions, since the excitation may be considered as pure electric quadrupole and the decay as mixed $M1-E2$ γ emission.

Even though magnetic excitation is similarly im-

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[1] Biedenharn, McHale, and Thaler, Phys. Rev. **100**, 376 (1955); Biedenharn, Goldstein, McHale, and Thaler, Phys. Rev. **101**, 662 (1956); Thaler, Goldstein, McHale, and Biedenharn, Phys. Rev. **103**, 1567 (1956). We refer to these papers as I, II, and III, respectively.

[2] This was brought to our attention by Dr. J. P. Schiffer in connection with his work on the proton bombardment of Ni (unpublished).

[3] J. H. Bjerregaard and T. Huus, Phys. Rev. **94**, 204 (1954).

[4] L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. **25**, 729 (1953).

[5] P. H. Stelson and F. K. McGowan, Bull. Am. Phys. Soc. Ser. II, **1**, 164 (1956).

[6] E.g., $\frac{3}{2}\rightarrow\frac{3}{2}$ transitions where the entire anisotropy arises from mixture.

probable, the $M1$ case has been examined for the sake of completeness. It is found that, neglecting spin effects, when properly defined the classical approximation for the magnetic dipole excitation function is good over a very large range of its variables. Moreover, the angular correlation particle parameter a_2 is shown below to be equal to unity everywhere. The spin, however, can enter in a significant way. For protons, spin corrections of the order of 10–30% may be expected. It is shown below that further calculations to take account of the spin effects are unnecessary since the required parameters are proportional to the already calculated $E2$ parameters.

II. SUMMARY OF FORMULAS

The total cross section for magnetic dipole excitation, without consideration of spin, has been given in I, Eqs. (10), (12), and (87).[7] In the long wavelength approximation these results take the form:

$$\sigma_{M1}=\frac{1}{2}\left(\frac{k_2}{k_1}\right)\left(\frac{2J_f+1}{2J_i+1}\right)(f\|\mathbf{j}_N\cdot\boldsymbol{r}\mathbf{Y}_{11}\|i)^2 \times\left(\frac{8\pi Z_1e}{3k_1k_2\hbar c^2}\right)^2\sum_{l=1}^{\infty}l(l+1)(2l+1)I^2_{l,l}, \quad (1)$$

where we have used the definition of the Coulomb integrals, $I_{ll'}$, to be:

$$I_{ll'}\equiv\int_0^\infty drr^{-3}F_l(\eta_1,k_1r)F_{l'}(\eta_2,k_2r). \quad (2)$$

The inclusion of spin effects can be readily taken into account, although the treatment (see Appendix) is somewhat lengthy. The result is

$$\sigma_{M1}(\text{spin})=\frac{3}{4}\mu^2\left(\frac{k_2}{k_1}\right)\left(\frac{2J_f+1}{2J_i+1}\right)(f\|\mathbf{j}_N\cdot\boldsymbol{r}\mathbf{Y}_{11}\|i)^2 \times\left(\frac{8\pi Z_1e}{3k_1k_2\hbar c^2}\right)\sum_l\left\{\frac{l(l+1)(2l+1)}{(2l-1)(2l+3)}I^2_{l,l}+\frac{3}{2}\frac{l(l-1)}{2l-1}I^2_{l-2,l}+\frac{3}{2}\frac{(l+1)(l+2)}{2l+3}I^2_{l+2,l}\right\}. \quad (3)$$

Here μ represents the spin magnetic moment of the incident particle in nuclear magnetons.

It should be noted that the term in curly brackets in Eq. (3) above is exactly the same as the sum that occurs for the total cross section in $E2$ excitation [II, Eq. (1)], where it was designated as $b_0(E2)$. The relevant quantity for $M1$ excitation, including spin effects, is then

$$b_0(M1)=b_0'(M1)+\tfrac{3}{2}\mu^2b_0(E2),\quad b_0'(M1)=\sum_{l=1}l(l+1)(2l+1)I^2_{l,l}. \quad (4)$$

As shown in the appendix, the calculation of the directional correlation particle parameters yields a very similar result, namely,

$$b_2(M1)=b_2'(M1)-3\mu^2b_2(E2), \quad (5)$$

$$b_2'(M1)=b_0'(M1). \quad (6)$$

The primed terms represent the contribution of the convection currents alone. Thus if the incident particles have no spin, the value of $a_2=b_2/b_0$ is exactly unity, and the (p,γ) direction correlation differs not at all from a $\gamma\gamma$ correlation between the same nuclear states. On the other hand, if the spin magnetic moment is nonzero, it is clear from Eqs. (4) and (5) that the general case for nonvanishing magnetic moment requires only the computation of $b_0'(M1)$, since the $b_\nu(E2)$ are given in II.

To consider the possibility of mixed $M1-E2$ excitation in any generality is quite involved[8] but, for-

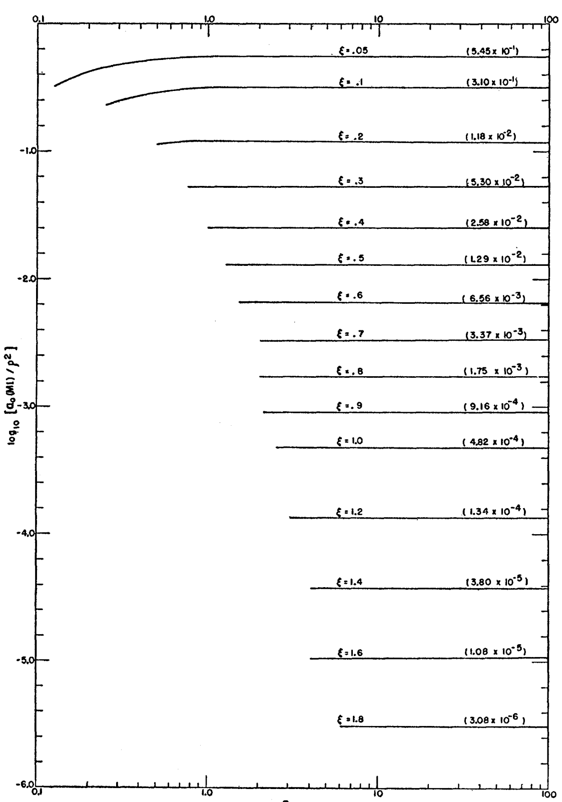

FIG. 1. Plot of $\log_{10}[a_0(M1)/\rho^2]$ *vs* η, for various typical values of ξ. The numbers in parentheses represent the values of the limit: $\lim_{\eta\to\infty}a_0(M1)/\rho^2=9/64\pi^2f_{M1}(\xi)$.

[7] It is useful to note at this point, that—despite the discussion of I, page 381—there are *no* center-of-mass effects for the total cross section in Coulomb excitation, and for the gamma correlation such effects are negligible for even the lightest target nuclei.

[8] See reference 4, pp. 745 ff.

tunately, is unnecessary. This circumstance is a consequence of the fact that for the emitted γ ray the $M1$ and $E2$ competition is such that all terms must be considered, while for the excitation, the $E2$ process greatly exceeds the $M1$ (for typical values by 10^3). Hence one need consider only the cross terms, and these enter only in the directional correlation. Moreover, the spin magnetic moment can make no contribution to the correlation for unpolarized beams. In this approximation we may write

$$W(\theta)=W_I(\theta)+2\delta(\hbar k/Mc)\lambda A P_2(\cos\theta). \qquad (7)$$

Here $W_I(\theta)$ is the directional correlation for a pure $E2$ excitation and a mixed $E2-M1$ decay, which involves only the $b_\nu(E2)$ previously given. The $W_I(\theta)$ is normalized so that the angle-independent term is unity.

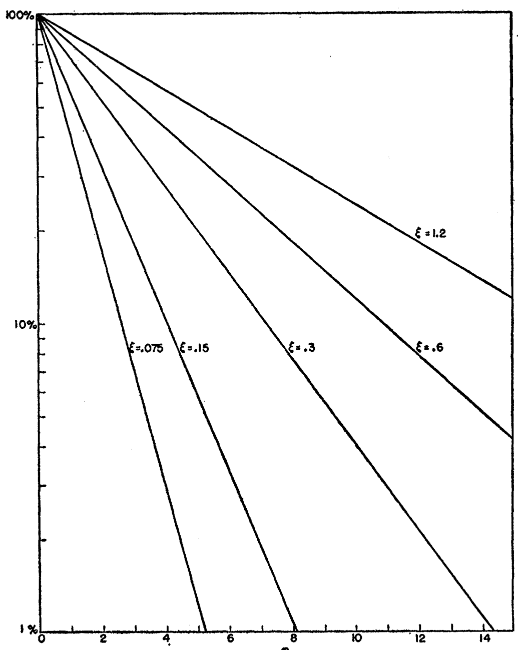

FIG. 2. Percent spin contribution to the magnetic dipole cross section for protons plotted against η for various values of ξ.

The "mixed excitation" term in (7) is a product of several factors. The factor δ is the square root of the the probability of $M1$ to $E2$ decay, and the sign of the square root is the sign of the ratio of reduced matrix elements for this decay. The factor A is related to the details of the γ emission, but is of the order of unity in size. In detail it is given by

$$A=C^{122}{}_{1-1}W(1J_f2J_f;J_i2)\{C^{222}{}_{1-1}W(2J_f2J_f;J_i2) +2\delta C^{122}{}_{1-1}W(2J_f2J_f;J_i2) +\delta^2 C^{112}{}_{1-1}W(1J_f1J_f;J_i2)\}. \qquad (8)$$

The numerical factor $\hbar k/Mc=E_\gamma/Mc^2$ ($\sim 10^{-4}$ to 10^{-3}) contributes greatly to reducing the importance of this cross term. Finally the factor λ, which alone depends upon the Coulomb excitation process, is given

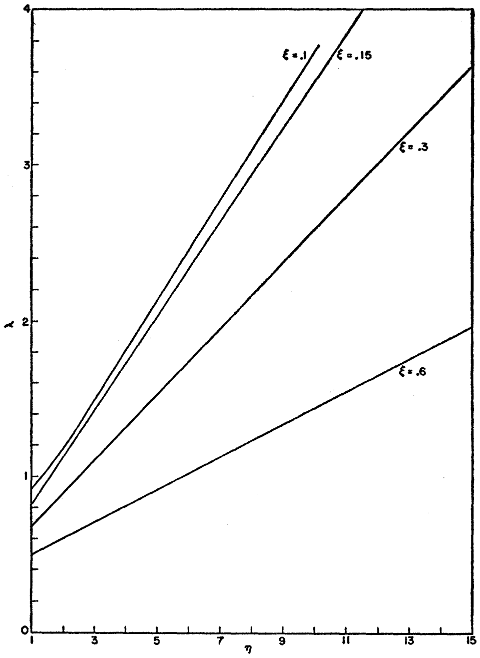

FIG. 3. The $E2$-$M1$ mixture function λ [see Eq. (9)] *vs* η plotted for various values of ξ. The $\xi=0$ curve lies below the $\xi=0.15$ curve.

by the following formulas as derived in the Appendix:

$$\lambda=[b_0(E2)]^{-1}\sum_l\left\{\frac{l(l+1)(2l+1)}{(2l-1)(2l+3)}I^2{}_{ll} +\frac{l(l+1)(l+2)}{2l+3}\cos(\sigma_l-\sigma_{l+2})I_{ll}I_{l+2,\,l} -\frac{(l-1)(l)(l+1)}{2l-1}\cos(\sigma_l-\sigma_{l-2})I_{ll}I_{l-2,\,l}\right\}. \qquad (9)$$

III. NUMERICAL RESULTS

In view of the fact that electric processes are so greatly favored over magnetic processes in Coulomb excitation (for particles of nucleonic mass), the numerical results are presented in abbreviated form. For magnetic dipole excitation, only the function $b_0'(M1)$ is required. This function is shown in Fig. 1. In many respects this function is similar to the $b_0(E1)$ and shows the same logarithmic infinity for small excitations. (The latter may be seen from the fact that $I_{ll}\sim l^{-2}$ for $\xi\to 0$ and hence $b_0'(M1)$ behaves like Σl^{-1}.)

Calculation of the functions $b_0'(M1)$ and λ is facilitated by the fact that the integrals that appear in the sums are the same integrals required for the $E2$ case; see II, Eq. (2), etc.

The magnetic dipole case is of interest in that it illustrates the magnitude of spin effects in a favorable case. Figure 2 shows that contribution of the spin magnetic moment to $M1$ Coulomb excitation.

Finally, Fig. 3 shows the function λ, which describes the mixing of $M1$ and $E2$ excitation. It is clear from this figure that $\lambda \sim 1$, so that the mixed excitation process need not be considered in analyzing the experiments, since $\hbar k/Mc \lesssim 10^{-3}$.

IV. CLASSICAL APPROXIMATION

Calculations have been presented of the $E1$, $E2$, and $M1$ excitation functions as obtained in a quantum treatment. It is of interest to survey these results and inquire as to just how much they deviate from the classical results. This question is, however, not very well-defined, for, although there exists a unique classical *limit*, generally one means not this but rather some sort of classical *approximation*. The point at issue is that the classical limit fails to distinguish k_f from k_i, and this, in turn, is rather inaccurate.

An example of this situation is afforded by the original formulation of Ter-Martirosyan,[9] which was later extended numerically by Alder and Winther.[10] These authors treated the classical cross section in terms of the parameter $\xi = (Z_1Z_2e^2/2\hbar v)(\Delta E/E)$. It was soon noticed, however, at many laboratories,[11] that the more symmetric variable $\xi = \eta_f - \eta_i = \eta(\rho - 1)$, which reduces to the above in the classical limit, improved agreement with experiment by a significant amount ($\sim 35\%$ in one instance). Although the justification for the symmetric variable $\xi = \eta_f - \eta_i$ was later given in many papers and is implicit in the exact dipole-bremsstrahlung results of Sommerfeld, it is of interest to note that this variable was explicitly given in the early work of Landau,[12] and later by Guth.[13]

By considering the reciprocity theorem, one can easily arrive at a more satisfactory form for the classical excitation function. That is, one has in general the relation:

$$(2J_i+1)k_i^2\sigma_{i\to f} = (2J_f+1)k_f^2\sigma_{f\to i}.$$

From this one sees that the only asymmetry between k_i and k_f contained in the cross section occurs through the k_2/k_1 factor in evidence in Eq. (1).[14] The excitation function is therefore *symmetric* in the variables 1 and 2. This leads to a reasonable prescription for defining the excitation function. Thus one ought to use in place of the classical limit for the (m,n,l) given in I, Eq. (81) the symmetrized form:

$$(m,n,l) \cong \tfrac{1}{4}(k_1k_2/\eta_1\eta_2)^{(n-1)/2} I_{n-1,\,m}(\xi,\epsilon), \tag{10}$$

where

$$I_{n-1,\,m}(\xi,\epsilon) \equiv \int_{-\infty}^{\infty} dt \exp[-i\xi(\epsilon \sinh t + t)] \times[\epsilon+\cosh t + i(\epsilon^2-1)^{\frac{1}{2}} \sinh t]^m \times[\epsilon \cosh t + 1]^{1-n-m}, \tag{11}$$

and

$$\epsilon^2 \equiv 1 + l^2/\eta_1\eta_2. \tag{12}$$

If one introduces this symmetrized limit into the general definition for the $b_0(mL)$ and $b_0(eL)$, that is,

$$b_0(eL) \equiv \sum_{l,m} (C^{l\,L\,l+m}{}_{000})^2(2l+1)(m,\,L+1,\,l)^2, \tag{13}$$

and,

$$b_0(mL) \equiv L(L+1)(2L+1)\sum_{l,m}(l+1)(2l+1)^2 \times(2l+3)(C^{l+1\,L\,l+m}{}_{000})^2 W^2(L\,L\,l\,l+1;1\,l+m) \times(m,\,L+2,\,l)^2, \tag{14}$$

a suitable form for the classical excitation function can be obtained as follows.

The classical limit requires that l be considered large. Thus one must also obtain the classical limit for the various angular momentum functions that appear in Eqs. (13) and (14). From the explicit formulas for these functions[15] it is readily seen that

$$C^{l\,L\,l+m}{}_{000} \sim \frac{[(L+m)!(L-m)!]^{\frac{1}{2}}}{2^{L/2}[\frac{1}{2}(L+m)]![\frac{1}{2}(L-m)]!} \text{ for } L+m = \text{even integer}, \tag{15a}$$

$$= 0 \text{ for } L+m = \text{odd integer}, \tag{15b}$$

and

$$W(L\,L\,l\,l+1;1\,l+m) \sim (-)^{L-m} \times\left[\frac{(L+m)(L+1-m)}{4lL(L+1)(2L+1)}\right]^{\frac{1}{2}}. \tag{16}$$

While the explicit results given in Eqs. (15) and (16) are well suited for actual calculations, a more elegant form utilizing the normalized spherical harmonics can be given. With some manipulation, it can be shown that:

$$C^{l\,L\,l+m}{}_{000} \sim (4\pi/2L+1)^{\frac{1}{2}}(-)^L Y_L{}^M(\pi/2,0). \tag{17}$$

By utilizing this result, the classical limit for the product

$$C^{l+1\,L\,l+m}{}_{000} W(L\,L\,l\,l+1;1\,l+m)$$

can be simplified, upon noting that the classical limit of the Racah function is proportional to $C^{L1L}{}_{m-1\,1m}$,

[9] K. A. Ter-Martirosyan, J. Exptl. Theoret. Phys. (U.S.S.R.) 22, 284 (1952).
[10] K. Alder and A. Winther, Phys. Rev. 91, 1578 (1953); CERN Report T/KA-AW-1, October, 1954 (unpublished).
[11] Sherr and Christy, Williamson and Goldberg, Temmer and Heydenburg, Class and Cook, Stelson and McGowan, among others.
[12] L. Landau, Physik Z. U.S.S.R. 1, 88 (1932).
[13] E. Guth, Phys. Rev. 68, 280 (1945).
[14] It should be noted in this connection that the reduced matrix elements used in this work are, aside from phase, *not* symmetric under interchange of J_i and J_f.
[15] G. Racah, Phys. Rev. 62, 438 (1943).

that is

$$C^{l+1\,L\,l+m}{}_{000}W(L\,L\,l\,l+1;1\,l+m)$$
$$\sim(-)^m[2\pi/l(2L+1)]^{\frac{1}{2}}C^{L1L}{}_{1-1}Y_L{}^{m-1}(\pi/2,0) \quad (18a)$$
$$=(-)^m[\pi/l\,L\,(L+1)(2L+1)^2]^{\frac{1}{2}} \times[(\partial/\partial\theta)Y_L{}^m(\theta,0)]_{\theta=\pi/2}. \quad (18b)$$

The last step utilized the relation between the angular momentum operator L_τ and the vector addition coefficient $C^{L1L}{}_{m-\tau\;\tau}$, when one operates on the spherical harmonics.

A classical approximation for the exact excitation function, b_0, then takes the form

$$b_0(eL)\cong\tfrac{1}{2}\pi(2L+1)^{-1}(\eta_1\eta_2)^{1-L}(k_1k_2)^L \times\sum_m[Y_L{}^m(\pi/2,0)]^2\int_1^\infty \epsilon d\epsilon I^2{}_{L,m}(\xi,\epsilon), \quad (19)$$

and

$$b_0(mL)\cong\tfrac{1}{2}\pi(2L+1)^{-1}(\eta_1\eta_2)^{1-L}(k_1k_2)^{L+1} \times\sum_m\left(\frac{\partial}{\partial\theta}Y_L{}^m\right)^2_{\theta=\pi/2}\int_1^\infty \epsilon(\epsilon^2-1)d\epsilon I^2{}_{L+1,m}(\xi,\epsilon). \quad (20)$$

It is interesting to note that parity conservation, which was expressed by the properties of the vector addition coefficients in Eqs. (13) and (14), has carried over very nicely into the properties of the spherical harmonics evaluated at 90°. The occurrence of the angle $\pi/2$ is readily interpreted from the classical orbit picture, since the motion takes place in a plane so that the components perpendicular can be set equal to zero (i.e., $\theta=\pi/2$).

The classical functions defined by the summands of Eqs. (19) and (20) (functions only of ξ, it should be noted) have, aside from a normalization, been given previously by Ter-Martirosyan,[9] by Alder and Winther,[10] and by Osborne and Rose.[16] Utilizing the notation of the Copenhagen group, we find that a symmetrized classical approximation for the excitation function is given by

$$b_0(eL)\cong\left(\frac{8\pi}{2L+1}\right)^{-2}(\eta_1\eta_2)^{1-L}(k_1k_2)^L f_{eL}(\xi), \quad (21)$$

$$b_0(mL)\cong\left(\frac{8\pi}{2L+1}\right)^{-2}(\eta_1\eta_2)^{1-L}(k_1k_2)^{L+1} f_{mL}(\xi), \quad (22)$$

with

$$f_{eL}(\xi)\equiv\frac{1}{2}\left(\frac{4\pi}{2L+1}\right)^3\sum_m[Y_L{}^m(\tfrac{1}{2}\pi,0)]^2 \times\int_1^\infty \epsilon d\epsilon I^2{}_{L,m}(\xi,\epsilon), \quad (23)$$

$$f_{mL}(\xi)\equiv\frac{1}{2}\left(\frac{4\pi}{2L+1}\right)^3\sum_m\left(\frac{\partial}{\partial\theta}Y_L{}^m(\theta,0)\right)_{\theta=\frac{1}{2}\pi} \times\int_1^\infty \epsilon(\epsilon^2-1)d\epsilon I^2{}_{L+1,m}(\xi,\epsilon). \quad (24)$$

The approximation given by the formulas above is expected to be good only for $\eta\gg2$, in accordance with Bohr's discussion of the classical limit.[17] The fact is, however, that this symmetrized form is really extraordinarily accurate, even for $\eta=1$, for the $E1$, $M1$, and $E2$ cases explicitly calculated.

For the ($E2$) zero-energy-loss case the accuracy of the classical limit for the excitation function was noted earlier.[18] However, for this case the excitation function is already in a symmetric form, since $k_1=k_2$. For $\xi\neq0$ ($k_1\neq k_2$), the symmetrization greatly improves the approximation; in fact the $\xi\sim1$ and $\xi=0$ cases are then comparable in accuracy.

Another important property of this symmetrized classical approximation is the fact that the ratio of the exact quantum excitation function to this classical excitation function is insensitive to the value of ξ, and hence is primarily a function of η.[19] Thus the case $\xi=0$, which is quite easy to treat exactly, can provide a further improvement to the above classical approximation.

This situation is illustrated in Fig. 4. In this figure we have plotted the ratio, $R(\eta)$, of the quantum excitation function to the "classical" excitation function [i.e., the ratio of the left to the right-hand sides of Eqs. (21) or (22)] for typical values of ξ in the $E1$, $E2$, and $M1$ cases as taken from the exact quantum mechanical calculations.

For the $E2$ case, we have plotted the function $R_{E2}(\eta,\xi)$ *vs* η for $\xi=0$. The curves for $\xi>0$ are indistinguishable from the $\xi=0$ curve for $\eta\gtrsim\frac{3}{2}$. While the

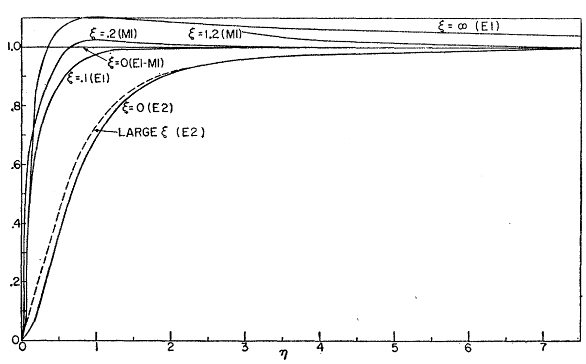

FIG. 4. The ratio, $R(\eta)$, of the quantum mechanical excitation function to its classical limit plotted against η for the $E1$, $E2$, and $M1$ cases. Parametric values of ξ are indicated in the curves. The dotted curve is an extrapolated $E2$ result for large ξ.

[16] R. K. Osborne and M. E. Rose, Oak Ridge National Laboratory Report No. 1685, 1954 (unpublished).

[17] N. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **18**, No. 8 (1948).

[18] G. Breit and P. B. Daitch, Phys. Rev. **96**, 1447 (1954); L. C. Biedenharn and C. M. Class, Phys. Rev. **98**, 691 (1955).

[19] For $E2$ transitions, this was noted earlier by K. Alder and A. Winther, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd **29**, No. 19 (1955).

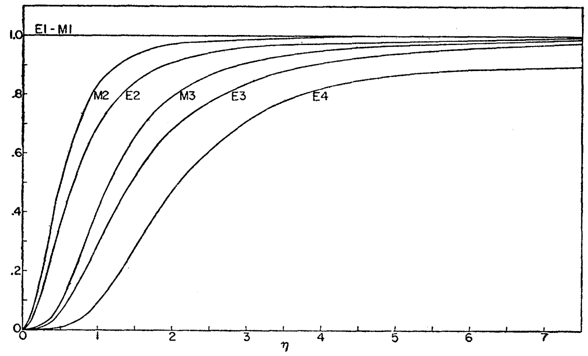

FIG. 5. The ratio function $R(\eta)$ *vs* η in the limit $\xi=0$ for various multipoles. These functions can be used to estimate the quantum mechanical excitation functions; see Eqs. (28)–(29).

deviation from the $\xi=0$ curve is not negligible for $\eta<\frac{3}{2}$ the spread nevertheless does not appear to be very great. Extrapolation of the available numerical data to large values of ξ yields the dotted curve[20] shown in Fig. 4. It is believed that for $\xi \gtrsim 4$, the dotted curve is an upper limit to the function $R_{E2}(\eta,\xi)$. All the published numerical results lie between this curve and the $\xi=0$ curve.

In the electric and magnetic dipole cases, $R_{E1}(\eta,\xi)=1$ for all values of η, for $\xi=0$. For the electric dipole case, the $\xi=0.1$ curve also appears. For all practical purposes this curve is indistinguishable from the curves for all values of ξ for which the exact calculations were performed, except for the region $\eta \lesssim 0.2$, $\xi \lesssim 0.01$. This behavior is a consequence of the logarithmic divergence of the $E1$ result as $\xi \to 0$. It can be shown that for small values of ξ the ratio approaches the limit

$$R_{E1}(\eta,\infty)\sim 1+\frac{\log\eta - \mathrm{Re}\,\psi(1+i\eta)}{\psi(1)-\log(|\xi|/2)}. \tag{25}$$

The limit at $\xi=0$ is therefore unity. However, this result is only weakly dependent on ξ, so that the $\xi=0.1$ curve yields a much better approximation to the function $R_{E1}(\eta,\xi)$ in the experimental region than does the $\xi=0$ curve.

The summation formula[21] for the $E1$ case allows one to obtain a closed form for the limit as $\xi\to\infty$ of $R_{E1}(\eta,\xi)$. The result is

$$\lim_{\xi\to\infty} R_{E1}(\eta,\xi)=2\sqrt{3}F_0(\eta,2\eta)F_0'(\eta,2\eta), \tag{26}$$

where $F_0(\eta,\rho)$ is the radial Coulomb wave function and $F_0'(\eta,\rho)$ is its derivative with respect to ρ. This curve is also plotted in Fig. 4. For values of $\eta>2$, this curve is given by the asymptotic formula[22]

$$R_{E1}(\eta,\infty)\sim 1+0.1728\eta^{-2/3}-0.0496\eta^{-4/3}+\cdots. \tag{27}$$

Unlike the $E2$ curve for large ξ, this limit is of little practical consequence, since all values of experimental interest center about the $\xi=0.1$ curve. The magnetic dipole case is similar in many respects to the electric dipole. The excitation function for $\xi=0$ diverges logarithmically, and $R_{M1}(\eta,0)=1$. Unfortunately, in this case no results in closed form are available. However, it appears that the spread is more marked for the $M1$ case than for the $E1$.

It appears reasonable from the results presented in Fig. 4 that an improved approximation to the exact excitation function can be obtained from

$$b_0(eL)\cong\left(\frac{2L+1}{8\pi}\right)^2(\eta_1\eta_2)^{1-L}(k_1k_2)^L R_{eL}(\eta,0)f_{eL}(\xi), \tag{28}$$

$$b_0(mL)\cong\left(\frac{2L+1}{8\pi}\right)^2(\eta_1\eta_2)^{1-L}(k_1k_2)^{L+1}\times R_{mL}(\eta,0)f_{mL}(\xi), \tag{29}$$

where $f_{eL}(\xi)$ and $f_{mL}(\xi)$ are the classical functions tabulated by Alder and Winther.[10] To this end, we have calculated the $\xi=0$ ratio function, $R(\eta,0)$, for electric multipoles with $L\leq 4$ and magnetic multipoles $L\leq 3$. These results appear in Fig. 5.

Finally it should be remarked that the correlation parameters $b_\nu(\nu>0)$ are much less well represented by the classical results. The principal reasons for this appear to be that the effects of the lower angular momenta (where the classical results are poorest) predominate, and that the sums contain terms of differing sign (unlike the excitation functions) so that cancellation among terms tends to emphasize the error.

In order to increase the usefulness of the results obtained here and previously, the equivalence between this notation and that of A. Bohr and collaborators is given. The reduced electric multipole matrix elements used throughout this work are normalized to unit charge and have the dimensions of (length)^L. The equivalent matrix element used by Bohr is

$$B(eL)=\left(\frac{2J_f+1}{2J_i+1}\right)(f\|r^LY_L\|i)^2(Z_2e)^2. \tag{30}$$

Similarly, for the reduced magnetic multipole, one has

$$B(mL)=\left(\frac{2J_f+1}{2J_i+1}\right)(f\|\mathbf{j}_N\cdot r^L\mathbf{Y}_{LL}\|i)^2\frac{e^2}{L(L+1)}. \tag{31}$$

The total cross section for the general multipole assumes then the two forms[23]

$$\sigma(eL)=\left(\frac{k_2}{k_1}\right)\left[\frac{2J_f+1}{2J_i+1}(f\|r^LY_L^m\|i)^2\right]\left(\frac{8\pi}{2L+1}\right)^2\left(\frac{\eta_1\eta_2}{k_1k_2}\right)b_0(eL)$$
$$=\left(\frac{k_2}{k_1}\right)\frac{B(eL)}{(Z_2e)^2}(\eta_1\eta_2)^{2-L}(k_1k_2)^{L-1}f_{eL}, \tag{32}$$

[20] Unlike the $E1$ case treated below, we have not succeeded in obtaining any limit $\xi\to\infty$ for $R_{E2}(\eta,\xi)$.

[21] L. C. Biedenharn, Phys. Rev. **102**, 262 (1956).

[22] Biedenharn, Gluckstern, Hull, and Breit, Phys. Rev. **97**, 542 (1955).

[23] K. Alder and A. Winther [CERN Report T/AW-1 (unpublished)] have given the classical approximation for the general multipole cross section. Equations (32) and (33) differ from their results in two respects: (*a*) the factor $(v_i/v_f)=(k_1/k_2)$ is misprinted and should be inverted, (*b*) a factor k_2/k_1 is missing for $\sigma(mL)$. For the $E1$ and $E2$ cases, Eq. (32) agrees exactly with their later results [K. Alder and A. Winther, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 19 (1955)].

and

$$\sigma(mL)=\left(\frac{k_2}{k_1}\right)\left[\left(\frac{2J_f+1}{2J_i+1}\right)(f\|j_N\cdot r^L Y_{LL}\|i)^2\right]\left(\frac{Z_1e^2}{\hbar ck_1k_2}\right)^2\left(\frac{8\pi}{2L+1}\right)^2 b_0(mL)=\left(\frac{k_2}{k_1}\right)\left(\frac{B(mL)}{e^2}\right)\left(\frac{Z_1e^2}{\hbar c}\right)^2(\eta_1\eta_2)^{1-L}(k_1k_2)^{L-1}f_{mL}. \quad (33)$$

The correlation parameters $a_\nu=b_\nu/b_0$ are pure numbers so that there exists no difficulty in comparing these results.

APPENDIX

The differential cross section for Coulomb excitation may be written as

$$\frac{d\sigma(\theta)}{d\Omega}=\frac{k_f}{k_i}\left(\frac{2M}{\hbar^2}\right)^2 \mathfrak{S}|H_{if}|^2, \quad \text{(A-1)}$$

where $\mathfrak{S}$ denotes an average over $\hat{k}_f$ and all unobserved spins and H_{if} is the energy density matrix element between initial and final Coulomb distorted plane wave states $\Psi(\hat{k}_i)$ and $\Psi(\hat{k}_f)$, where the $\Psi(\hat{k})$ are as given in I, Eqs. (8), (9), and $\hat{k}=\mathbf{k}/k$. Explicitly, one has

$$H_{if}=\sum i^{-J-\epsilon}D_{M,P}{}^J(\gamma)\langle j_3m_3|\mathbf{j}_N\cdot\mathbf{A}^M{}_{J'\pi'}(<)|j_2m_2\rangle\langle j_2m_2|\mathbf{j}_N \cdot\mathbf{A}^{M'}{}_{J\pi}(<)^\dagger|j_1m_1\rangle\langle\Psi(\hat{k}_f)\chi_{\frac{1}{2},\mu f}|\mathbf{j}_p\cdot\mathbf{A}^{M'}{}_{J\pi}(>)|\Psi(\hat{k}_i)\chi_{\frac{1}{2},\mu i}\rangle, \quad \text{(A-2)}$$

where the sum is over J, M, π, J', M', π', m_2,

$$\mathbf{A}^M{}_{Jm}=[J(J+1)]^{-\frac{1}{2}}f_j(kr)\mathbf{L}Y_{J,M}(\theta,\phi)] \quad \text{(A-3a)}$$

for magnetic multipoles,

$$A^M{}_{Je}=ik^{-1}\nabla\times\mathbf{A}^M{}_{Jm} \quad \text{(A-3b)}$$

for electric multipoles, $f_J(kr)$ represents either the regular spherical Bessel function $f_J(kr)$ or the irregular function $h_J(kr)$ for outgoing waves as indicated by the symbols $<$ and $>$ respectively, and $\epsilon=0, 1$ for electric and magnetic multipoles, respectively. The symbol $\mathbf{j}_N$ stands for the total (spin and convection) nuclear current operator, $\mathbf{j}_p$ for the impinging particle current operator. The symbol γ in Eq. (A-2) denotes the angles of the gamma-ray direction.

The reduced nuclear matrix elements $(j_am_a\|\mathbf{j}_N\cdot\mathbf{A}^M{}_{J\pi}(<)\|j_bm_b)$ are as defined in I, Eqs. (6) and (7).

Before introducing the Coulomb reduced matrix elements, it is necessary formally to couple the spin and orbital angular momentum through the relation

$$\chi_{\frac{1}{2},\sigma}Y_{l,\mu-\sigma}=\sum_\kappa(l(\kappa)\tfrac{1}{2}\ j(\kappa)\mu\,|\,l(\kappa)\tfrac{1}{2}\ \sigma\ \mu-\sigma)\chi_{\kappa,\mu}. \quad \text{(A-4)}$$

The reduced matrix elements are then defined by

$$\langle\kappa_f\mu_f|\mathbf{j}_p\cdot\mathbf{A}^M{}_{J\pi}{}^\dagger(>)|\kappa_i\mu_i\rangle=(-)^{M+\epsilon}(\kappa_f\|J\pi\|\kappa_i)(j(\kappa_i)J\ j(\kappa_f)\mu_f\,|\,j(\kappa_i)J\ \mu_i-M), \quad \text{(A-5)}$$

similarly to I, Eqs. (10) and (11). The differential cross section may then be expressed as

$$\frac{d\sigma(\theta)}{d\Omega}=\frac{k_f}{k_i}\left(\frac{2M}{\hbar^2}\right)^2\sum(-)^\tau(JJ'\nu0|JJ'1\ -1)W(Jj_2J'j_2;\,j_3\nu)i^{J'-J+\epsilon'-\epsilon}(j_3\|J\pi\|j_2)(j_3\|J'\pi'\|j_2)^* \times(\bar{J}\bar{J}'\nu0|\bar{J}\bar{J}'1\ -1)W(j_2\bar{J}j_2\bar{J}';\,j_1\nu)(j_2\|\bar{J}\bar{\pi}\|j_1)(j_2\|\bar{J}'\bar{\pi}'\|j_1)a_\nu(\bar{J}\bar{\pi}\bar{J}'\bar{\pi}')P_\nu(\cos\theta), \quad \text{(A-6)}$$

where $\sum$ indicates a sum over $J\pi$, $J'\pi'$, $\bar{J}\bar{\pi}$, $\bar{J}',\bar{\pi}'$, and ν (even). Here τ denotes $\bar{\epsilon}-\bar{\epsilon}'$. The particle parameters, a_ν, are given by

$$(\bar{J}\bar{J}'\nu0|\bar{J}\bar{J}'1\ -1)a_\nu(\bar{J}\bar{\pi}\bar{J}'\bar{\pi}')\equiv\sum_{(\kappa\kappa'\kappa'')}i^{l-l'}\exp[i(\sigma_l-\sigma_{l'})](ll\nu0|ll00)[(2l+1)(2l'+1)]^{\frac{1}{2}} \times(\kappa''\|\bar{J}\bar{\pi}\|\kappa)(\kappa''\|\bar{J}'\bar{\pi}'\|\kappa')^*W(j\bar{J}j'\bar{J}';\,j''\nu)(2j''+1)[(2j+1)(2j'+1)]^{\frac{1}{2}}(-)^{\frac{1}{2}-j''}W(jlj'l';\,\tfrac{1}{2}\nu). \quad \text{(A-7)}$$

For electric multipoles, omitting effects due to radial magnetization and convection currents as negligible, the calculation proceeds as in I and yields the result that

$$(\kappa'\|Je\|\kappa)=(-ie)\left[\frac{(2l+1)(2j+1)(2J+1)}{4\pi J(J+1)}\right]^{\frac{1}{2}}(lJl'0|lJ00)(-1)^{l+j'-J-\frac{1}{2}}W(ll'jj';\,J\tfrac{1}{2})(k_1k_2)^{-1} \times\int_0^\infty dr\,F_l(\eta_1,k_1r)F_{l'}(\eta_2,k_2r)\frac{d}{dr}(rh_J(kr)). \quad \text{(A-8)}$$

For magnetic multipoles, considering first convection currents alone, the Coulomb reduced matrix element may be expressed as

$$\langle\kappa'\mu'|\mathbf{j}_N\cdot\mathbf{A}_{Jm}{}^{-M}|\kappa\mu\rangle_{\text{no spin}}=-\left(\frac{e\hbar}{Mc}\right)[J(J+1)]^{-\frac{1}{2}}\langle(k_2r)^{-1}F_{l'}\chi_{\kappa',\mu'}|r^{-2}h_J\mathbf{L}Y_{J,-M}\cdot(\mathbf{r}\times\mathbf{L})|(k_1r)^{-1}F_l\chi_{\kappa,\mu}\rangle, \quad \text{(A-9)}$$

through the use of Eq. (A-3) and by replacing the gradient operator in $\mathbf{j}_N$ by $-ir^{-2}\mathbf{r}\times\mathbf{L}$. Using the inverse of of Eq. (A-4) and the properties of the vector spherical harmonics[24] the indicated integration may be performed to give, for the reduced magnetic matrix element for convection terms alone, the result

$$(\kappa'\|Jm\|\kappa)=i(4\pi)^{-\frac{1}{2}}(e\hbar/Mc)(2J+1)(2l+1)[(l+1)(2l+3)(2j+1)]^{\frac{1}{2}}(-)^{l'+\frac{1}{2}-j'}(l+1\;Jl'0|l+1\;J00)$$
$$\times W(JJ\;l+1\;l;1l')W(jlj'l';\tfrac{1}{2}J)(k_1k_2)^{-1}\int_0^\infty drF_l(\eta_1,k_1r)F_{l'}(\eta_2,k_2r)\frac{h_j}{r}. \quad \text{(A-10)}$$

To evaluate the spin contributions, one introduces the spin magnetization current $\mathbf{j}=(e\hbar\mu/2Mc)\nabla\times\langle\boldsymbol{\sigma}\rangle$; and again utilizing the properties of the vector spherical harmonics, one obtains

$$\langle\kappa'\mu'|\mathbf{j}_{\text{spin}}\cdot\mathbf{A}^M{}_{Jm}|\kappa\mu\rangle=(e\hbar k\mu/2Mc)[J(J+1)]^{-\frac{1}{2}}\langle(k_2r)^{-1}F_{l'}\chi_{\kappa',\mu'}|\{(h'_J+(kr)^{-1}h_J)\boldsymbol{\sigma}\cdot(\mathbf{r}\times\mathbf{L})Y_{J,-M}$$
$$+(J)(J+1)(kr)^{-1}\boldsymbol{\sigma}\cdot\hat{r}Y_{J,-M}\}|(k_1r)^{-1}F_l\chi_{\kappa,\mu}\rangle. \quad \text{(A-11)}$$

The properties of the spin-angle functions, $\chi_{\kappa,\mu}$, greatly facilitate this integration, yielding the final result that

$$(\kappa'\|Jm\|\kappa)_{\text{spin only}}=(-ie\hbar k\mu/2Mc)[4\pi J(J+1)]^{-\frac{1}{2}}(-)^{j'-\frac{1}{2}+J}[(2l'+1)(2j+1)(2l(-\kappa)+1)]^{\frac{1}{2}}(l(-\kappa)l'J0|l(-\kappa)l'00)$$
$$\times W(l'j'l(-\kappa)j;\tfrac{1}{2}J)\cdot(k_1k_2)^{-1}\left\{(\kappa+\kappa')\int_0^\infty drF_{l'}(\eta_1,k_2r)F_l(\eta_1,k_1r)\left(h_J+\frac{h_J}{kr}\right)\right.$$
$$\left.-J(J+1)\int_0^\infty drF_{l'}(\eta_2,k_2r)F_l(\eta_1,k_1r)\frac{h_J}{kr}\right\}. \quad \text{(A-12)}$$

The complete reduced matrix elements for magnetic multipoles is the sum of Eqs. (A-10) and (A-12). In the text the long-wavelength approximation is employed for the Coulomb integrals in the reduced matrix elements, i.e., $h_J(kr)\to -i(2J-1)!!(kr)^{-L-1}$. In this limit the Coulomb reduced matrix elements are explicitly real, and the radial integrals for $E2$ and $M1$ are all of the same form, *viz.*:

$$I_{ll'}=\int_0^\infty drr^{-3}F_l(\eta_1,k_1r)F_{l'}(\eta_2,k_2r).$$

The particle parameters that must be calculated for mixed $E2-M1$ excitation are $a_\nu(2e,2e)$, $a_\nu(1m,1m)$ and the mixed terms $a_\nu(2e,1m)$ and $a_\nu(1m,2e)$. The $a_\nu(2e,2e)$ are calculated as in I and are given by I, Eqs. (20)–(22). For convection currents alone the $a_\nu(1m,1m)$ are easily evaluated to be

$$(11\nu0|111\;-1)a_\nu(1m1m)=\left(\frac{e\hbar}{Mc}\right)^2\left(\frac{9}{4\pi}\right)(k_1k_2k^2)^{-2}\sum_l(2l+1)^3(l+1)(2l+3)[(l+1\;1l0|l+1\;100)]^2$$
$$\times W^2(11\;l+1l;1l)I_{ll}{}^2(ll\nu0|ll00)\{\cdots\}, \quad \text{(A-13)}$$

where

$$\{\cdots\}=\sum_{jj'j''}(-1)^{\frac{1}{2}-j''}(2j''+1)(2j+1)(2j'+1)W(jlj'l;\tfrac{1}{2}\nu)W(jlj'l;\tfrac{1}{2}\nu)W(jlj''l;\tfrac{1}{2}1)W(j'lj''l;\tfrac{1}{2}\,1)$$
$$=2(-)^{l+1}W(1\nu ll;1l).$$

The sum in curly brackets can be carried out by using the orthonormality of the Racah coefficients and the sum rule quoted below.[25] Thus for convection currents alone:

$$a_0(1m1m)=a_2(1m1m)=-\left(\frac{e\hbar}{Mc}\right)^2\left(\frac{3}{4\pi}\right)(k_1k_2k^2)^{-2}\sum_l l(l+1)(2l+1)I^2{}_{ll}. \quad \text{(A-14)}$$

[24] H. C. Corben and J. Schwinger, Phys. Rev. 58, 967 (1940). J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Appendix B. See also M. E. Rose, *Multipole Fields* (John Wiley and Sons, Inc., New York, 1955).

[25] L. C. Biedenharn, J. Math. Phys. 31, 287 (1953).

In order to evaluate the spin contribution to the magnetic dipole particle parameters, it is necessary to evaluate spin-convection current cross terms as well as spin-spin contributions. The former turn out to be zero, since terms linear in the spin cannot arise from unpolarized beams. The spin-spin contribution is

$$(11\nu 0|111\ -1)a_\nu(1m1m)_{\text{spin only}}=\left(\frac{e\hbar\mu}{Mc}\right)^2(k_1k_2k^2)^{-2}\left(\frac{1}{32\pi}\right)\sum_{\kappa\kappa'\kappa''}i^{l-l'}\exp[i(\sigma_l-\sigma_{l'})](ll'\nu 0|ll'00)$$
$$\times[(2l+1)(2l'+1)]^{\frac{1}{2}}(2j''+1)(2l''+1)(2j+1)(2j'+1)(-)^{j''-\frac{1}{2}}$$
$$\times[2l(-\kappa)+1]^{\frac{1}{2}}[2l(-\kappa')+1]^{\frac{1}{2}}(l(-\kappa)l''10|l(-\kappa)l''00)(l(-\kappa')l''10|l(-\kappa')l''00)$$
$$\times(\kappa+\kappa''+2)(\kappa'+\kappa''+2)I_{ll''}I_{l'l''}W(l''j''l(-\kappa)j;\tfrac{1}{2}\,1),$$
$$W(l''j''l(-\kappa')j';\tfrac{1}{2}\,1)W(j1j'1;j''\nu)W(jlj'l';\tfrac{1}{2}\nu)=(e\hbar\mu/Mc)^2(k_1k_2k^2)^{-2}(9/8\pi)2^{-\nu}(11\nu 0|1100)$$
$$\times\{(22\nu 0|221\ -1)^{-1}\sum_{ll'l''}(2l+1)(2l'+1)i^{l-l'}(-)^{l+1}\exp[i(\sigma_l-\sigma_{l'})](l2l''0|l200)(l'2l0|l'200)$$
$$\times W(ll''\nu 2;2l')I_{ll''}I_{l'l''}\}.\qquad\text{(A-15)}$$

The reduction implicit in the right hand side of Eq. (A-15) is obtained through the use of the identity

$$\sum_{jj'j''}(2j''+1)(2j+1)(2j'+1)(-)^{j''-\frac{1}{2}}[2l(-\kappa)+1]^{\frac{1}{2}}[2l(-\kappa')+1]^{\frac{1}{2}}(l(-\kappa)l''10|l(-\kappa)l''00)$$
$$\times(l(-\kappa')l''10|l(-\kappa')l''00)(\kappa+\kappa''+2)(\kappa'+\kappa''+2)W(j1j'1;j''\nu)W(ljl'j';\tfrac{1}{2}\nu)W(l''j''l(-\kappa)j;\tfrac{1}{2}\,1)$$
$$\times W(l''j''l(-\kappa')j';\tfrac{1}{2}\,1)=(-)^{l+1}(36)(2^{-\nu})\left[\frac{(2l+1)(2l'+1)}{(2l''+1)^2}\right]^{\frac{1}{2}}[(22\nu 0|221\ -1)]^{-1}$$
$$\times(111\nu 0|1100)(l2l''0|l200)(l'2l''0|l'200)W(ll''\nu 2;2l').\qquad\text{(A-16)}$$

Identifying the right-hand side of Eq. (A-15) with I—Eqs. (20)–(21) yields the result quoted in Eqs. (4)–(6) of the text.

Similarly the mixed particle parameters $a_2(1m,2e)=a_2^*(2e,1m)$ (in the limit of zero retardation) may be calculated to be

$$(1220|121-1)b_2(1m2e)=\left[\frac{3(30)^{\frac{1}{2}}}{4\pi}\left(\frac{e^2\hbar k}{Mc}\right)\right](k_1k_2k^3)^{-2}\sum\exp[i(\sigma_l-\sigma_{l'})]i^{l-l'}(ll'20|ll'00)(2j''+1)(2j+1)$$
$$\times(2j'+1)(l'2l''0|l'200)(-)^{l'-\frac{1}{2}+j''+l''}W(l''l'j''j';2\,\tfrac{1}{2})I_{l'l''}[(l+1)(2l+3)(2l+1)]^{\frac{1}{2}}$$
$$\times(1l+1\ l''0|1l+1\ 00)W(11l+1l;1l'')W(jlj''l'';\tfrac{1}{2}\,1)I_{ll''}(2l+1)(2l'+1)$$
$$\times W(j1j'2;j''2)W(jlj'l';\tfrac{1}{2}\,2),\qquad\text{(A-17)}$$

where the sum is taken over all the indices.

Using the Racah function identity quoted earlier,[25] one obtains the result that

$$\sum_{jj'j''}(2j+1)(2j'+1)(2j''+1)(-)^{j''-\frac{1}{2}}W(j1j'2;j''2)W(jlj'l';\tfrac{1}{2}\,2)W(l''l'j''j';2\,\tfrac{1}{2})W(jlj''l'';\tfrac{1}{2}\,1)$$
$$=2(-)^{l'+1}W(2l''2l;l'1).\qquad\text{(A-18)}$$

The explicit final result is then

$$b_2(1m2e)=\left[\left(\frac{3[15]^{\frac{1}{2}}}{4\pi}\right)\left(\frac{e^2\hbar k}{Mc}\right)(k_1k_2k^3)^{-2}\right]\sum_l\left\{\frac{l(l+1)(2l+1)}{(2l-1)(2l+3)}I^2{}_{ll}+\frac{l(l+1)(l+2)}{(2l+3)}\exp[i(\sigma_l-\sigma_{l+2})]I_{ll}I_{l+2,\,l}\right.$$
$$\left.-\frac{(l-1)l(l+1)}{(2l-1)}\exp[i(\sigma_l-\sigma_{l-2})]I_{ll}I_{l-2,\,l}\right\},\qquad\text{(A-19)}$$

which gives Eq. (9) in the text.