PHYSICAL REVIEW VOLUME 104, NUMBER 5 DECEMBER 1, 1956

Angular Momentum Coupling and the Internucleon Interaction in the Calcium Isotopes*

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(Received July 2, 1956)

A partial analysis is made of the results of the Massachusetts Institute of Technology Van de Graaff group on (d,p) reactions leading to the levels of Ca^{41}, Ca^{43}, and Ca^{45}. The results demonstrate that the coupling scheme is rather close to the jj limit; more precisely they give small values for the amplitudes of certain minor components of the wave functions for low-lying odd-parity states (e.g., the $(f_{7/2})^2{}_0\, p_{3/2}$ component of the lowest $J=3/2^-$ level in Ca^{43}). The amplitudes determined in this way are used to deduce some features of the effective internucleon interaction. It turns out that the (d,p) amplitudes for the low-lying "multi-particle" states are essentially proportional to a matrix element which vanishes identically for any spin-dependent nucleon-nucleon interaction (of either $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, tensor or vector type) and thus we obtain directly an approximate range *vs* depth relationship for the spin-independent interaction. Finally, an approximate spin-dependent central interaction is determined by considering the level structures. A fairly satisfactory interaction between identical nucleons is found to be $H_{12}\simeq -3[3-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\exp(-r^2/r_0^2)$ Mev with $r_0\sim 2.7\times 10^{-13}$ cm. The general features of the results are similar to those of Levinson and Ford, who consider level structures and magnetic moments, but the exchange nature of the interaction is different.

I. INTRODUCTION

DURING the past few years several authors[1] have stressed the fact that deuteron pickup or stripping reactions may be used to give information about the structure of nuclei and in particular about the angular momentum coupling scheme. Interest has mainly centered on the information which may be derived by considering relative cross sections to different levels of the same final nucleus and in the nuclear $1p$ shell, in particular, there has been a considerable amount of detailed analysis.

In most cases which have been examined to date, the use of deuteron cross sections has simply supplied experimental data which then are compared with the values predicted by using definite configurations and internucleon interactions. Some time ago the present authors[2] pointed out that in certain cases measurement of relative deuteron cross sections would give immediate information concerning the coupling scheme. The essential point here was that, in any stripping or pickup reaction beginning with a spin zero target, the transferred nucleon has a definite l and a definite j value. This immediately suggests that such cases will be most simply described if we use a jj-coupling representation (including, of course, mixed jj configurations); but more important is the fact that, if the wave functions are reasonably well described by pure jj coupling, then the measurement of relative cross sections will supply a quantitative measure of the departures from this scheme.

The original note[2] concerned itself with relative deuteron cross sections for reactions with the same l value, since then the "barrier" effects are the same except for small, more or less calculable, corrections. It will become apparent that this restriction can be largely removed provided that measurements of relative cross sections in a nearby "closed shell plus one" nucleus are available, for with the use of these results we are enabled to make meaningful comparisons of cross sections with different l values. In this way, the applicability of the procedure is very greatly extended.

In the present paper we concern ourselves specifically with angular momentum coupling in the isotopes of calcium and make a partial analysis of a very excellent set of (d,p) measurements made by the M.I.T. Van de Graaff group.[3] It will become clear that such measurements can be considered in two stages: first, making use only of the concept of angular momentum coupling along with a simple picture of the stripping process, we can find a measure of certain amplitudes in the nuclear wave functions involved; the second stage consists of taking these amplitudes as given quantities and, by the conventional techniques of nuclear spectroscopy, determining a configuration and an internucleon interaction which will be consistent with these and other available data. In the present case there is fortunately much information available concerning the level schemes.

Finally, we should say that the spectroscopy of the

* This work was supported in part by the U. S. Atomic Energy Commission.

[1] F. L. Friedman and W. Tobocman, Phys. Rev. **92**, 93 (1953); R. G. Thomas, Phys. Rev. **91**, 453(A) (1953); R. Huby, Progr. Nuclear Phys. **3**, 177 (1953); Fujimoto, Kikuchi, and Yoshida, Progr. Theoret. Phys. (Japan) **11**, 264 (1954); A. M. Lane, Proc. Phys. Soc. (London) **A66**, 977 (1953); T. Auerbach and J. B. French, Phys. Rev. **98**, 1276 (1955); J. E. Bowcock, Proc. Phys. Soc. (London) **A67**, 981 (1954); W. Tobocman and M. H. Kalos, Phys. Rev. **97**, 132 (1955); P. M. Endt and C. M. Braams, Physica **21**, 839 (1955); and W. Tobocman, Phys. Rev. **102**, 588 (1956); see also H. A. Bethe and S. T. Butler, Phys. Rev. **85**, 1045 (1952); S. Okai and M. Sano, Progr. Theoret. Phys. (Japan) **14**, 399 (1955) and **15**, 203 (1956). The basic theory of the stripping reaction is due to S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951).

[2] J. B. French and B. J. Raz, Phys. Rev. **98**, 1523 (1955).

[3] Bockelman, Braams, Browne, Buechner, Cobb, Guthe, and Sperduto; this work is as yet almost entirely unpublished.

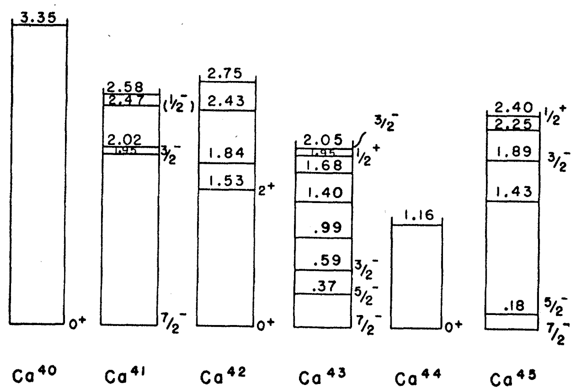

FIG. 1. The low-lying levels of the lighter calcium isotopes.[3,5]

calcium isotopes has been recently considered by Levinson and Ford,[4] who focus their attention on level structures and magnetic moments but do not consider the deuteron reactions. Their conclusions are similar to, but by no means identical with, those of the present paper and we shall briefly discuss the differences.

II. EXPERIMENTAL RESULTS AND QUALITATIVE CONSIDERATIONS

To begin with, we remind the reader that one expects $_{20}Ca^{40}$ to be represented by the closed shells $1s$, $1p$, $1d$, $2s$. Then in pure jj coupling the calcium isotopes are described in terms of the configuration $(f_{7/2})^n$ of identical nucleons. This then is expected to be perhaps the simplest region from the standpoint of the nuclear shell model, for the $f_{7/2}$ single particle level is well isolated. The other single-particle levels which one might expect to be important for the low states of the calcium isotopes are $f_{5/2}$, $p_{3/2}$, $p_{1/2}$, and $g_{9/2}$.

The available pertinent information concerning the levels of Ca^{41}-Ca^{45} is displayed in Fig. 1. In Ca^{41} we observe the single particle $f_{7/2}$, $p_{3/2}$, and $p_{1/2}$ but note that the $f_{5/2}$ and $g_{9/2}$ levels are not identified. It is, in fact, unfortunate for our analysis that the f doublet splitting is not accurately known. In Ca^{43} a level at 0.82 Mev has been reported, but there now seems considerable evidence that this level does not exist[5] and we shall make this assumption.

Table I gives, among other things, the l values and the relative peak cross sections[6] for the lower levels of $Ca^{41,43,45}$ as given by the M.I.T. group[3] for deuteron energy $E_d = 7$ Mev. The $Ca^{40}(d,p)Ca^{41}$ experiment has been done for $E_d = 8$ Mev with less resolution but with a measurement of the absolute cross sections by Holt and Marsham.[7] The two sets of results are in satisfactory agreement. For the 0.99-Mev level of Ca^{43} the l value is ambiguous. However, it will turn out that the analysis will favor $l=2$ for this level. In addition, the angular distribution is identical with that of the 2.014-Mev level in Ca^{41} which we also tentatively regard as having positive parity. The $Ca^{42}(d,p)$ cross sections are known relative to the $Ca^{40}(d,p)$ (as given in Table I) but the $Ca^{44}(d,p)$ are not known, and therefore, we list $x\sigma$ for $Ca^{44}(d,p)$ with x experimentally undetermined. The angular distributions for the weak first excited states of Ca^{43} and Ca^{45} are more or less isotropic and presumably mainly result from compound-nucleus formation. Among the higher levels observed by the M.I.T. group there are many isotropic levels, and the cross sections for these levels are all about equal.

The fact that no stripping is observed to the first excited states $(5/2^-)$ of Ca^{43} and Ca^{45} means that these levels have only a small component which may be described by the coupling of an $f_{5/2}$ particle to the ground states of Ca^{42} and Ca^{44}. This strongly suggests that we are rather close to pure jj coupling for the low levels of Ca^{43} and Ca^{45}. We proceed then on this basis and assume the same for the other isotopes too.

These, in fact, are two of the cases discussed earlier by the present authors.[2] Unfortunately, in each case, the presence of the compound-nucleus contribution obscures any stripping which may exist so that we can give only an upper limit to the $(f_{7/2})^n{}_0 f_{5/2}$ component of the excited-state wave functions. We recall that the primary reason why one normally restricts oneself to comparisons of stripping cross sections with the same l value is that there exists no satisfactory procedure for calculating the relative "single-particle" cross sections for different l values. However, the $Ca^{40}(d,p)$ results of Table I give the relative $l=1$, 3 cross sections and since Ca^{40} is a closed-shell nucleus, these should give a quite satisfactory measure of the basic cross sections. This calibration then enables us to consider the $l=1$ reactions in the higher isotopes, and in particular, we can decide that the 0.59-Mev level in Ca^{43} is predominantly $(f_{7/2})^3$ but has a 4% (probability) admixture of $(f_{7/2})^2{}_0 p_{3/2}$. The reader should note that this small admixture is easily measurable though a similar admixture of $(f_{7/2})^2{}_0 f_{5/2}$ in an $(f_{7/2})^3$ wave function is not, because the $l=1$ single-particle cross section is larger by an order of magnitude than that for $l=3$.

The conclusion that we are close to jj coupling enables us (as in Sec. III) to calculate approximately, with no assumptions about nuclear forces, the relative ground state reactions, and we find $\sigma(Ca^{41}):\sigma(Ca^{43})$:

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[4] C. Levinson and K. W. Ford, Phys. Rev. **100**, 13 (1955).

[5] For experimental information see Way, King, McGinnis, and Van Lieshout, U. S. Atomic Energy Commission Report T.I.D.-5300, 1955 (unpublished); also P. M. Endt and J. C. Kluyver, Revs. Modern Phys. **26**, 95 (1954).

[6] In principle the cross sections should be corrected for the purely kinematic factors in the stripping theory. We have verified that these corrections, *as calculated by Butler's equations*, are quite small, as we would expect since the outgoing protons are quite energetic (the ground state Q values are about 6 Mev); we therefore ignore them. Concerning the over-all accuracy of the relative reduced widths, S, we would feel that, if we ignore possible errors in the Butler theory itself, they should, in most cases, be correct to ±20% though often, for arithmetical consistency, we quote values with greater precision.

[7] J. R. Holt and T. N. Marsham, Proc. Phys. Soc. (London) **A66**, 565 (1953).

$\sigma(\mathrm{Ca}^{45})=4/3:1:2/3$. $\sigma(\mathrm{Ca}^{41}):\sigma(\mathrm{Ca}^{43})$ is given experimentally as $1.3\pm 10\%$. The relative $\sigma(\mathrm{Ca}^{45})$ is unknown.

Finally, we point out that the smallness of the mixed amplitudes suggests that there are higher levels in Ca^{43} and Ca^{45} which have predominantly a "single particle" character—i.e., they are mainly $(f_{7/2})^n{}_0\, f_{5/2}$ or $(f_{7/2})^n{}_0\, p_{3/2}$, etc. These would be identifiable by their strong (d,p) reactions; the 2.05-Mev level in Ca^{43} and the 1.89-Mev level in Ca^{45} both have $l=1$ cross sections with a magnitude about equal to the single particle $p_{3/2}$ cross section. They may therefore be safely identified as $3/2^-$, described primarily by the $(f_{7/2})^n{}_0\, p_{3/2}$ ($n=2, 4$) configuration.

Before continuing with the analysis, we emphasize the fact that in $\mathrm{Ca}^{40}(d,p)$ no $l=3$ excited-state reaction with cross section comparable to the ground state (as we would expect for the single-particle $f_{5/2}$ level) has been observed up to 5.7 Mev either by the M.I.T. group[3] or by Holt and Marsham.[7] This fact is of direct concern for our analysis[8]; we see three possible explanations for it. (1) The splitting may be larger than 5.7 Mev; our subsequent analysis of the spectroscopy assumes both 2 and 4 Mev for this splitting, but in fact would be little changed by a higher value; there is also the fact that we need not assume that the "effective splitting" in Ca^{43} is identical with that in Ca^{41}. (2) The single-particle $f_{5/2}$ level may not exist; this would imply that the configurations involving core excitation (promotion of some of the first 40 nucleons to higher orbits) interact strongly with this single-particle configuration. This type of effect could perhaps be important for states of relatively high excitation. On the other hand, the single-particle $p_{3/2}$ level shows up in all three odd isotopes. (3) The 2.01-Mev level in Ca^{41} may have $l=3$ and be the single-particle level, but for some reason the single-particle $f_{5/2}$ Butler cross section is smaller by an order of magnitude than the $f_{7/2}$ cross section. This would be seriously disturbing to our analysis, implying, as it does, that our understanding of the Butler mechanism is at fault.[9]

It is not easy to settle which, if any, of these explanations is near to the truth. A definite determination of the parity of the 2.01-Mev level in Ca^{41} (and the 0.99-Mev level in Ca^{43}) would be very helpful. On the optimistic side is the fact that our *quantitative* analysis mainly involves stripping with $f_{7/2}$ and $p_{3/2}$ particles.

TABLE I. This table gives the M.I.T. results.[3] We list, in an arbitrary unit, cross sections at the peak of the stripping curve (the experimental error is less than $\pm 10\%$). The $\mathrm{Ca}^{40}(d,p)$ results are used to determine the relative single-particle cross sections ϕ_l which we renormalize to $\phi_3=1$. The $\mathrm{Ca}^{42}(d,p)$ and $\mathrm{Ca}^{44}(d,p)$ results are used to determine the factor S which measures the co-operative effect of the nucleons involved (for a "single-particle level" $S=1$; for a $(j)^n{}_{J=j}$ level S is of order of magnitude unity; for any other "multiparticle level" $S\ll 1$). S is then used to calculate the probability amplitudes $|K|^2$.

$\mathrm{Ca}^{40}(d,p)\mathrm{Ca}^{41}$

E (Mev)	J	l	$\sigma_{\max}(\theta)$	ϕ_l	Notes
0	$7/2^-$	3	1	1.0	
1.947	$3/2^-$	1	6.7	13	a
2.014	···	2	0.11		b
2.469	$1/2^-$	1	2.2	9	c
2.582	···	···	0.04		d

$\mathrm{Ca}^{42}(d,p)\mathrm{Ca}^{43}$

E	J	l	$\sigma_{\max}(\theta)$	S	$\lvert K[(7/2)^2{}_0\, j]\rvert^2$	Notes
0	$7/2^-$	3	0.79	~0.79	~1.0	e
0.37	$5/2^-$	···	<0.02	<0.03	<0.03	d
0.59	$3/2^-$	1	0.31	0.047	0.047	
0.99	$3/2^+$?	2	0.12	0.07	0.07	f,g
1.40	···	···	weak	···	···	
1.68	···	···	weak	···	···	
1.95	$1/2^+$	0	0.8 ($\theta=10°$)	0.07	0.07	f
2.05	$3/2^-$	1	6.3	0.96	~1.0	h

$\mathrm{Ca}^{44}(d,p)\mathrm{Ca}^{45}$

E	J	l	$x\sigma_{\max}(\theta)$	S	$\lvert K[(7/2)^4{}_0\, j]\rvert^2$	Notes
0	$7/2^-$	3	1	0.50	~1.0	i, j
0.18	$5/2^-$	···	<0.05	<0.03	<0.03	d
1.43	{$1/2^-$ {$3/2^-$	1	1.95	0.30 0.15	0.30} 0.15}	k
1.89	$3/2^-$	1	11.3	0.88	0.88	h
2.25	{$1/2^-$ {$3/2^-$	1	1.87	0.29 0.14	0.29} 0.14}	k
2.40	$1/2^+$	0	1.83 ($\theta=7.5°$)	0.08	0.08	f

[a] Following Holt and Marsham the 1.95-Mev level is identified as the single-particle $p_{3/2}$ level. Their data taken at $E_d=8$ Mev give $\phi_1\simeq 12$. We choose the value $\phi_1=13$ for our analysis.

[b] The distribution here could be fitted with $l=3$. The intensity is too low by a factor ~7 for the level to be identified as single particle $f_{5/2}$.

[c] Holt and Marsham[7] identify this as the $p_{1/2}$ level; their result gives $\phi_1\simeq 10$. It could turn out (Bockelman[21]) that this level cannot be classified in this way. However, this point is unimportant for our analysis. One of the most curious features of the (d,p) results (not discussed in the text) is the occurrence of a third strong $l=1$ reaction. This appears to be seen in $\mathrm{Ca}^{40}(d,p)$,[7] and is quite clearly seen in the $\mathrm{Cr}^{52}(d,p)$ data referred to above.[9] See also R. H. Nussbaum *et al.* [Phys. Rev. **101**, 905 (1956)] for the levels of Ni^{61}.

[d] Shows no stripping.

[e] The calculated S is 3/4 for pure jj coupling and thus we have consistency in assuming that the ground state probability for $(7/2)^3$ is essentially 1.

[f] For the purpose of deducing S values and amplitudes for the even-parity states, we assume $\phi_{l+1}{}^2=\phi_l\phi_{l+2}$.

[g] This distribution also could be fitted with $l=3$. The intensity is too low by a factor ~7 for the level to be identified as the "single-particle" $f_{5/2}$ level, so we choose the even parity assignment of Lindqvist and Mitchell[19] and exclude this level from our analysis. Note, however, the considerable controversy about this level.[5]

[h] This level is a "single-particle" $p_{3/2}$ level.

[i] It has sometimes been suggested that the ground state has spin $5/2^-$. It seems entirely certain from the present analysis that this is not the case.

[j] The cross sections in Ca^{45} are not known relative to the other isotopes. The calculated ratio of the ground state reactions in Ca^{45} to Ca^{41} is 1:2 in pure jj coupling and we assume this value, i.e., we take $x=2$. *Note added in proof.*—The Ca^{45} cross sections have recently been determined relative to Ca^{41} (C. K. Bockelman, private communication). This ratio (i.e., Ca^{45} to Ca^{41}) is measured to be 0.39 ± 0.03, thus leading to a value of $2.5\pm 10\%$ for x.

[k] The J value is unknown, so we consider both possibilities.

III. ANALYSIS OF THE (d,p) RESULTS

We begin by deriving expressions which will enable us to convert the (d,p) results into statements concern-

[8] The elusive nature of the single-particle $f_{5/2}$ level has been noted also by R. H. Nussbaum (to be published).

[9] There is also the possibility (suggested by W. Tobocman in a private communication to C. Levinson) that distortion effects as considered by Tobocman and Kalos[1] could lead to such differences between the two $l=3$ cross sections. Recently the single-particle $f_{5/2}$ level itself has almost certainly been seen by A. J. Elwyn and F. B. Shull in $\mathrm{Cr}^{52}(d,p)$ [Bull. Am. Phys. Soc. Ser. II, **1**, 281 (1956)]. The cross-section magnitude results (private communication from A. J. Elwyn) are completely consistent with the assumption that the $f_{7/2}$, $f_{5/2}$ single-particle widths are identical. We have also been informed by R. Hayward that there is now no valid evidence for a negative-parity level in Ca^{43} near 1.0 Mev. We may thus safely regard the 0.99-Mev level as +. Indirectly then this favors a + assignment for the 2.01-Mev level in Ca^{41} [which has an identical (d,p) angular distribution]. The difficulty about the relative reduced widths for $f_{7/2}$, $f_{5/2}$ would then disappear.

ing amplitudes. The procedure is identical with that of Auerbach and French[1] except that we use a jj picture. Then, considering only the case of all identical nucleons (and thus ignoring isotopic spin), we have, for an initial nucleus J_0 and final J with a single l value,

$$\frac{d\sigma}{d\omega} \equiv \sigma(\theta) = \frac{(2J+1)}{(2J_0+1)} S\phi_l(\theta), \qquad (1)$$

where ϕ_l may be called the intrinsic Butler single-particle cross section and S the factor (previously called $n\sum_z \beta_z^2$) which measures the cooperative effect of all the nucleons; S is simply the reduced width for the reaction in units of the single-particle reduced width. We find easily that

$$S^{\frac{1}{2}} = n^{\frac{1}{2}} \sum_{\alpha,\beta,j_n} (-1)^{J_0+j_n-J} K_\alpha(n) K_\beta(n-1) \times \langle \psi_\alpha(n) | \psi_\beta(n-1) \times j_n \rangle, \quad (2)$$

where n = number of nucleons considered in the final state; α, β are the various possible components of the final and initial wave functions (determined, of course, by the configurations we elect to consider) and K_α, K_β are the corresponding amplitudes; $\langle \psi_\alpha | \psi_\beta \times j_n \rangle$ is the spin and angular overlap integral between ψ_α and the state formed by vector-coupling particle number n (with angular momentum j_n) to ψ_β. The symbol $\times$ signifies vector coupling.

For the latter quantity there are two cases of interest to us:

(a) The target and final state have only equivalent nucleons. Then $\langle \psi_\alpha | \psi_\beta \times j_n \rangle$ = coefficient of fractional parentage $= [(2j+2-n)/(2j+1)n]^{\frac{1}{2}}$, where the last equality is true for the seniority-one states and[10] $J_0 = 0$ (the only case we need).

(b) The target has all equivalent nucleons and the final state has one nonequivalent nucleon. Then, provided that the angular momenta are compatible, we have $\langle \psi_\alpha | \psi_\beta \times j_n \rangle = n^{-\frac{1}{2}}$.

We now examine the experimental results using, to begin with, the configurations $(f_{7/2})^n{}_0$ for Ca^{42} and Ca^{44}, and for Ca^{43} the configurations $(f_{7/2})^3$, $(f_{7/2})^2{}_0\, p_{3/2}$, $(f_{7/2})^2{}_0\, f_{5/2}$, and $(f_{5/2})^2{}_0\, f_{7/2}$, where the subscript 0 refers to the angular momentum of the coupled pair. We start with this restricted vector space but (as it is always logically necessary) we shall extend it to examine the corrections. These will be quite negligible for the (d,p) results. We have now two allowed states each for $J = 3/2$, $5/2$, $7/2$ and one for $J = 9/2$, $11/2$, $15/2$.

By applying Eq. (1), we use the $Ca^{40}(d,p)$ results to compare ϕ_1 and ϕ_3, assuming these reactions to be proper single-particle reactions. The $l=3$ reaction is intrinsically weaker than the $l=1$ by about a factor 13.

These determinations are now used to give the experimental S values for the $Ca^{42}(d,p)$ and $Ca^{44}(d,p)$ reactions as recorded in Table I and these in turn, by Eq. (2), lead to the probability amplitudes of Table I. The $Ca^{44}(d,p)$ cross sections are not known relative to $Ca^{40}(d,p)$ but we assume (and the entire analysis would appear to make this a fair assumption) that the ground-state reaction is half as strong as the $Ca^{40}(d,p)$ ground-state reaction. In two cases for $Ca^{44}(d,p)$ we record two values of S, the J values being unknown.

All the reactions observed in Ca^{43} and Ca^{45} are weak (small S value) except the ground-state reactions, the 2.05-Mev level in Ca^{43}, and the 1.89-Mev level in Ca^{45}, the last two of which have $S \sim 1$ and are thereby demonstrated to be single-particle levels. Their major components would be $(f_{7/2})^2{}_0\, p_{3/2}$ and $(f_{7/2})^4{}_0\, p_{3/2}$. It seems entirely clear that the Ca^{45} ground-state level has $J = 7/2^-$ and the 0.18-Mev level has $J = 5/2^-$ (the opposite assignment has sometimes been given). As was mentioned before, for the 0.99-Mev level in Ca^{43} our analysis favors an even-parity assignment. Finally, we point out that the relative size of the amplitudes demonstrates that in Ca^{45}, the coupling is not as simple as in Ca^{43}; this may be due to the increase in the number of nucleons or simply to the fact that the energy levels are higher.

Our problem now is, by resort to the standard techniques of spectroscopy, to determine, if possible, the internucleon interaction. For reasons discussed later we do not consider, in the present paper, the analysis of the Ca^{45} spectrum. We ignore all perturbations of the closed shell core and thus disregard the even-parity levels of Ca^{43}. As is customary, we consider a central two-nucleon interaction for identical particles[11]

$$H_{12} = H_{12}^{(0)} + H_{12}^{(1)} = [\beta + \epsilon(1-\beta)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2] J(r_{12}), \quad \epsilon = \pm 1, \quad 0 \leq \beta \leq 1. \qquad (3)$$

The spin-orbit dependence is accounted for by a single-nucleon term whose parameters are then the $p_{3/2} - f_{7/2}$ single-particle difference and the p and f doublet splittings. The first two parameters we take from Ca^{41}; for the major calculations we take the f doublet splitting as 4 Mev, though we find very similar results for a 2-Mev value. For $J(r)$ we have used a Gaussian form,[12] since we can then make good use of the results of Kurath[13] for the $(f_{7/2})^n$ configuration, i.e., we have

$$J(r) = V \exp(-r^2/r_0^2). \qquad (4)$$

We take a harmonic-oscillator dependence for the radial wave function and define $\lambda = r_0/r_1$ where the exponential factor in the wave function is $\exp(-r^2/r_1^2)$.

We now construct the Hamiltonian matrices with the

[10] C. Schwartz and A. de-Shalit, Phys. Rev. 94, 1257 (1954).

[11] But note below that the results of this section are quite unchanged if we include in H_{12} both the tensor and vector interaction. This will come about because our direct analysis of the (d,p) results will separate out the spin-independent central part of H_{12}.

[12] Some results for a Yukawa radial dependence (calculated by D. C. Choudhury) are given in Appendix II.

[13] D. Kurath, Phys. Rev. 91, 1430 (1953).

representation defined above. Diagonalization then produces for Ca^{43} wave functions and energy values, two each for $J=3/2$, $5/2$, $7/2$ and one for $J=9/2$, $11/2$, and $15/2$ (we consider Ca^{42} later). We do this for varying choices of β and V. We make the primary demand of the interaction that it predict the correct relative intensity of the $Ca^{42}(d,p)$ reaction to the $3/2^-$ level at 0.59 Mev; we place major emphasis on this because it appears to us that, since this reaction involves only a simple angular momentum coupling, this datum may be essentially simpler than, for example, a magnetic moment or energy level.

Let us first see what approximate features of the interaction we can deduce directly from the (d,p) reactions without considering the energy-level scheme in detail. One point of view here is that the (d,p) reactions roughly determine the wave functions so that we encounter a Hamiltonian problem with experimentally determined solution and eigenvalues (the latter are the energy levels).

Consider the $J=3/2$ matrix for Ca^{43}; we label the $(f_{7/2})^3$ state as 1 and the $(f_{7/2})^2{}_0\, p_{3/2}$ as 2. Then from Table I the eigenfunction is $(1, \pm 0.22)$ and we have

$$\begin{pmatrix} \mathfrak{H}_{11} & \mathfrak{H}_{12} \\ \mathfrak{H}_{12} & \mathfrak{H}_{22} \end{pmatrix} \begin{pmatrix} 1 \\ \pm 0.22 \end{pmatrix} = E_{\frac{3}{2}}{}^{(1)} \begin{pmatrix} 1 \\ \pm 0.22 \end{pmatrix}, \qquad (5)$$

where $\mathfrak{H}_{22}$ includes the $p_{3/2}-f_{7/2}$ single-particle difference; $E_{3/2}{}^{(1)}$ is the position of the $J=3/2$ "multiparticle" level; $E_{3/2}{}^{(2)}$, the other eigenvalue, gives the position of the $J=3/2$ "single-particle" level. Because we are rather close to jj coupling we can solve (5) by perturbation theory and can believe moreover that the results will not be seriously changed when the configuration is enlarged. We thus have roughly

$$|\langle f_{7/2})^3 | H_{12} | (f_{7/2})^2{}_0\, p_{3/2}\rangle| \simeq 0.22[E_{3/2}{}^{(2)} - E_{3/2}{}^{(1)}], \qquad (6)$$

and we have now made an approximate determination of a single matrix element of the internucleon interaction. Moreover, as discussed in Appendix I, spin-dependent interactions (of either $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$, tensor or vector types) make no contribution to this matrix element and consequently Eq. (6) gives us a measure of the spin-independent central interaction $H_{12}{}^{(0)} = \beta J(r_{12})$ of Eq. (3). More precisely, Eq. (6) gives, for any given type of radial dependence, an approximate depth-range relationship for the spin-independent interaction. The matrix element in (6) clearly vanishes in the long-range limit; it vanishes also in the short-range limit [for in this limit $J(r_{12}) \equiv -\frac{1}{3}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 J(r_{12})$, whose matrix elements *do* vanish]. Taking the experimental energy separation to be 1.46 Mev we have, for the Gaussian interaction (4) with $\lambda = 0$, 0.16, 0.7, 1, 1.25, respectively, βV (in Mev) $= \infty$, 3.9×10^5, 35, 14, 11.

Precisely the same technique can be applied to the $Ca^{44}(d,p)$ results, assuming that the 1.43-Mev level is the $3/2^-$ multinucleon level. The essential off-diagonal matrix element here is found, for any interaction, to be $(2/3)^{\frac{1}{2}}$ times the corresponding three-nucleon matrix element (see Appendix I). Then the $Ca^{44}(d,p)$ experiment, assuming the 1.43-Mev level is the multiparticle $3/2^-$ level, would demand a spin-independent interaction weaker by a factor 1.9 than deduced above. This determination however is not trustworthy; for one thing, the perturbation technique is less accurate (the minor components are larger); more importantly, we do not have an experimental determination of the $Ca^{44}(d,p)$ normalization (see Table I, note (j)).

Since the (d,p) reaction cross sections to the $5/2^-$ first-excited states of $Ca^{43,45}$ are not distinguishable from the compound-nucleus background we can, from these results, give only a rough upper limit to βV for a given λ. Even this is of doubtful value because of the unknown behavior of the $f_{5/2}$ single-particle level, as discussed above, and we therefore do not give numerical results. Instead we have verified that for the quite satisfactory range, $\lambda \geq 0.7$, the (d,p) stripping reaction to both first excited states will not be observable, in agreement with experiment.

A final point about the (d,p) reaction results is worth mentioning. Presumably the small isotropic cross sections, for example, with the first excited states of Ca^{43} and Ca^{45}, give a measure of the compound nucleus effect. We would expect that this be insensitive to small changes of the final-state spin and that we can confidently use the same value in separating out the compound-nucleus contribution to a reaction which shows a good stripping curve. A simple procedure here would be to assume an arbitrary phase difference between the two amplitudes[14]; the subtraction technique would then replace the measured curve by a band in which the stripping curve proper should reside. Among the reactions considered here the single-particle $l=1$ cross sections have peak cross sections 300–400 times the nonstripping cross sections; in these cases in particular we should gain valuable information about the stripping cross section itself. We are unable to carry out this step now because, as yet, large-angle cross sections are not available.

IV. SPECTROSCOPY

We now undertake a more formal treatment considering the level spectrum of Ca^{43}.[15] We briefly consider also Ca^{42} but shall not treat the spectroscopy of Ca^{45}, since beyond the (d,p) results there is scarcely sufficient experimental information to justify this at present. Since the spin-independent interaction has been largely determined above, our essential aim here is to fix the spin-dependent part; we therefore calculate the

[14] For a discussion of combined stripping and compound-nucleus effects see R. G. Thomas, Phys. Rev. **100**, 25 (1955).

[15] The details of the theoretical spectroscopy are contained in Appendix I. The numerical results are contained in J. B. French and B. J. Raz, U. S. Atomic Energy Commission Report NYO-7460 (unpublished).

TABLE II. The magnitude V of the effective potential tabulated against the exchange parameters ϵ and β. For the range parameter, we have $\lambda=1$. [See Eqs. (3) and (4).]

	$\epsilon=+1$				$\epsilon=-1$				
$\beta=$	0.7	0.8	0.9	1.0	0.9	0.8	0.7	0.6	0.5
V (Mev) $=$	-22.4	-16.8	-13.5	-11.2	-11.6	-12.0	-12.5	-13.0	-13.5

spectrum of Ca^{43} as a function of β, λ, and ϵ [Eqs. (3) and (4)]. We impose the primary condition that the wave functions produced should properly give the (d,p) cross section to the first $3/2^-$ level (and also be consistent with the negative result for the first $5/2^-$ level and with the large cross section to the second $3/2^-$ level).

We emphasize that potentials satisfying the range-depth relationship of Sec. III will give the relative cross section in question only if the Hamiltonian properly gives the experimental separation of the $3/2^-$ levels. There is, in fact, no guarantee that such a potential combined with the spin-dependent term $H_{12}^{(1)}$ and the assumed single-particle splittings will produce the desired separation and other features of the spectrum. To examine this it is convenient to proceed differently; for given λ, ϵ, β we calculate from Eq. (5) the potential depth V. Because of the sign ambiguity in the measured amplitude each parameter set determines two values of V, but in each case one of these may be eliminated directly (e.g., it may give the lowest $3/2^-$ state to be the single-particle state). Table II shows the values of V thus obtained. Using the chosen V, we construct and diagonalize the matrices for each J value. The $p_{3/2}-f_{7/2}$ single-particle splitting we take as 1.95 Mev (from Ca^{41}).

Figure 2 shows the spectrum which results with $\lambda=1$ and $\Delta_f=E(f_{5/2})-E(f_{7/2})=2$ Mev. Figure 3 shows for one force mixture ($\epsilon=-1, \beta=0.83$) the variation of the level structure with range. A range parameter $\lambda\simeq1$ appears, in this case and in others, to give the most satisfactory fit and from now on we use only this value. This range is quite long. If we associate a nuclear radius (taken as $1.40A^{\frac{1}{3}}\times10^{-13}$ cm) with the parameter r_1 by the method of Swiatecki,[16] we find for $\lambda=1, r_0=2.7\times10^{-13}$ cm. The same procedure applied to the nuclear $1p$ shell and assuming as usual[17] $L/K=6$ would give $r_0\simeq1.5\times10^{-13}$ cm. The analysis of Levinson and Ford[4] also favors a long range, $\lambda\sim1.2$. This long-range assumption was first made by Kurath[18] in his study of the j^n configurations and for the $f_{7/2}$ shell his results[13] favor $\lambda\sim1.1$.

Figure 4 shows the spectrum which results with $\lambda=1$ and $\Delta_f=4$ Mev. In this case we have enlarged the configurations to include $(p_{3/2})^2{}_0 f_{7/2}$, $(f_{7/2})^2{}_2 f_{5/2}$, and $(f_{5/2})^2{}_2 f_{7/2}$. We exclude g particles because in this region of A no information is available about them and also $p_{1/2}$ particles because they do not affect the results. Figure 5 shows a spectrum for Ca^{42} and Table III shows the composition, for various interaction parameters, of some of the wave functions.

In the Ca^{43} plots we do not show results for small β; since βV_0 is essentially constant, a small β implies large V_0 and the energy levels then all become quite high. In particular, we exclude the Rosenfeld mixture ($\epsilon=+1, \beta=0.3$).

The two calculated spectra for Ca^{43} are about the same except for the position of the second $5/2^-$ level. The levels between 1 and 2 Mev are not experimentally

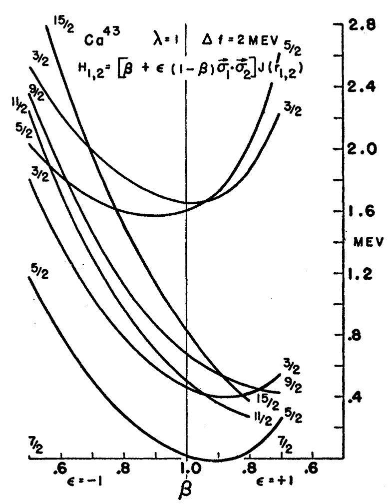

FIG. 2. The calculated spectra of Ca^{43} *versus* the exchange parameters β, ϵ for a Gaussian potential, where $H_{12}=[\beta+\epsilon(1-\beta)\sigma_1\cdot\sigma_2]\times V\exp(-r^2/r_0^2)$, assuming that the f splitting is 2 Mev and the range parameter λ is 1.

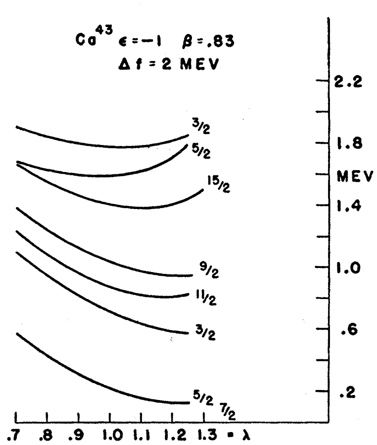

FIG. 3. The calculated spectra of Ca^{43} *versus* the range parameter λ, assuming that $\epsilon=-1, \beta=0.83$, and the f splitting is 2 Mev. H_{12} is the same as in Fig. 2.

[16] W. J. Swiatecki, Proc. Roy. Soc. (London) **A205**, 238 (1951).
[17] See, for example, D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).
[18] D. Kurath, Phys. Rev. **80**, 98 (1950).

TABLE III. The wave-function composition for the lowest states of Ca^{42} and Ca^{43}. Tabulated against the exchange parameters ϵ and β are the squares of the amplitude of the various components.[a] The f doublet splitting is 4 Mev and the range parameter $\lambda=1$.

Ca^{42}

Comp. \ ϵ, β	$J=0$: −1, 0.5	$J=0$: −1, 0.8	$J=0$: +1, 0.8	Comp. \ ϵ, β	$J=2$: −1, 0.5	$J=2$: −1, 0.8
$(7/2)^2{}_0$	0.873	0.962	0.981	$(7/2)^2{}_2$	0.945	0.994
$(3/2)^2{}_0$	0.039	0.017	0.003	$(5/2,7/2)_2$	0.026*	0.002*
$(1/2)^2{}_0$	0.022	0.005	0.001*	$(5/2)^2{}_2$	0.028	0.004
$(5/2)^2{}_0$	0.066	0.015	0.015*			

Ca^{43}

Comp. \ ϵ, β	$J=7/2$: −1, 0.5	$J=7/2$: −1, 0.8	$J=7/2$: +1, 0.8	Comp. \ ϵ, β	$J=5/2$: −1, 0.6	$J=5/2$: −1, 0.8	$J=5/2$: +1, 0.8	Comp. \ ϵ, β	$J=3/2$: −1, 0.5	$J=3/2$: −1, 0.8	$J=3/2$: +1, 0.8
$(7/2)^3{}_{7/2}$	0.909	0.966	0.982	$(7/2)^3{}_{5/2}$	0.938	0.982	0.968	$(7/2)^3{}_{3/2}$	0.942	0.955	0.948
$(3/2)^2{}_0\ 7/2$	0.019	0.011	0.003	$(7/2)^2{}_0\ 5/2$	0.000	0.003	0.008	$(7/2)^2{}_0\ 3/2$	0.034	0.042	0.039
$(7/2)^2{}_2\ 5/2$	0.015*	0.010*	0.001*	$(7/2)^2{}_2\ 5/2$	0.038*	0.008*	0.011	$(7/2)^2{}_2\ 5/2$	0.006	0.000	0.009*
$(5/2)^2{}_0\ 7/2$	0.054	0.011	0.011*	$(5/2)^2{}_2\ 7/2$	0.023	0.006	0.013*	$(5/2)^2{}_2\ 7/2$	0.018	0.003	0.004*
$(5/2)^2{}_2\ 7/2$	0.004	0.001	0.003								

[a] The Asterisk indicates a negative amplitude.

identified (though there is a tentative assignment by Lindqvist and Mitchell[19] of positive parity); the figures predict levels $J=9/2^-$, $11/2^-$, and possibly also $15/2^-$ between these limits. Such high-spin levels would show no stripping and there is the possibility that the compound-nucleus contribution would be quite small. A preliminary report,[3] however, suggests that no extra levels show up in the inelastic proton scattering. [*Note added in proof*.—Recently, Braams (C. M. Braams, thesis, Utrecht, 1956 (unpublished)) has reported three other levels in Ca^{43} below 2 Mev, at 1.904, 1.932, and 1.985 Mev. He also reports that the 1.68-Mev level is seen very strongly in $Ca^{43}(p,p')$ and suggests that it might be one of the high-spin levels from the configuration $(7/2)^3$.] The analysis of Levinson and Ford also predicts these low-lying high spin levels. The best fit to the Ca^{43} spectrum seems to be at about $\epsilon=-1$, $\beta=0.75$ thus giving for the effective interaction between identical nucleons (assuming Gaussian dependence)

$$H_{12}(\mathbf{r})\simeq -3[3-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\exp(-r^2/r_0^2)\text{ Mev}, \quad (7)$$

with $r_0=2.7\times10^{-13}$ cm.

The agreement with the Ca^{42} spectrum is mediocre. It should be remembered however that in this case (and in Ca^{43} also) the ground state, in particular, may be depressed by the inclusion of zero-coupled pairs from higher single-particle levels. Levinson and Ford[4] in fact make use of this by adding in $(g_{9/2})^2{}_0$ pairs assuming a quite low $g_{9/2}$ single-particle level. For certain exchange mixtures and particularly when the f doublet splitting is taken as 2 Mev, the Ca^{42} ground state wave function has an appreciable $(f_{5/2})^2{}_0$ component and this may be expected to have an effect on the analysis of the $Ca^{42}(d,p)$ results.[20] We have verified that these effects are satisfactorily small.

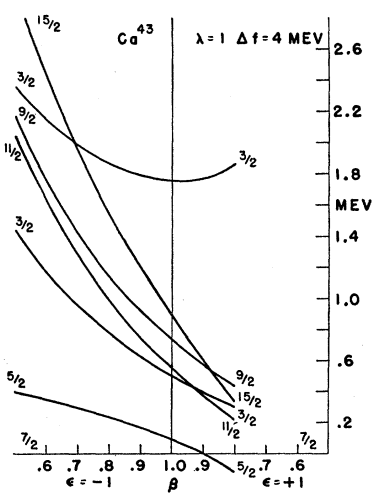

FIG 4. The calculated spectra of Ca^{43} *versus* β, ϵ assuming that the f splitting is 4 Mev. H_{12} and λ are the same as in Fig. 2.

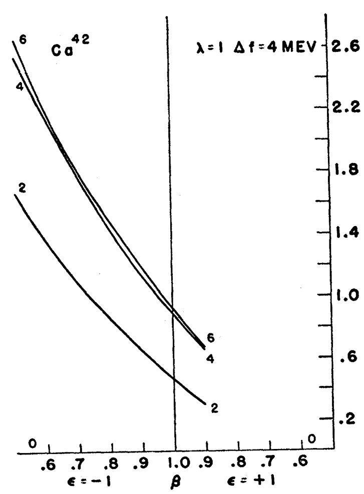

FIG. 5. The calculated spectra of Ca^{43} *versus* β, ϵ, assuming the same parameters as in Fig. 4.

[19] T. Lindqvist and A. C. G. Mitchell, Phys. Rev. 95, 1535 (1954).

[20] We are indebted to Dr. C. Levinson for pointing this out.

We add here some final comments concerning the comparison of our work with that of Levinson and Ford.[4] These authors place primary emphasis on fitting the level structures and the Ca^{43} magnetic moment; they use a long-range singlet interaction

$$H_{12}(\mathbf{r})\simeq -1.2[3-3\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\exp(-r^2/r_0^2),$$

with $r_0\simeq 3.4\times 10^{-13}$ cm. Assuming, as we have done, that the Butler analysis gives a satisfactory treatment for our relative reduced widths, we see that this interaction has a spin-independent part too small by about a factor 2 to explain the $Ca^{42}(d,p)$ result for the 0.59-Mev level; it would predict a cross section small by a factor 4. With respect to the magnetic moment we add only that since we ignore states of high seniority [e.g., $(f_{7/2})^2{}_4 f_{5/2}$] which are of little consequence for energy levels and (d,p) reactions but important for magnetic moments (as shown by Levinson and Ford[4]) we are unable to make a meaningful calculation. There is, of course, no guarantee that, should we include the missing states, we would get an agreement with the measured moment.

V. CONCLUSION

As a result of our analysis it appears that in the lower Ca isotopes the angular-momentum coupling scheme, at least for the lower levels, may be classed as close to jj coupling. The effective nucleon-nucleon interaction has a long range and a relatively large spin-independent part; the corresponding determination of the spin-dependent interaction, in part because of experimental ambiguities, has been quite rough. We have excluded higher states from consideration, but a cursory examination of the available data[21] shows that for levels above, say, 2 Mev the situation is more complicated than is describable by our relatively simple configurations.

It appears certain that a great deal of further experimental work can profitably be done. We may in fact distinguish two kinds of experimental projects. The first of these would devote itself to a study of the stripping process itself with an aim to improving empirically the technique of extracting relative reduced widths from experimental cross sections; uncertainties about this perhaps constitute the major weakness of the present type of analysis. Of particular value here would be measurements of the energy variation of stripping cross sections. Of great help too would be further calculations of corrections to the Butler theory as examined particularly by Tobocman and Kalos.[1] The second experimental project would involve further measurements of the type considered here, preferably at the same time measuring absolute cross sections. If the nuclei involved in such reactions are susceptible to theoretical spectroscopic analysis such measurements will be of great value.[22] Besides the nuclear p shell this applies in particular to the atomic number region $A=30-55$.[23]

VI. ACKNOWLEDGMENTS

We thank W. W. Buechner and the other members of the M.I.T. group (C. K. Bockelman, C. M. Braams, C. P. Browne, W. R. Cobb, D. B. Guthe, and A. Sperduto) for giving us free access to their experimental results and for allowing us to quote them before publication. We are particularly indebted to C. K. Bockelman for much discussion and correspondence concerning the experimental results and their interpretation and significance. We thank C. Levinson for discussion about his theoretical work done with K. W. Ford and for several valuable criticisms of the present work, R. W. Hayward and R. van Lieshout for discussion concerning their experiments with Ca^{43}, and R. van Lieshout for a critical reading of this work. N. S. Wall and R. H. Nussbaum, J. M. Kennedy, and C. Schwartz have helped us by discussing the experimental and theoretical results. S. P. Pandya has been very helpful with the theoretical spectroscopy. We are indebted to D. Kurath for many helpful conversations and for giving us a computing code for use with the AVIDAC at Argonne National Laboratory; we finally thank the AVIDAC group for much of the computing.

APPENDIX I. TWO- AND THREE-NUCLEON SPECTROSCOPY WITH jj WAVE FUNCTIONS

Two-nucleon spectroscopy is of course completely well known; for three nucleons we may in many cases use the coefficient of fractional parentage (c.f.p.) tables of Flowers and Edmonds[24] if all three nucleons are equivalent, while otherwise we may proceed by a direct application of an antisymmetrizer to a wave function for three coupled particles. Despite this, and at the risk of belaboring a well-known subject, we give

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[21] In particular see the $Ca^{40}(d,p)$ experiment by Holt and Marsham (reference 7) and by the M.I.T. group. The results have been discussed in detail by C. K. Bockelman, Bull. Am. Phys. Soc. Ser. II, **1**, 223 (1956). It is important too that the magnitudes of the reduced widths be very small even for the low-lying single-particle levels, as emphasized by Fujimoto *et al.* [see *Proceedings of the International Conference on Theoretical Physics*, Kyoto and Tokyo, September, 1953 (Science Council of Japan, Tokyo, 1954).

[22] J. P. Schiffer, Phys. Rev. **97**, 428 (1955) has measured the $Ca^{43}(d,p)Ca^{44}$ reaction at 90°. He finds that the cross section to the first three levels is roughly 1/50 of the $Ca^{40}(d,p)Ca^{41}$ ground state reaction at this angle. Unfortunately, Butler theory may not be valid at angles away from the first maximum so that our analysis cannot be applied to these data. If jj coupling were valid, one would expect the following ratios for $\sigma_{\max}(\theta)$ compared to the $Ca^{40}(d,p)Ca^{41}$ ground state reaction:

$$\frac{\sigma_{\max}(\theta):\ Ca^{43}(d,p)Ca^{44}{}_J\ \text{level}}{\sigma_{\max}(\theta):\ Ca^{40}(d,p)Ca^{41}\ \text{ground state}}=\begin{array}{cccc} J=0 & 2 & 4 & 6\\ 3/48 & 5/48 & 9/48 & 13/48\end{array}$$

This problem has been discussed also by Endt and Braams.[1]

[23] A review of possible experiments in this region is given by J. B. French and S. P. Pandya, U. S. Atomic Energy Commission Report NYO-7671 (unpublished).

[24] B. H. Flowers, Proc. Roy. Soc. (London) **A212**, 248 (1952); A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A214**, 515 (1952); A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A215**, 120 (1952).

a different formulation than one usually encounters. It will have the advantage that most of the matrix elements which we need will be given in simple explicit form.

Two-Nucleon Spectroscopy

For two nucleons we divide the central interaction into a spin-independent and a singlet part. The general matrix element for the first part may be written down at sight, as done by Racah.[25] For the second part we transform to the LS representation but note that, because of the singlet nature of the interaction, the LS sum reduces to a single term and the $(9j)$ symbols[26] of the transformation reduce to simple Racah coefficients since they have one zero argument. Specifically, for matrix elements between states of definite isotopic spin T we write the general two-nucleon charge-independent central interaction as

$$H_c=[A_T+\tfrac{1}{2}B_T(1-B)]J(r_{12}), \tag{A1}$$

where $B=\frac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ is the spin-exchange operator and A_T, B_T are constants. Then, carrying out the steps above, we have immediately

$$\begin{aligned}\langle l_1'j_1':l_2'j_2'|H_c|l_1j_1:l_2j_2\rangle_{TJ}&=2aa'[[j_1][j_1'][j_2][j_2']]^{\frac{1}{2}}(-1)^J\\
&\times\{(-1)^{j_2+j_2'}A_T\textstyle\sum_k\epsilon(l_1l_1'k)C_{j_1j_1'k}C_{j_2j_2'k}W(j_1j_2j_1'j_2':Jk)F^k(l_1'l_2':l_1l_2)[k]^{-1}\\
&+\tfrac{1}{2}(-1)^{j_1+j_1'+1}W(l_1j_1l_2j_2:\tfrac{1}{2}J)W(l_1'j_1'l_2'j_2':\tfrac{1}{2}J)B_T\textstyle\sum_kD_{l_1'l_1k}D_{l_2'l_2k}W(l_1'l_2'l_1l_2:Jk)F^k(l_1'l_2':l_1l_2)\}\\
&+(-1)^{T+j_1+j_2-J}[\text{same with } l_1,j_1\rightleftarrows l_2,j_2]\end{aligned}\tag{A2}$$

$$\begin{aligned}&\xrightarrow{\text{(short-range limit)}}aa'[[j_1][j_1'][j_2][j_2']]^{\frac{1}{2}}(-1)^{j_1+j_1'+l_1+l_1'+1}[J]^{-1}F^0\\
&\times\{(A_T+\epsilon(l_1l_2J)B_T)C_{j_1j_2J}C_{j_1'j_2'J}+(-1)^{j_2+j_2'+l_2+l_2'+1}A_TC_{\frac{1}{2},\frac{1}{2}}{}^{j_1j_2J}C_{\frac{1}{2},\frac{1}{2}}{}^{j_1'j_2'J}\}\\
&+(-1)^{T+j_1+j_2-J}[\text{same with } l_1,j_1\rightleftarrows l_2,j_2].\end{aligned}\tag{A3}$$

Here we have

$$\begin{aligned}&[j]=(2j+1),\\
&a=\tfrac{1}{2}\text{ if } j_1,\ j_2 \text{ are equivalent, } =1/\sqrt{2}\text{ if nonequivalent,}\\
&D_{ll'k}=D_{l'lk}=[[l][l']]^{\frac{1}{2}}[k]^{-\frac{1}{2}}C_{00}{}^{ll'k},\\
&\epsilon(abc\cdots)=\begin{cases}1 & \text{if}\quad a+b+c+\cdots=\text{even}\\ 0 & \text{if}\quad a+b+c+\cdots=\text{odd},\end{cases}\\
&C_{abc}=C_{bac}=C_{00}{}^{abc}\quad\text{if } a \text{ and } b \text{ are integers,}\\
&\qquad=C_{\frac{1}{2},-\frac{1}{2}}{}^{abc}\quad\text{if } a \text{ and } b \text{ are half integers,}\\
&F^k(l_1'l_2':l_1l_2)=\int_0^\infty r_1^2dr_1\int_0^\infty r_2^2dr_2[\mathfrak{R}_{l_1'}(r_1)\mathfrak{R}_{l_2'}(r_2)\mathfrak{R}_{l_1}(r_1)\mathfrak{R}_{l_2}(r_2)]J_k(r_1,r_2),\end{aligned}\tag{A4}$$

where the $\mathfrak{R}_l$ are the single-particle radial wave functions (the principal quantum number n being always understood) and as usual[25] $J(r_{12})=\sum_kJ_k(r_1,r_2)[C^k(1)\cdot C^k(2)]$.

In writing the formulas we have used

$$\begin{aligned}[k]^{\frac{1}{2}}\langle l\tfrac{1}{2}j\|C^k\|l'\tfrac{1}{2}j'\rangle&=(-1)^{k+\frac{1}{2}-j'}[[j][j']]^{\frac{1}{2}}D_{ll'k}W(ljl'j':\tfrac{1}{2}k)\\
&=(-1)^{k+\frac{1}{2}-j}[[j][j']]^{\frac{1}{2}}\epsilon(ll'k)C_{jj'k},\end{aligned}\tag{A5}$$

and we have finally written the short-range-limit expression using here a relationship between $(C_{j_1j_2J})^2$ and $(C_{\frac{1}{2},\frac{1}{2}}{}^{j_1j_2J})^2$ as given by de-Shalit.[27] The above expressions (modified in the obvious way when we do not use the isotopic spin formulation) contain, as special cases, the formulas derived by de-Shalit[27] for his study of the two-

[25] G. Racah, Phys. Rev. **62**, 438 (1942).
[26] E. P. Wigner (unpublished manuscript, 1951). See also U. Fano, National Bureau of Standards Report 1214, 1951 (unpublished).
[27] A. de-Shalit, Phys. Rev. **91**, 1479 (1953).

nucleon model of odd-odd nuclei and the formulas of Talmi[28] for the corresponding model of the even-even nuclei. A table of $C_{jj'J}$ given by de-Shalit[27] is particularly convenient.

Three-Nucleon Spectroscopy

We now give expressions for the central-force matrix elements for three nucleons, omitting however consideration of states where all three nucleons are inequivalent (this case is simple but of no interest to us). Consider the 3-nucleon wave function $\Psi_{TJ}((j^2)_{T_0J_0}j')$ which is formed by acting with the operator $(1-P_{13}-P_{23})$ on the function formed by vector-coupling nucleon No. 3 (j') to the antisymmetric state $\Psi_{T_0J_0}(j^2)$ of nucleons No. 1, 2, the resultant being then normalized without changing its phase. We have then

$$\Psi_{TJ}((j^2)_{T_0J_0}j')=N^{-\frac{1}{2}}\{\Psi_{T_0J_0}(j^2)\times\phi_{j'}(3)+a^{-1}(-1)^{j+\frac{1}{2}+T+J}\sum_{T_1J_1}(-1)^{T_1+J_1}U(jjJj':J_0J_1)$$
$$\times U(T:T_0T_1)\Psi_{T_1J_1}(jj')\times\phi_j(3)\},\quad\text{(A6)}$$

where $\times$ signifies vector coupling to resultant T, J. U is a normalized Racah coefficient and $U(T:T_0T_1)\equiv U(\frac{1}{2}\frac{1}{2}T\frac{1}{2}:T_0T_1)$. The normalization N is given by

$$N=3\{1-2(-1)^{j+\frac{1}{2}+T+J}U(jjJj:J_0J_0)U(T:T_0T_0)\delta_{nn'}\delta_{ll'}\delta_{jj'}\},\quad\text{(A7)}$$

and $a\equiv a_{jj'}$ [see Eq. (A4)].

The general matrix element is now

$$\langle\Psi[(j^2)_{T_0J_0}j']H_c\Psi[(j_1^2)_{T_0'J_0'}j_1']\rangle_{TJ}=3(NN_1)^{-\frac{1}{2}}\{\delta_{J_0J_0'}\delta_{j'j_1'}\langle j^2|H_c|j_1^2\rangle_{T_0J_0}$$
$$-\delta_{j'j_1}(-1)^{j_1+\frac{1}{2}+T+J}(a_1)^{-1}U(j_1j_1Jj_1':J_0'J_0)U(T:T_0'T_0)\langle j^2|H_c|j_1j_1'\rangle_{T_0J_0}$$
$$-\delta_{j_1'j}(-1)^{j+\frac{1}{2}+T+J}(a)^{-1}U(jjJj':J_0J_0')U(T:T_0T_0')\langle jj'|H_c|j_1^2\rangle_{T_0'J_0'}$$
$$+\delta_{jj_1}(aa_1)^{-1}\sum_{T_1J_1}U(jjJj':J_0J_1)U(jjJj_1':J_0'J_1)U(T:T_0T_1)U(T:T_0'T_1)\langle jj'|H_c|jj_1'\rangle_{T_1J_1}\},\quad\text{(A8)}$$

where we understand by H_c appearing in an n-nucleon matrix element the interaction between all n nucleons.

For the simple case $j\neq j_1$, this result, combined with the two-nucleon result [Eq. (A2)], supplies an explicit form for the matrix element. For the more interesting case $j=j_1$, we shall perform the J_1T_1 sum. Clearly for this purpose it is advantageous to isolate the J_1 dependence of the two-nucleon matrix element and for this the method used by de-Shalit,[27] of working with tensor products of irreducible tensors, is most convenient.

For this purpose we rewrite the central interaction as

$$H_c^T=[D_T^0+D_T^1\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]J(r_{12})=\sum_n D_T^nH_c^n,\quad\text{(A9)}$$

where $D_T^0=A_T+\frac{1}{4}B_T$, $D_T^1=-\frac{1}{4}B_T$. Then

$$\langle j_1'j_2'|H_c|j_1j_2\rangle_{TJ}=4aa'[[j_1][j_1'][j_2][j_2']]^{\frac{1}{2}}\sum_{n,k,r}(-1)^{j_1'+j_2-J+n+l_1+l_2+k}$$
$$\times[n]D_T^nD_{l_1'l_1k}D_{l_2'l_2k}[r](-1)^{-r}\begin{Bmatrix}l_1' & \frac{1}{2} & j_1'\\ l_1 & \frac{1}{2} & j_1\\ k & n & r\end{Bmatrix}\begin{Bmatrix}l_2' & \frac{1}{2} & j_2'\\ l_2 & \frac{1}{2} & j_2\\ k & n & r\end{Bmatrix}$$
$$\times W(j_1'j_2'j_1j_2:Jr)F^k(l_1'l_2':l_1l_2)+(-1)^{T+j_1+j_2-J}[\text{same with } l_1, j_1\rightleftharpoons l_2, j_2].\quad\text{(A10)}$$

We now introduce the "constants" $E^n(T:T_0T_0')$, $G^n(T:T_0T_0')$ defined by

$$E^n(T:T_0T_0')=E^n(T:T_0'T_0)=\sum_{T_1}U(T:T_0T_1)U(T:T_0'T_1)D_{T_1}^n,$$
$$G^n(T:T_0T_0')=G^n(T:T_0'T_0)=\sum_{T_1}(-1)^{T_1}U(T:T_0T_1)U(T:T_0'T_1)D_{T_1}^n\quad\text{(A11)}$$
$$=-E^n(T:T_0T_0')+(-1)^{T_0+T_0'}[[T_0][T_0']]^{-\frac{1}{2}}[T]^{-1}D_0^n,$$

[28] I. Talmi, Phys. Rev. 90, 1001 (1953).

and now we have for the last term in the bracket of (A8), which we temporarily label M,

$$M=4\delta_{jj_1}[j][[j'][j_1'][J_0][J_0']]^{\frac{1}{2}}\Bigg\{(-1)^{l+l_1'-j_1'+J_0+J_0'+J}\sum_{n,k}D_{ll k}D_{l'l_1'k}F^k(ll':ll_1')$$

$$\times[n]E^n(T:T_0T_0')\sum_r[r]\begin{Bmatrix} l & \frac{1}{2} & j \\ l & \frac{1}{2} & j \\ k & n & r\end{Bmatrix}\begin{Bmatrix} l' & \frac{1}{2} & j' \\ l_1' & \frac{1}{2} & j_1' \\ k & n & r\end{Bmatrix}W(jJ_0jJ_0':jr)W(j'J_0j_1'J_0':Jr)$$

$$+(-1)^{J_0+J_0'+J+j'}\sum_{n,k}D_{ll_1'k}D_{ll'k}F^k(ll':l_1'l)[n](-1)^nG^n(T:T_0T_0')$$

$$\times\sum_r[r]\begin{Bmatrix} l & \frac{1}{2} & j \\ l_1' & \frac{1}{2} & j_1' \\ k & n & r\end{Bmatrix}\begin{Bmatrix} l' & \frac{1}{2} & j' \\ l & \frac{1}{2} & j \\ k & n & r\end{Bmatrix}\begin{Bmatrix} J_0 & j' & J \\ j & r & j_1' \\ j & j & J_0'\end{Bmatrix}\Bigg\}. \qquad \text{(A12)}$$

We record too the values of the quantities E^n.

$$4E^n(\tfrac{1}{2}:00)=D_0{}^n+3D_1{}^n;\quad 4E^n(\tfrac{1}{2}:01)=-\sqrt{3}[D_0{}^n-D_1{}^n];$$
$$4E^n(\tfrac{1}{2}:11)=3D_0{}^n+D_1{}^n;\quad 4E^n(\tfrac{3}{2}:11)=4D_1{}^n. \qquad \text{(A13)}$$

The G^n are given in terms of E^n by (A11) or by $G^n=E^n(D_1{}^n\rightarrow -D_1{}^n)$.

The quantities represented by the r sums above belong to the class of higher $(3n-j)$ symbols[26] and of course are not tabulated. (We stress the fact, however, that a good tabulation of the $LS\rightleftharpoons jj$ transformation coefficients is available[29] so that in particular the first of the two foregoing symbols is simple to evaluate.) We are particularly interested, however, in special cases, the results for which we now give.

Matrix Element $\langle(\mathrm{j}^2)_{10}\mathrm{j}'|H_c|\mathrm{j}^3\rangle_{Tj'}$

$$\langle(j^2)_{10}j'|H_c|(j^2)_{T_0'J_0'}j\rangle_{T,j'}=3(NN')^{-\frac{1}{2}}\Bigg\{\delta_{jj'}\langle j^2|H_c|j^2\rangle_{10}[\delta_{J_0',0}+2(-1)^{j+\frac{1}{2}+J+T+J_0'}[J_0']^{\frac{1}{2}}[j]^{-1}U(T:T_0'1)]$$

$$+(-1)^{J_0'}(a_{jj'})^{-1}[J_0']^{\frac{1}{2}}[j]^{-\frac{1}{2}}[j']^{-\frac{1}{2}}\sum_n\langle jj'|H_c{}^n|j^2\rangle_{T_0'J_0'}[(-1)^{T-\frac{1}{2}}U(T:1T_0')D_{T_0'}{}^n-(-1)^{n+T_0'}G^n(T:1T_0')]$$

$$+4(-1)^{j+j'+1}[j][J_0']^{\frac{1}{2}}\sum_{n,k}[n]D_{llk}D_{ll'k}\begin{Bmatrix} l & \frac{1}{2} & j \\ l & \frac{1}{2} & j \\ k & n & J_0'\end{Bmatrix}\begin{Bmatrix} l' & \frac{1}{2} & j' \\ l & \frac{1}{2} & j \\ k & n & J_0'\end{Bmatrix}F^k(ll':ll')E^n(T:1T_0')\Bigg\}. \qquad \text{(A14)}$$

This formula combined with (A2) and the tabulated $9j$ symbols[29] is completely explicit. It displays some important selection rules. Its third term vanishes if $n+J_0'=$ odd (or $n+T_0'=$ even), since $9j$ symbols with 2 identical rows vanish if the sum of the elements in the third row is odd. The second term vanishes when

$$\sum_n[(-1)^{T-\frac{1}{2}}U(T:1T_0')D_{T_0'}{}^n-(-1)^{n+T_0'}G^n(T:1T_0')]H^{n,T_0'}=0, \qquad \text{(A15)}$$

one solution of which is afforded by $H^0=0$, $T=3/2$. The first term vanishes if $j\neq j'$. In particular we have then, for the $T=3/2$, $j\neq j'$ case discussed in the text, that the $n=1$ matrix element vanishes and thus, as we have stressed, the (d,p) reactions which we have considered effectively measure the spin-independent part of the interaction. The actual matrix element in this case is quite simple. Other selection rules which would be important in the analysis of reactions with $T=1/2$ are imbedded in (A15), but we do not write them out in detail. We point out later that by using the concept of seniority the selection rule found here may be extended to a much more general case.

[29] J. M. Kennedy and M. J. Cliff, Atomic Energy of Canada Report CRT-609, 1955 (unpublished).

Diagonal Matrix Element for $(j^3)_{Tj}$

We write Eq. (A14) explicitly for this case with $J_0=J_0'=0$. (This includes the important seniority-1 case.) Then we have for this special case:

$$\begin{aligned}
&\text{(a)}\quad \frac{[j]}{[j]+1}\{([j]+\tfrac{3}{2})D_1{}^0+\tfrac{3}{2}D_0{}^0\}\sum_k (C_{jjk})^2[k]^{-1}F^k+\frac{[j]}{[j]+1}(\tfrac{3}{2}D_0{}^0+\tfrac{1}{2}D_1{}^0)F^0 \quad \text{if}\quad n=0,\ T=\tfrac{1}{2},\\
&\text{(b)}\quad \frac{[j]}{[j]+1}\{([j]+5/2)D_1{}^1-\tfrac{3}{2}D_0{}^1\}\sum_k \{(C_{jjk})^2-2(C_{llk})^2\}[k]^{-1}F^k \quad \text{if}\quad n=1,\ T=\tfrac{1}{2},\\
&\text{(c)}\quad \frac{[j]}{[j]-2}\{([j]-6)D_1{}^0\}\sum_k (C_{jjk})^2[k]^{-1}F^k+\frac{[j]}{[j]-2}(2D_1{}^0)F^0 \quad \text{if}\quad n=0,\ T=\tfrac{3}{2},\\
&\text{(d)}\quad [j]D_1{}^1\sum_k\{(C_{jjk})^2-2(C_{llk})^2\}[k]^{-1}F^k \quad \text{if}\quad n=1,\ T=\tfrac{3}{2}.
\end{aligned}\tag{A16}$$

The Diagonal Matrix Element for $[(j^2)_{10}j']_{Tj'}$ with $j\neq j'$

This is the other case of particular interest to us; it measures (to first order) the coupling of an inequivalent nucleon to a zero-coupled pair. We find immediately from Eqs. (A8), (A12)

$$\begin{aligned}
\langle (j^2)_{10}j'|H_c|(j^2)_{10}j'\rangle_{T,j'(j\neq j')}&=\langle j^2|H_c|j^2\rangle_{10}+2E^0(T:11)F^0(ll':ll')\\
&+2\sum_k\{[G^0(T:11)-G^1(T:11)](C_{jj'k})^2\epsilon(ll'k)+2G^1(T:11)(C_{ll'k})^2\}[k]^{-1}F^k(ll':l'l).
\end{aligned}\tag{A17}$$

Matrix Elements in Terms of Coefficients of Fractional Parentage

Finally we outline the connection of the above three-nucleon formulas with those based on the explicit use of coefficients of fractional parentage (c.f.p.) for the wave functions involving three equivalent nucleons. In this case we write (instead of A6)

$$\Psi_{TJ}(j^3)=\sum_{T_1J_1}\langle TJ|T_1J_1\rangle\Psi_{T_1J_1}(j^2)\times\phi_j(3),\tag{A18}$$

and then the formulas which replace (A8) when either or both wave functions have three equivalent nucleons are obvious. By comparing (A6) and (A18), it is clear that we have in fact been using for j^3 c.f.p. given by

$$\langle TJ|T_1J_1\rangle=N^{-\frac{1}{2}}\{\delta_{J_1J_0}\delta_{T_1T_0}-2(-1)^{j+\frac{1}{2}+T+J}U(jjJj:J_0J_1)U(T:T_0T_1)\},\tag{A19}$$

with N given by (A7). This explicit representation has been used by many authors.[30] We may choose any set of T_0J_0 values consistent with the nonvanishing of $\langle TJ|T_1J_1\rangle$ but we must remember that the wave functions for the same TJ but different T_0J_0 in general are not orthogonal and do not make a linearly independent set. (For example, if $T=3/2$ and $1/2<j<9/2$, there is only one allowed wave function but more than one pair T_0J_0.) However, if we are dealing with configurations which have more than one allowed state of given TJ, we may evaluate the matrix elements for a sufficient set of T_0J_0 values and then make the necessary corrections for lack of orthogonality. This procedure is not elegant but is quite simple.

There is an additional point to be noted concerning the phases of the c.f.p. If, for example, we evaluate off-diagonal matrix elements between two states by using tabulated c.f.p. in some cases and the above formulas in others, it will be important to compensate for any phase difference between the c.f.p. tabulated and given by (A19).

Seniority Considerations[31]

The reader may suspect that the selection rule discussed above concerning a matrix element of the spin-dependent interaction is in fact a special case of a more general result. This is indeed so, as may be seen by considering the concept of seniority.[32] We sketch the derivation.

Consider $\langle j^n|H|(j^{n-1})_0j'\rangle$ for identical nucleons. We assume $j\neq j'$ and have easily

$$H\equiv\tfrac{1}{2}\sum_r(\sum_iT_i{}^r)^2=\tfrac{1}{2}\sum_r(T^r)^2,\tag{A20}$$

[30] See Schwartz and de-Shalit.[10] This type of representation was used also by one of the present authors (J.B.F.) (unpublished lectures, 1953) and by P. J. Redmond (referred to by Schwartz and de-Shalit[10]).

[31] The results and procedures of this section are due to S. P. Pandya.

[32] G. Racah, Phys. Rev. 63, 367 (1943).

where T_i^r is a tensor operator of rank r involving particle i. Then

$$2\langle j^n|H^r|(j^{n-1})_0 j'\rangle=\sum_{j''}\langle (j^n)_{j'}|T^r|(j^n)_{j''}\rangle\langle (j^n)_{j''}|T^r|(j^{n-1})_0 j'\rangle$$
$$+\sum_{j''}\langle (j^n)_{j'}|T^r|(j^{n-1})_0 j''\rangle\langle (j^{n-1})_0 j''|T^r|(j^{n-1})_0 j'\rangle. \quad \text{(A21)}$$

The second term vanishes since $(j^n)_{j'}$ does not have $(j^{n-1})_0$ as a parent. In the first term we encounter a matrix element between only equivalent particles and thus can assert (see A10) that $n+k+r=$ even, and thus for *any* spin dependence r must be odd for a nonvanishing matrix element. However, for this case, Racah's theorem[32] that odd tensor operators are diagonal in seniority coupled with the fact that in the first matrix element the seniorities are necessarily different proves that for spin-dependent H the matrix element vanishes.

The same type of argument using Racah's[32] Eqs. (58b) and (67) serves to evaluate many of the matrix elements as functions of the number of particles.

APPENDIX II. CA[43] SPECTROSCOPY WITH A YUKAWA POTENTIAL[33]

The levels of Ca^{43} have been calculated as above by using a Yukawa potential:

$$J(r)=V(r/r_0)\exp(-r/r_0).$$

The $p_{3/2}-f_{7/2}$ and the $f_{5/2}-f_{7/2}$ single-particle splittings were both taken to be 2 Mev. As in the foregoing, the primary demand was made that the (d,p) reaction to the first $3/2^-$ state should be properly given. Use was made of Talmi's method[34] for evaluating Slater integrals.

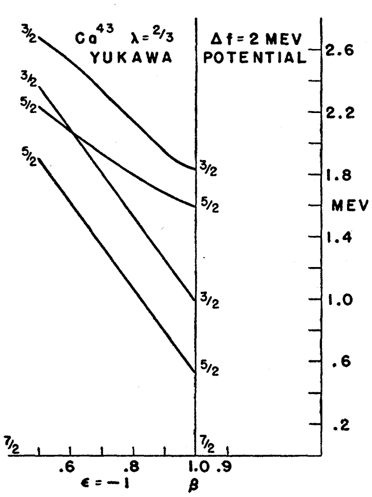

FIG. 6. The 7/2, 5/2, and 3/2 levels of Ca^{43} *versus* β, $\epsilon=-1$, for a Yukawa potential, where $H_{12}=[\beta+\epsilon(1-\beta)\sigma_1\cdot\sigma_2]\times Ve^{-r/r_0}/(r/r_0)$, assuming that the f splitting is 2 Mev and λ is 2/3.

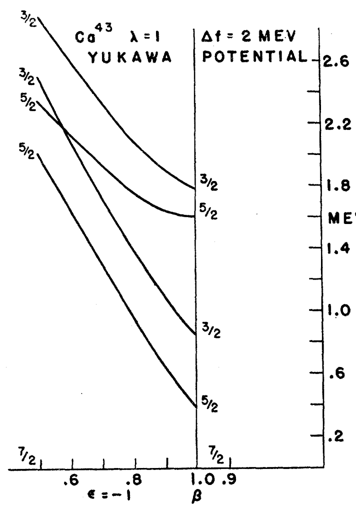

FIG. 7. The same as Fig. 6 except that $\lambda=1$.

With $\lambda\equiv(r_0/r_1)=\frac{2}{3}$ ($r_0\simeq 1.8\times 10^{-13}$ cm) and $\lambda=1$ ($r_0\simeq 2.7\times 10^{-13}$ cm), the results are given in Figs. 6, 7 and these may be compared directly with Fig. 2 for the Gaussian potential. Agreement with experiment is not improved.

The values obtained for V are given in Table IV.

TABLE IV. The magnitude V of the effective potential tabulated against the exchange parameter β ($\epsilon=-1$) used in Appendix II for the Yukawa potential.

	$\epsilon=-1$					
$\beta=$	0.5	0.67	0.79	0.88	0.94	1.0
V (Mev) for $\lambda=\frac{2}{3}$	−15.0	−15.9	−16.7	−17.3	−17.8	−18.2
for $\lambda=1$	−7.9	−9.1	−9.2	−9.4	−9.5	−9.6

[33] The calculations reported in Appendix II were made by D. C. Choudhury.

[34] I. Talmi, Helv. Phys. Acta 25, 185 (1952).