Rayleigh Wave Propagation on Anisotropic (Cubic) Media

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The essential features of surface-wave propagation for elastic waves on anisotropic media are delineated by consideration of the solution for cubic crystals. In using the coordinate system defined by the surface and the direction normal to it, transformation of the elastic coefficient tensor is required. Conventional means for doing this can be prohibitively laborious; but by invoking the isotropy condition, the calculation becomes quite amenable. Detailed elaboration is given for propagation in the (100) and (110) planes. The set of relations from which the damping coefficient and the Rayleigh wave velocity can be evaluated is derived.

INTRODUCTION

WHILE the propagation of surface waves has been characterized for isotropic media, the more general problem for anisotropic media apparently has not been considered beyond possibly the special case of propagation along the cubic axis of a crystal[1]; as we shall show, this isolated solution is not correct and the analysis fails to bring to light the basic features of the anisotropy. Other extensions of Rayleigh waves may be found in the literature such as Sveklo's[2] treatment for orthotropic materials and Fu's[3] interesting investigation of spherical-type surface waves.[4,5]

Our approach consists of utilizing a coordinate system containing the planar surface and the orthogonal direction of damping. This means that the tensor for the elastic constants (here taken as the stiffness coefficients c_{jk}) must be transformed to these generalized coordinates. The transformation usually entails laborious details[6]; but we introduce a technique based upon the isotropy condition which considerably facilitates this operation.

For simplicity we have confined our treatment to cubic crystals and have elaborated the calculation for propagation in the (110) and (100) planes. Other crystal systems can be dealt with in similar manner but with necessarily more cumbersome algebraic detail.[4]

As might perhaps be anticipated, we find that the expression for the penetration depth involves square roots of algebraic terms, which make it very awkward to extract an explicit solution for the Rayleigh wave velocities. Nevertheless, the velocities can be found by graphical means. Also, as we shall see, once having found the essential relations for propagation in the (100) plane, we can immediately deduce those for the (110) plane.

TRANSFORMATION SYSTEM

We initially designate our coordinate system by primes X_1', X_2', X_3' with the cubic reference frame specified by X_1, X_2, X_3; the plane of propagation is X_1', X_2' with damping along X_3'. Thus, for the two planes we are considering, the transformation matrices are

	(100) plane				(110) plane		
	X_1	X_2	X_3		X_1	X_2	X_3
X_1'	$\cos\theta$	$\sin\theta$	0	X_1'	$(1/\sqrt{2})\sin\theta$	$(1/\sqrt{2})\sin\theta$	$\cos\theta$
X_2'	$-\sin\theta$	$\cos\theta$	0	X_2'	$(1/\sqrt{2})\cos\theta$	$(1/\sqrt{2})\cos\theta$	$-\sin\theta$
X_3'	0	0	1	X_3'	$-1/\sqrt{2}$	$1/\sqrt{2}$	0

For the (100) plane, θ is the angle between direction of propagation and cube axis $(X_1'X_1)$; the (110) case has θ the angle $(X_3'X_3)$.

The fourth rank tensor for the stiffness coefficients of a cubic crystal transform via the well-known relation (actually the definition of the tensor) which reads symbolically

$$\bar{A}^{ijkl}=A^{ijkl}\frac{\partial x^\alpha}{\partial\bar{x}_i}\frac{\partial x^\beta}{\partial\bar{x}_j}\frac{\partial x^\gamma}{\partial\bar{x}_k}\frac{\partial x^\delta}{\partial\bar{x}_l}, \tag{1}$$

whereby we find[7] the transformed tensors of Table I.

In general the transformed coefficients are of the form

$$c_{jk}'=c_{11}f_1(\theta)+c_{12}f_2(\theta)+c_{44}f_3(\theta). \tag{2}$$

Operationally, relation (1) means

$$A_{pq}{}'^{mn}=\sum_r\sum_s\sum_v\sum_w A_{vw}{}^{rs}\cos(X_rX_m')\times\cos X_sX_n'\cos(X_vX_p')\cos(X_wX_q'), \tag{3}$$

with the c_{jk} written in four-index style and the $A_{pq}{}'^{mn}$ and $A_{vw}{}^{rs}$ reduced to two indices via 11→1, 22→2, 33→3, 12→6, 13→5, 23→4. If we expand (3) only for terms having all indices $A_{vw}{}^{rs}$ identical, then we readily secure $f_1(\theta)$; obtaining $f_2(\theta)$ and $f_3(\theta)$ by direct ex-

[1] G. Garcia, Univ. nac. La Plata. Publs. Fac. cien. fismat. **11**, No. 2 (1941).
[2] V. A. Sveklo, Doklady Akad. Nauk U.S.S.R. **59**, 871 (1948).
[3] C. Y. Fu, Geophys. **12**, 57 (1947).
[4] Recently, Stoneley[5] has formulated the problem of propagation in a (100) plane and has explicitly solved for velocities along a [100] and [110] direction. The approach offered in the present paper does not restrict solutions to special high-symmetry directions because of the implementation of a reduced transformation scheme that greatly minimizes the algebra.
[5] R. Stoneley, Proc. Roy. Soc. (London) **232**, 447 (1955).
[6] W. G. Cady, *Piezoelectricity* (McGraw-Hill Book Company, Inc., New York, 1946), p. 69.

[7] See Appendix for true tensor form which is always symmetrical. The unsymmetrical forms above are due to reduced notation for the original 9×9 tensor.

pansion is tedious, particularly where only two of the indices m, n, p, q are repeated.

If we invoke the isotropy condition $c_{44}=\frac{1}{2}(c_{11}-c_{12})$ for which the c_{jk}' reduce to certain c_{jk}, then

$$c_{jk}=c_{jk}'=c_{11}f_1+c_{12}f_2+\tfrac{1}{2}(c_{11}-c_{12})f_3 =c_{11}(f_1+\tfrac{1}{2}f_3)+c_{12}(f_2-\tfrac{1}{2}f_3). \quad (4)$$

Now c_{jk} in the original tensor has four possible values, each of which defines a different set of f_2 and f_3 functions in terms of the known function f_1. Table II summarizes the situation.

In order to determine f_2 and f_3 and thus the c_{jk}' relations, we tabulate the expression for f_1 in Table III, which combined with Table II, gives us the following results:

Tensor components for (100) plane

$$\begin{aligned}
c_{11}'&=c_{11}(\cos^4\theta+\sin^4\theta)+2(c_{12}+2c_{44})\sin^2\theta\cos^2\theta,\\
c_{12}'&=2(c_{11}-2c_{44})\sin^2\theta\cos^2\theta+c_{12}(\sin^4\theta+\cos^4\theta),\\
c_{16}'&=(c_{11}-c_{12}-2c_{44})\sin\theta\cos\theta(\sin^2\theta-\cos^2\theta),\\
c_{22}'&=c_{11}(\cos^4\theta+\sin^4\theta)+2(c_{12}+2c_{44})\sin^2\theta\cos^2\theta=c_{11}',\\
c_{26}'&=-c_{16}',\\
c_{66}'&=4(c_{11}-c_{12})\sin^2\theta\cos^2\theta+c_{44}(1-8\sin^2\theta\cos^2\theta).
\end{aligned} \quad (5)$$

Tensor components for (110) plane

$$\begin{aligned}
c_{11}'&=c_{11}(\tfrac{1}{2}\sin^4\theta+\cos^4\theta)+(c_{12}+2c_{44})\\
&\qquad\times(\tfrac{1}{2}\sin^4\theta+2\sin^2\theta\cos^2\theta),\\
c_{12}'&=3(\tfrac{1}{2}c_{11}-c_{44})\sin^2\theta\cos^2\theta\\
&\qquad+c_{12}(\cos^4\theta+\sin^4\theta+\tfrac{1}{2}\sin^2\theta\cos^2\theta),\\
c_{13}'&=(\tfrac{1}{2}c_{11}-c_{44})\sin^2\theta+c_{12}(1-\tfrac{1}{2}\sin^2\theta),\\
c_{16}'&=(c_{11}-c_{12}-2c_{44})\sin\theta\cos\theta(\tfrac{1}{2}\sin^2\theta-\cos^2\theta),\\
c_{22}'&=c_{11}(\tfrac{1}{2}\cos^4\theta+\sin^4\theta)+(c_{12}+2c_{44})\\
&\qquad\times(\tfrac{1}{2}\cos^4\theta+2\sin^2\theta\cos^2\theta),\\
c_{23}'&=(\tfrac{1}{2}c_{11}-c_{44})\cos^2\theta+c_{12}(1-\tfrac{1}{2}\cos^2\theta),\\
c_{26}'&=(c_{11}-c_{12}-2c_{44})\sin\theta\cos\theta(\tfrac{1}{2}\cos^2\theta-\sin^2\theta),\\
c_{33}'&=\tfrac{1}{2}(c_{11}+c_{12}+2c_{44}),\\
c_{36}'&=[\tfrac{1}{2}(c_{11}-c_{12})-c_{44}]\sin\theta\cos\theta,\\
c_{44}'&=(c_{11}-c_{12})\cos^2\theta+c_{44}(1-2\cos^2\theta),\\
c_{45}'&=c_{36}',\\
c_{55}'&=(c_{11}-c_{12})\sin^2\theta+c_{44}(1-2\sin^2\theta),\\
c_{66}'&=3(c_{11}-c_{12})\sin^2\theta\cos^2\theta+c_{44}(1-6\sin^2\theta\cos^2\theta).
\end{aligned} \quad (6)$$

SURFACE WAVE SOLUTION FOR THE (100) PLANE

We now specify our primed coordinate as x, y, and z. The displacements along the direction of propagation x and the direction of damping z are taken as u and w, respectively; for Rayleigh waves $v=0$. Thus for Hooke's law we find

TABLE I. Transformed tensors for the stiffness coefficients in the (100) and (110) planes.

$$\begin{vmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0\\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0\\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{vmatrix} \quad \text{cubic reference frame}$$

(100)

$$\begin{vmatrix} c_{11}' & c_{12}' & c_{12} & 0 & 0 & 2c_{16}'\\ c_{12}' & c_{11}' & c_{12} & 0 & 0 & 2c_{26}'\\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0\\ c_{16}' & c_{26}' & 0 & 0 & 0 & c_{66}' \end{vmatrix}$$

(110)

$$\begin{vmatrix} c_{11}' & c_{12}' & c_{13}' & 0 & 0 & 2c_{16}'\\ c_{12}' & c_{22}' & c_{33}' & 0 & 0 & 2c_{26}'\\ c_{13}' & c_{23}' & c_{33}' & 0 & 0 & 2c_{36}'\\ 0 & 0 & 0 & c_{44}' & 2c_{36}' & 0\\ 0 & 0 & 0 & 2c_{36}' & c_{55}' & 0\\ c_{16}' & c_{26}' & c_{36}' & 0 & 0 & c_{66}' \end{vmatrix}$$

$$\begin{aligned}
\sigma_x&=c_{11}'\frac{\partial u}{\partial x}+c_{12}\frac{\partial w}{\partial z}, & \tau_{yz}&=0,\\
\sigma_y&=c_{12}'\frac{\partial u}{\partial x}+c_{12}\frac{\partial w}{\partial z}, & \tau_{xz}&=c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\\
\sigma_z&=c_{12}\frac{\partial u}{\partial x}+c_{11}\frac{\partial w}{\partial z}, & \tau_{xy}&=c_{16}'\frac{\partial u}{\partial x}.
\end{aligned} \quad (7)$$

For the equation of equilibrium

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}=\rho\frac{\partial^2u}{\partial t^2},\quad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}=\rho\frac{\partial^2w}{\partial t^2}, \quad (8)$$

we assume damped wave solutions of the form

$$\begin{aligned}
u&=u_0e^{-qz}e^{i\omega(t-x/V)},\\
v&=0,\\
w&=w_0e^{-qz}e^{i\omega(t-x/V)}.
\end{aligned} \quad (9)$$

The wave equations are

$$\begin{aligned}
c_{11}'\frac{\partial^2u}{\partial x^2}+(c_{12}+c_{44})\frac{\partial^2w}{\partial z\partial x}+c_{44}\frac{\partial^2u}{\partial z^2}&=-\rho\omega^2u,\\
c_{44}\frac{\partial^2w}{\partial x^2}+(c_{12}+c_{44})\frac{\partial^2u}{\partial x\partial z}+c_{11}\frac{\partial^2w}{\partial z^2}&=-\rho\omega^2w,
\end{aligned} \quad (10)$$

TABLE II. The angular functions f_2 and f_3 as related to f_1 for $c_{jk}=c_{11}$, c_{12}, $\frac{1}{2}c_{44}$, and 0.

c_{jk}	f_2	f_3
c_{11}	$1-f_1$	$2(1-f_1)$
c_{12}	$1-f_1$	$-2f_1$
$\frac{1}{2}c_{44}$	$-f_1$	$2(\frac{1}{4}-f_1)$
0	$-f_1$	$-2f_1$

TABLE III. The angular function f_1 for the twenty-one c_{jk}' that appear in the transformed tensor for propagation in the (100) and (110) planes.

c_{jk}'	f_1 (100)	f_1 (110)
c_{11}'	$\cos^4\theta+\sin^4\theta$	$\frac{1}{2}\sin^4\theta+\cos^4\theta$
c_{12}'	$2\sin^2\theta\cos^2\theta$	$\frac{3}{2}\sin^2\theta\cos^2\theta$
c_{13}'	0	$\frac{1}{2}\sin^2\theta$
c_{14}'	0	0
c_{15}'	0	0
c_{16}'	$\sin\theta\cos\theta(\sin^2\theta-\cos^2\theta)$	$\sin\theta\cos\theta(\frac{1}{2}\sin^2\theta-\cos^2\theta)$
c_{22}'	$\cos^4\theta+\sin^4\theta=c_{11}'$	$\frac{1}{2}\cos^4\theta+\sin^4\theta$
c_{23}'	0	$\frac{1}{2}\cos^2\theta$
c_{24}'	0	0
c_{25}'	0	0
c_{26}'	$\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)=-c_{16}'$	$\sin\theta\cos\theta(\frac{1}{2}\cos^2\theta-\sin^2\theta)$
c_{33}'	1	$\frac{1}{2}$
c_{34}'	0	0
c_{35}'	0	0
c_{36}'	0	$\frac{1}{2}\sin\theta\cos\theta$
c_{44}'	0	$\frac{1}{2}\cos^2\theta=c_{23}'$
c_{45}'	0	$\frac{1}{2}\sin\theta\cos\theta=c_{36}'$
c_{46}'	0	0
c_{55}'	0	$\frac{1}{2}\sin^2\theta=c_{13}'$
c_{56}'	0	0
c_{66}'	$2\sin^2\theta\cos^2\theta=c_{12}'$	$\frac{3}{2}\sin^2\theta\cos^2\theta=c_{12}'$

which from (9) yield the secular equation

$$\begin{vmatrix} \left(c_{44}q^2-\dfrac{\omega^2}{V^2}c_{11}'+\rho\omega^2\right) & \dfrac{i\omega q}{V}(c_{12}+c_{44}) \\ \dfrac{i\omega q}{V}(c_{12}+c_{44}) & \left(c_{11}q^2-\dfrac{\omega^2}{V^2}c_{44}+\rho\omega^2\right) \end{vmatrix}=0. \quad (11)$$

In general there will be two roots for q (reciprocal of the penetration depth):

$$S_3\frac{q_{1,2}^2}{\omega^2}=-\frac{1}{2}\left(S_1\rho+\frac{S_2}{V^2}\right)^2 \pm\left[(\tfrac{1}{4}S_1^2-S_3)\rho^2+(\tfrac{1}{2}S_1S_2+S_4)\frac{\rho}{V^2}+(\tfrac{1}{4}S_2^2-S_5)\frac{1}{V^4}\right]^{\frac{1}{2}} \quad (12a)$$

where the abbreviations are

$$S_1=c_{11}+c_{44},\quad S_2=c_{12}(c_{12}+2c_{44})-c_{11}c_{11}',\quad S_3=c_{11}c_{44},$$
$$S_4=c_{11}c_{44}(c_{11}'+c_{44}),\quad S_5=c_{11}c_{11}'c_{44}^2. \quad (12b)$$

Our result for q clearly shows that the penetration depth varies inversely with the frequency of the elastic wave. Explicit solution for q requires evaluation of the Rayleigh wave velocity V. Actually we need another relation between q and V which indeed derives from the boundary condition, namely, that all stresses associated with the directional normal to the surface vanish at the surface, i.e., $\sigma_z=\tau_{xz}=0$ at $z=0$.

Thus from

$$c_{12}\frac{\partial u}{\partial x}+c_{11}\frac{\partial w}{\partial z}=0,\quad \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0, \quad (13)$$

and the composite wave solutions

$$U=U_1+U_2=(U_0e^{-q_1z}+U_0'e^{-q_2z})e^{i\omega(t-x/V)},$$
$$w=w_1+w_2=(w_0e^{-q_1z}+w_0'e^{-q_2z})e^{i\omega(t-x/V)}, \quad (14)$$

we arrive at

$$-c_{12}(i\omega/V)(U_0+U_0')-c_{11}(q_1w_0+q_2w_0')=0,$$
$$-(U_0q_1+U_0'q_2)-(i\omega/V)(\omega_0+\omega_0')=0. \quad (15)$$

We eliminate w_0 and w_0' by observing that U_0 and w_0 and U_0' and w_0' are related as indicated in the secular Eq. (11), i.e.,

$$\frac{U_0}{w_0}=-\frac{(c_{12}+c_{44})(i\omega q_1/V)}{\rho\omega^2-(\omega^2/V^2)c_{11}'+c_{44}q_1^2}, \quad (16)$$

with a similar expression for U_0'/w_0', q_2 replacing q_1. Hence we find that

$$a_{11}U_0+a_{12}U_0'=0,\quad a_{21}U_0+a_{22}U_0'=0, \quad (17a)$$

with

$$a_{11}=-ic_{12}\frac{\omega}{V}+\left(\frac{c_{11}}{c_{12}+c_{44}}\right)\left(\frac{\rho\omega^2-(\omega^2/V^2)c_{11}'+c_{44}q_1^2}{i\omega/V}\right),$$

$$a_{12}=-ic_{12}\frac{\omega}{V}+\left(\frac{c_{11}}{c_{12}+c_{44}}\right)\left(\frac{\rho\omega^2-(\omega^2/V^2)c_{11}'+c_{44}q_2^2}{i\omega/V}\right), \quad (17b)$$

$$a_{21}=-q_1+\left(\frac{1}{c_{12}+c_{44}}\right)\left(\frac{1}{q_1}\left(\rho\omega^2-\frac{\omega^2}{V^2}c_{11}'+c_{44}q_1^2\right)\right),$$

$$a_{22}=-q_2+\left(\frac{1}{c_{12}+c_{44}}\right)\left(\frac{1}{q_2}\left(\rho\omega^2-\frac{\omega^2}{V^2}c_{11}'+c_{44}q_2^2\right)\right).$$

This gives for our second relation involving q and V:

$$(q_1^2/\omega^2)\{\ \}_{\mathrm{I}}^2\{\ \}_{\mathrm{II}}^2=(q_2^2/\omega^2)\{\ \}_{\mathrm{III}}^2\{\ \}_{\mathrm{IV}}^2, \quad (18a)$$

where

$$\{\ \}_{\mathrm{I}}=c_{11}\rho+(c_{12}^2+c_{12}c_{44}-c_{11}c_{11}')/V^2+c_{11}c_{44}q_1^2/\omega^2,$$
$$\{\ \}_{\mathrm{II}}=\rho-(c_{11}'/V^2)-c_{12}(q_2^2/\omega^2),$$
$$\{\ \}_{\mathrm{III}}=c_{11}\rho+(c_{12}^2+c_{12}c_{44}-c_{11}c_{11}')/V^2+c_{11}c_{44}q_2^2/\omega^2, \quad (18b)$$
$$\{\ \}_{\mathrm{IV}}=\rho-(c_{11}'/V^2)-c_{12}(q_1^2/\omega^2).$$

Since the q^2/ω^2 contain only c_{ik}, V, and ρ, Eq. (18a) properly does not really contain ω; insertion of (12a) into (18a) actually gives the equation for the Rayleigh wave velocity. Solution for V may best be found by plotting $q_{1,2}^2/\omega^2$ *vs* V and then plotting the left- and right-hand members of Eq. (18a) to find the crossover points.[8]

A check of our central relations (12a) and (18a) is afforded by noting how these reduce for the isotropic case. Letting $c_{44}=\frac{1}{2}(c_{11}-c_{12})$, we obtain $c_{11}'=c_{11}$, whence

$$c_{11}c_{44}q_{1,2}^2/\omega^2=-\tfrac{1}{2}[\tfrac{1}{2}(3c_{11}-c_{12})\rho+c_{11}(c_{12}-c_{11})/V^2]\pm\tfrac{1}{4}(c_{11}+c_{12})\rho.$$

The two solutions are then

$$\frac{q_1^2}{\omega^2}=\frac{1}{V^2}-\frac{\rho}{c_{11}},\quad \frac{q_2^2}{\omega^2}=\frac{1}{V^2}-\frac{\rho}{\frac{1}{2}(c_{11}-c_{12})}. \tag{19}$$

We can see that these results are correct for isotropy since, for $q=0$, Eq. (19) yields

$$V_1=(c_{11}/\rho)^{\frac{1}{2}},\quad V_2=[\tfrac{1}{2}(c_{11}-c_{12})/\rho]^{\frac{1}{2}}, \tag{20}$$

which are the correct expressions for the body-wave velocities for longitudinal and transverse modes, respectively. If we had put $q=0$ in (12a) instead, the body-wave velocities for propagation in some direction of the (100) plane would result. Indeed,

$$V^4-\frac{S_4}{S_3\rho}V^2+\frac{S_5}{S_3\rho^2}=0,$$

or

$$V^2=\frac{1}{S_3\rho}[\tfrac{1}{2}S_4\pm(\tfrac{1}{4}S_4^2-S_3S_5)^{\frac{1}{2}}],$$

which reduces to

$$\rho V^2=\tfrac{1}{2}(c_{11}'+c_{44})\pm\tfrac{1}{2}(c_{11}'-c_{44}), \tag{21}$$

with the compressional and shear velocities in turn

$$\rho V_1^2=c_{11}',\quad \rho V_2^2=c_{44}, \tag{22}$$

The shear velocity is independent of θ, but for the compressional wave we observe that the angular dependence derives from the listed value of c_{11}' in (5).

Continuing now with our isotropy check, we find that the expressions for (18b) become

$$\begin{aligned}\{\ \}_{\rm I}&=\tfrac{1}{2}(c_{11}+c_{12})[\rho-(c_{11}-c_{12})/V^2],\\ \{\ \}_{\rm II}&=(c_{11}+c_{12})\left(\frac{\rho}{c_{11}-c_{12}}-\frac{1}{V^2}\right),\\ \{\ \}_{\rm III}&=-\tfrac{1}{2}(c_{11}-c_{12})/V^2,\\ \{\ \}_{\rm IV}&=(c_{11}+c_{12})\left(\frac{\rho}{c_{11}}-\frac{1}{V^2}\right).\end{aligned} \tag{23}$$

When we insert these along with (19) into our Eq. (18a), it turns out that the $1/V^8$ term vanishes and (after multiplying through by V^6/ρ^4) we get the anticipated sixth degree equation for the Rayleigh wave velocity, *viz.*

$$V^6-\frac{4(c_{11}-c_{12})}{\rho}V^4+\frac{2(c_{11}-c_{12})^2}{\rho^2}\left(3-\frac{c_{11}-c_{12}}{c_{11}}\right)V^2+\frac{(c_{11}-c_{12})^3}{\rho^3}\left(\frac{c_{11}-c_{12}}{c_{11}}-2\right)=0. \tag{24}$$

Garcia's[1] solution for propagation along cubic axis is now obviously incorrect since his equation for the velocity is only of the fifth degree; in any event, it does not properly reduce to the well-known isotropic result[9]

$$\frac{V^6}{V_2^6}-\frac{8V^4}{V_2^4}+\left(\frac{24}{V_2^2}-\frac{16}{V_1^2}\right)-16\left(1-\frac{V_2^2}{V_1^2}\right)=0, \tag{25}$$

which we get from (24) by making use of (20).

TABLE IV. The actual 9×9 stress-strain tensor.

Stress \ Strain	XX	YY	ZZ	YZ	ZY	XZ	ZX	XY	YX
XX	A_{11}^{11}	A_{22}^{11}	A_{33}^{11}	A_{23}^{11}	A_{32}^{11}	A_{13}^{11}	A_{31}^{11}	A_{12}^{11}	A_{21}^{11}
YY	A_{11}^{22}	A_{22}^{22}	...	...	...	...	...	...	...
ZZ	A_{11}^{33}	...	A_{33}^{33}	...	...	...	...	...	...
YZ	A_{11}^{23}	...	...	A_{23}^{23}	...	...	...	...	...
ZY	A_{11}^{32}	...	...	...	A_{32}^{32}	...	...	...	...
XZ	A_{11}^{13}	...	...	...	...	A_{13}^{13}	...	...	...
ZX	A_{11}^{31}	...	...	...	...	...	A_{31}^{31}	...	...
XY	A_{11}^{12}	...	...	...	...	...	...	A_{12}^{12}	...
YX	A_{11}^{21}	...	...	...	...	...	...	...	A_{21}^{21}

SURFACE-WAVE SOLUTION FOR THE (110) PLANE

The following stress-strain relations replace those in (7):

$$\begin{aligned}\sigma_x&=c_{11}'\frac{\partial u}{\partial x}+c_{13}'\frac{\partial w}{\partial z}, & \tau_{yz}&=c_{36}'\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\\ \sigma_y&=c_{12}'\frac{\partial u}{\partial x}+c_{23}'\frac{\partial w}{\partial z}, & \tau_{xz}&=c_{55}'\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\\ \sigma_z&=c_{13}'\frac{\partial u}{\partial x}+c_{33}'\frac{\partial w}{\partial z}, & \tau_{xy}&=c_{16}'\frac{\partial u}{\partial x}+c_{36}'\frac{\partial w}{\partial z}.\end{aligned} \tag{26}$$

Thus it turns out that in Eqs. (12) and (18) we need only replace c_{12} by c_{13}' and c_{44} by c_{55}', and the solutions for q and V hence come out directly from our (100) plane analysis.

APPENDIX. ACTUAL FORM OF THE STRESS-STRAIN TENSOR AND THE TRACE INVARIANCE

By observing the actual form of the stress-strain tensor, we can make use of the trace invariance to

[8] Stoneley[5] points out that certain materials like aluminum and copper may not propagate Rayleigh waves, i.e., no real solutions for V may arise. Substances like rock salt, sylvine, and fluorspar have densities and elastic constants favorable for Rayleigh waves.[5]

[9] See, for example, P. Byerly, *Seismology* (Prentice-Hall, Inc., New York, 1942), p. 169.

check the transformations in the (100) and (110) planes. Schematically, the true appearance of the tensor is indicated in Table IV.

Since $\sigma_{ij}=\sigma_{ji}$ and $\tau_{ij}=\tau_{ji}$, the customary practice has been to reduce the 9×9 array to 6×6 by combining or eliminating the rows and columns indicated; use is made of $A_{23}{}^{11}=A_{32}{}^{11}=A_{11}{}^{23}=A_{11}{}^{32}$, etc. The conventional cubic array results when $A_{11}{}^{11}=A_{22}{}^{22}=A_{33}{}^{33}=c_{11}$, $A_{22}{}^{11}=A_{11}{}^{22}=A_{33}{}^{11}=A_{11}{}^{33}=c_{12}$ and $A_{23}{}^{23}=A_{13}{}^{13}=A_{12}{}^{12}=\frac{1}{2}c_{44}$. Combining the equivalent strain terms, $A_{23}{}^{23}+A_{32}{}^{32}=c_{44}$, etc.

Now the various transformations can be checked by noting the trace is $\mathrm{Tr}=3(c_{11}+c_{44})$. The (100) transformation has $\mathrm{Tr}=2c_{11}'+c_{11}+2c_{44}+c_{66}'$, whence it is required that $2c_{11}'+c_{66}'=2c_{11}+c_{44}$. Explicitly, it is found that

$$2c_{11}+c_{44}=c_{11}[2(\cos^4\theta+\sin^4\theta)+4\sin^2\theta\cos^2\theta]_1 +c_{12}[4\sin^2\theta\cos^2\theta-4\sin^2\theta\cos^2\theta]_2 +c_{44}[8\sin^2\theta\cos^2\theta+1-8\sin^2\theta\cos^2\theta]_3,$$

and $[\ ]_1=2$, $[\ ]_2=0$, $[\ ]_3=1$ as required. Next, for the (110) transformation, $\mathrm{Tr}=c_{11}'+c_{22}'+c_{33}'+c_{44}'+c_{55}'+c_{66}'$, with the result

$$\mathrm{Tr}=c_{11}\{\tfrac{3}{2}(\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta)+\tfrac{1}{2}+(\sin^2\theta+\cos^2\theta)\}_1+c_{12}\{\tfrac{1}{2}\sin^4\theta+\tfrac{1}{2}\cos^4\theta+4\sin^2\theta\cos^2\theta+\tfrac{1}{2}-(\cos^2\theta+\sin^2\theta)-3\sin^2\theta\cos^2\theta\}_2 +c_{44}\{\sin^4\theta+\cos^4\theta+8\sin^2\theta\cos^2\theta+1+2-2(\cos^2\theta+\sin^2\theta)+1-6\sin^2\theta\cos^2\theta\}_3.$$

Again, as required, $\{\ \}_1=3$, $\{\ \}_2=0$, $\{\ \}_3=3$.

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Diffusion in Ordered and Disordered Copper-Zinc*

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The diffusivities of Cu^{64}, Zn^{65}, and Sb^{124} in single crystals of 47–48 atomic percent zinc copper-zinc (beta brass) have been measured over the temperature range 265–817°C, by using sectioning techniques. The diffusion coefficients show a striking dependence on the degree of long-range order at temperatures below the critical temperature (468°C). A slight dependence of the diffusion coefficient on short-range order is noted above the critical temperature. The diffusion coefficients obey an Arrhenius equation only in the fully disordered phase, with temperature dependences given by $D_{Cu}=0.011\exp(-22\,000/RT)$ cm²/sec; $D_{Zn}=0.0035\exp(-18\,800/RT)$ cm²/sec; $D_{Sb}=0.08\exp(-23\,500/RT)$ cm²/sec. The variation of the diffusion coefficients with temperature in the ordered phase is considered in terms of a simple elastic model. Excellent agreement is obtained by using the measured elastic constants and assuming that the energy for motion of the imperfection is simply related to the smallest (110) shear modulus. In the disordered phase Sb diffuses faster than Zn or Cu, while in the ordered phase Sb diffuses at the same rate as Zn, which is faster than Cu. This result is shown to be inconsistent with an interchange, interstitial, or nearest-neighbor vacancy mechanism for diffusion. The result is consistent with an interstitialcy mechanism.

1. INTRODUCTION

Diffusion in solid materials is generally believed to result from the presence of point imperfections naturally present in the crystal lattice. In recent years, a considerable number of investigations[1] of diffusion phenomena in simple lattices have substantiated this mechanism, and in particular have provided firm evidence for the existence of vacant lattice sites, or vacancies, in solids.

Since most of the radioactive tracer diffusion experiments in metals have been concerned with simple monovalent elements, there still exists a considerable shortage of information on the role of imperfections in alloy systems. Many of the most interesting properties of alloys result from the possibility of achieving varying degrees of order among the constituents. The effects of order on the lattice are most strikingly illustrated by systems such as CuZn, Cu_3Au, CuAu, CoPt, etc., which exhibit order-disorder, or "superlattice" transitions.[2] Of the known superlattice systems, CuZn, or beta brass, has provided a particularly rich field for investigation, since the alloy equilibrates rapidly and the degree of order may be varied from nearly complete order to complete randomness, over a relatively small temperature range, without any discontinuous change in crystal structure.[3]

The present availability of radioisotopes of high

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[1] E.g., see F. Seitz, in *Phase Transformations in Solids* (John Wiley and Sons, Inc., New York, 1951), p. 77 ff.

[2] H. Lipson, *Progress in Metal Physics* (Interscience Publishers, Inc., New York, 1950), Vol. 2.

[3] F. W. Jones and C. Sykes, Proc. Roy. Soc. (London) **161**, 440 (1937).