tion in γ by a factor of two, the variation of γ with d at constant pd cannot be considered any more important than the state of the cathode in determining absolute values of the second Townsend coefficient. It is interesting to note that even though the measured values of γ in hydrogen depend on d, the Townsend condition for breakdown holds at any separation providing the value of γ inserted into the condition is measured at the corresponding value of d.

For both gases, the value of $e^{\alpha d}$ at breakdown is of the order of 10^3, and at 2% overvoltage is only of the order of 2×10^3. In the present work total multiplications of 10^5 (due to first and second ionization coefficients acting simultaneously) have been measured below breakdown. To attain this multiplication of 10^5 by a primary ionization process alone would require an overvoltage of about ten percent. In formative time lag work, two percent overvoltage in these gases has been shown to lead to a spark within a fraction of a microsecond.[7,8] Hence, it is concluded that a secondary mechanism is necessary not only to describe prebreakdown currents but is also necessary to describe the buildup process preceding a spark up to overvoltages of the order of 10% in these gases.

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Čerenkov Radiation of Neutral Particles with a Magnetic Moment*

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A simple expression is deduced for the Čerenkov radiation caused by magnetic and electric dipoles. The effect is very small. In the visible spectrum for a fast neutron, the energy loss per unit path per unit frequency range is about 10^{-15} times that of a fast electron. The expression for the energy loss does not contain explicitly the mass of the moving particle. Consequently, this method cannot be used to try to detect whether or not the neutrino has any trace of a magnetic moment.

I

IN this note we shall discuss the radiation caused by neutral particles endowed with a magnetic moment, which move through matter. As in the discussion of Čerenkov radiation, we shall be interested in the case when the velocity of the particle is constant, and exceeds that of light in the medium. Though it turns out that the effect is too small to be observed, we think the results are of interest.[1]

In Sec. II we give a rather intuitive derivation of the energy radiated per unit path length. In Sec. III we derive this expression from Maxwell's equations. In Sec. IV we give a brief numerical estimate.

II

If an electron moves with the constant velocity $v>c/n$, through a medium of index of refraction n, the energy lost per unit path length by radiation with frequency between ω and $\omega+d\omega$ will be given by[2]:

$$(e^2/n^2)(\beta^2n^2-1)(\omega/v)d\omega/v, \quad (\beta=v/c). \qquad (1)$$

We pose now the question: Can one derive from Eq. (1) by some simple intuitive manipulation the energy loss per unit path length for an electric dipole moving under similar conditions? (If one is able to do this for an electric dipole, one can do it immediately for a magnetic dipole as well.) The answer is yes.

For, observe first that (1) contains the time average of the Poynting vector, integrated over a surface. The Poynting vector in turn contains products of the field strengths. Now we can obtain the field of a dipole by taking the space derivatives of the field of a charge and multiply it by, say, q, the separation of the charge in the dipole. Now, in the expression for the field of the charge each coordinate will appear, for dimensional reasons, multiplied by a characteristic length s. Hence each operation on the field will bring in a factor q/s. Since the whole physical situation is stationary, (q/s) will be a constant, and will not be influenced by the time averaging. Thus we expect that (1), containing the products of fields, will acquire a factor $(q/s)^2$. Also, if the dipole is directed normal to the direction of motion, we expect the field to have an angular dependence $\cos\varphi$ and $\sin\varphi$ (φ being an angle in a plane normal to the direction of motion). The squares of this factor, averaged, will bring in another factor $\frac{1}{2}$. Let us find now s, the characteristic length. We expect that this will be the distance the electron has to travel (as seen by the electron) to emit a wave of frequency ω. The time to emit one wave of frequency ω is $1/\omega$; however, for the electron this duration will change into $1/[\omega(\beta^2n^2-1)^{\frac{1}{2}}]$

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[1] After completion of this work, my attention has been called to a paper by V. L. Ginsburg, J. Phys. (U.S.S.R.) 2, 441 (1940). In this article the author quotes the results of his calculations (by a different method) concerning this effect. His results are identical with ours for a dipole if the dipole axis is parallel to the direction of motion, but differ from ours if the axis is normal to it. Though Ginsburg does not give details, one suspects an error, since according to his expression the radiated energy would be proportional to the dipole moment and not to its square.

[2] See, e.g., L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 265.

[the factor $(\beta^2n^2-1)^{\frac{1}{2}}$ taking the place of the usual Lorentz factor in this case].[3] Consequently, the distance traveled by the electron (as seen by it) while emitting a wave of frequency ω, will be $v/[\omega(\beta^2n^2-1)^{\frac{1}{2}}]$. This is our characteristic length s. Putting the pieces together, we may conjecture that the energy loss per unit path length will be given by something like

$$\tfrac{1}{2}(b/n)^2(\beta^2n^2-1)^2(\omega/v)^3(d\omega/v), \quad (2)$$

b being the electric dipole moment of the moving dipole.

Then, for a magnetic dipole of strength g we get immediately

$$\tfrac{1}{2}g^2(\beta^2n^2-1)^2(\omega/v)^3(d\omega/v). \quad (3)$$

The factor $(1/n)^2$ is missing; since, while the effectiveness of the electric dipole moment is decreased by the electric polarization of the medium (expressed by the factor $1/n^2$), there is no magnetic polarization which would decrease the magnetic dipole moment.

If the moving particle has but a magnetic moment at rest, it will acquire an electric one as well (normal to the magnetic dipole), just as a purely magnetic field receives some electric components if it is viewed from a moving frame of reference. We suppose now that both dipole moments are normal to the direction of motion. Under such conditions the electric fields due to each dipole will be mutually orthogonal, and so will be the magnetic fields. If this be the case, the fields due to each moment will not interfere and the total energy loss will be simply the sum of the energy losses suffered by the field of each moment. Adding (2) and (3), we obtain finally

$$\tfrac{1}{2}(\beta^2n^2-1)^2[g^2+(b/n)^2](\omega/v)^3(d\omega/v). \quad (4)$$

In the next section, we shall show that this expression is actually exact.

III

We want to find solutions of Maxwell's equations representing the field of a moving particle which has only a magnetic moment if at rest. We assume that the particle moves in the positive z direction with a constant speed $v>c/n$, in a medium of index of refraction n. Its magnetic and electric moments (the latter induced by the motion) should point along the positive x and y axes, respectively. (We know from the Dirac theory of the electron that this is true for the expectation values of the electric and magnetic moments of a fast electron.)

We shall restrict our interest to those components of the fields which will be needed for the calculation of the energy loss due to radiation. As mentioned in Sec. II, we shall do that by differentiating the solutions obtained for the Čerenkov electron. Moreover, by a suitable interchange of the electric and magnetic field components, we shall get the field of the magnetic dipole as well.

The relevant field components in cylindrical coordinates for a Čerenkov electron are given as[4]

$$\begin{aligned} E_z^e&=\frac{ie}{2c^2}\int_{-\infty}^{\infty}e^{i\omega(t-z/v)}\left(\frac{1}{\beta^2n^2}-1\right)\omega a d\omega,\\ H_\varphi^e&=-\frac{e}{2c}\int_{-\infty}^{\infty}e^{i\omega(t-z/v)}\frac{\partial a}{\partial\rho}d\omega;\\ \beta n>1;\quad a&=-iH_0^{(2)}(\rho/s),\quad \omega>0;\\ a&=iH_0^{(1)}(\rho/s),\quad \omega<0;\\ s&=\omega/(\beta^2n^2-1)^{\frac{1}{2}}. \end{aligned} \quad (5)$$

$H_0^{(1)}$, $H_0^{(2)}$ are Hankel functions.

If we have an electric dipole along the $+y$ axis, the relevant field components will be

$$\begin{aligned} \mathcal{E}_z^e&=-q\frac{\partial E_z^e}{\partial y}=-\frac{iqe}{2c^2}\int_{-\infty}^{\infty}e^{i\omega(t-z/v)}\left(\frac{1}{\beta^2n^2}-1\right)\frac{\omega y}{\rho}\frac{\partial a}{\partial\rho}d\omega\\ &\equiv-(ib/2c^2)P(y); \quad (6)\\ \mathcal{H}_\varphi^e&=-q\frac{\partial H_\varphi^2}{\partial y}=\frac{qe}{2c^2}\int_{-\infty}^{\infty}e^{i\omega(t-z/v)}\frac{y}{\rho}\frac{\partial^2a}{\partial\rho^2}d\omega\equiv(b/2c)Q(y). \end{aligned}$$

Here $b=qe$ is the electric dipole moment of the *moving* dipole.

If we have a magnetic dipole pointing in the $+x$ direction, we obtain the relevant field quantities by performing the substitution

$$\mathcal{E}_\varphi^m=-\mathcal{H}_\varphi^e/n,\quad \mathcal{H}_z^m=n\mathcal{E}_z^e, \quad (7)$$

where in $\mathcal{E}_z^e$, $\mathcal{H}_\varphi^e$ we exchange e with np (p being the pole strength in electromagnetic units), and x with y. (This is an obvious generalization of the well-known transformation $E^e\rightarrow H^m$; $H^e\rightarrow -E^m$, valid in empty space.) If n is a function of ω we should think of performing this substitution for each Fourier component, which would bring the factors n inside the integral sign. Since we take, for simplicity, n constant, we may leave the n's where they are.

The substitution gives

$$\begin{aligned} \mathcal{E}_\varphi^m&=-(q/2c)Q(x),\\ \mathcal{H}_z^m&=-(in^2q/2c)P(x); \end{aligned} \quad (8)$$

$g=qp$ being the dipole moment of the *moving* magnetic dipole.

The complete field components, as far as they are needed for the computation of the energy losses, will be given by the superposition of (6) and (8). The

[3] This can be seen in many different ways. Compare, for example, the z component of the electric field in empty space due to a charge moving with uniform velocity v along the z axis, with that of a charge moving in the same way inside matter, if $v>c/n$. The first is proportional to $(1-\beta^2)(vt-z)/R^3$, where $R^2=(z-vt)^2-(y^2+x^2)(\beta^2-1)$; while the second is proportional to $(\beta^2n^2-1)\times(vt-z)/R'^3$, where $R'^2=(z-vt)^2-(y^2+x^2)(\beta^2n^2-1)$.

[4] I. Tamm, J. Sci. (U.S.S.R.) 1, 409 (1939).

amount of energy, dW, radiated per path length dl is given by

$$dW = dl\rho \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} S_\rho dt. \qquad (9)$$

S_ρ, the ρ component of the Poynting vector, is

$$S_\rho = (c/4\pi)(\mathcal{E}_\varphi \mathcal{H}_z - \mathcal{E}_z \mathcal{H}_\varphi)$$
$$= (c/4\pi)(\mathcal{E}_\varphi{}^m \mathcal{H}_z{}^m - \mathcal{E}_z{}^e \mathcal{H}_\varphi{}^e)$$
$$= (in^2/16\pi c^2)[g^2 P(x)Q(x) + (b^2/n^2)P(y)Q(y)]. \qquad (10)$$

Performing the necessary substitution and integrating over φ, we get

$$dW = -(i\rho n^2/16c^2)(1-\beta^{-2}n^{-2})(g^2+n^{-2}b^2)R\, dl, \qquad (11)$$

where R is given by the following string of expressions, each following rather obviously from the other:

$$R = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \left[\int_{-\infty}^{\infty} dt e^{i(\omega+\omega')t}\right] \times e^{-i(\omega+\omega')z/v}\omega' \frac{\partial a(\omega')}{\partial\rho}\frac{\partial^2 a(\omega)}{\partial\rho^2}$$
$$= -2\pi \int_{-\infty}^{\infty} \omega \frac{\partial a(-\omega)}{\partial\rho}\frac{\partial^2 a(\omega)}{\partial^2\rho} d\omega \qquad (12)$$
$$= -2\pi \int_0^{\infty} \left(\frac{\partial^2 a(\omega)}{\partial\rho^2}\frac{\partial a(-\omega)}{\partial\rho} - \frac{\partial^2 a(-\omega)}{\partial\rho^2}\frac{\partial a(\omega)}{\partial\rho}\right)\omega d\omega.$$

Substitute now from (5) the corresponding values for a. Then, making use of the recurrence formulas for Hankel functions, and finally of their asymptotic expansions. This gives, for R,

$$R = (8i/\rho c^2)n^2(1-\beta^{-2}n^{-2}) \int_0 \omega^3 d\omega; \qquad (13)$$

and for dW,

$$dW/dl = \tfrac{1}{2}(\beta^2 n^2 - 1)^2 (g^2 + b^2/n^2) \int_0 (\omega/v)^3 d\omega/v.$$

This expression is divergent since we have assumed no dispersion, which makes possible the radiation of arbitrarily high frequencies. We can now introduce a cutoff, say the Compton wavelength of the particle, or we can discuss the energy loss per unit frequency range, simply leaving off the integral sign. If we do this, we recover Eq. (4).

If the dipoles are normal to each other and normal to the direction of propagation, as we have assumed, one has the following relation between g, b, and g_0 (the latter is the magnitude of the magnetic moment if the particle is at rest and is equal to g in this case):

$$b = \beta g = \beta g_0. \qquad (14)$$

This gives our fundamental expression for the energy loss per unit path length, per unit frequency range:

$$(d^2W/dld\omega) = \tfrac{1}{2}(1-1/\beta^2 n^2)^2 (1+\beta^2/n^2) g_0{}^2 \omega^3 (n/c)^4. \qquad (15)$$

Observe that the term $(\beta/n)^2$ contains the effect of the electric dipole induced by the motion. This will be of about the order $\frac{1}{2}$ for fast particles in water. This shows that although the effect of the electric dipole is smaller than that of the magnetic one, it cannot be neglected.

For completeness we give also the energy losses for a magnetic dipole which moves in the direction of its moment. Under such conditions the electric moment is zero and we obtain

$$(d^2W/dld\omega) = \tfrac{1}{2}(\beta^2 n^2 - 1) g^2 (\omega/v)^3 (1/v).$$

Here g is the magnitude of the moving magnetic dipole pointing in the direction of motion.

However, we can see from the transformation formulas of the polarization tensor (of which g is a component) that for large velocities g will tend to zero if the magnetic moment is finite in the rest frame. This is readily understood if we realize that in this situation the separation between the poles goes to zero on account of the Lorentz contraction.

IV

Instead of evaluating expression (15) as it stands, let us take the ratio of (15) and (1). This will immediately show us the magnitude of the effect, and will also give some indication why it is so small. Furthermore, let us express g_0 in nuclear magnetons, and put $\omega \sim Rc/2\pi$, where R is the Rydberg constant. The required ratio is

$$\tfrac{1}{2}n^4(1-1/\beta^2 n^2)(1+\beta^2 n^2)(m/M)^2(e^2/c\hbar)^4,$$

where m is the rest mass of an electron, and M is the rest mass of a proton (entering through the nuclear magneton). As we see, the very serious and unfavorable factors are the last two. The fine structure constant raised to the fourth power gives about 2×10^{-9}, while the mass ratios squared will give a factor about 2×10^{-7}. Neglecting factors of order unity (if $\beta\sim1$), we finally get for the ratio about 10^{-15}. The expression is independent of the mass of the moving particle. Thus, even if a neutrino would have a small magnetic moment (it cannot have a large one, since then we would have observed it already), this method could not be used to detect it.

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