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High-Frequency Scattering*

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In recent years, there has been a revival of interest in the phenomena of scattering at high frequencies. The simplest problems of this sort, in which the obstacle is either a circular cylinder or a sphere, are treated here. The treatment is limited to the total scattering cross sections, even though the method is by no means restricted in this way. The specific problems treated here include the scattering of a plane electromagnetic wave by a perfectly conducting circular cylinder (two possible polarizations) or a perfectly conducting sphere, the acoustic scattering by a rigid sphere, and the quantum-mechanical scattering by an impenetrable sphere.

By considering the creeping waves as defined on the universal covering space, the scattering cross sections of a circular cylinder may be expressed by asymptotic expansions for vanishingly small wavelengths. Analogous calculations yield the corresponding results for a sphere. It turns out, as expected, that the resulting expressions are accurate even for fairly large wavelengths. The first six terms of the asymptotic series are explicitly found in each case.

In conclusion, the application of the method to the determination of the approximate current distribution on the obstacle is considered briefly; also some generalizations about the scattering cross section are conjectured.

1. INTRODUCTION

In the electromagnetic theory of light, two distinct disciplines are often invoked to find the scattered field. One is the theory developed from the Maxwellian field equations which are supposed to be valid for all frequencies. The other is the theory of geometrical optics, valid only for very high frequencies, i.e., for wavelengths that are short compared with the size of the obstacle. In this paper, the scattering cross sections as given by geometrical optics are modified to extend their region of validity to lower frequencies for a circular cylindrical obstacle and a spherical obstacle with special properties.

In connection with the study of this problem, the relevant sequence of papers consists of those of Mie, Debye, White, Rubinow and Wu, and Kear,[1] arranged in chronological order. Briefly, Debye showed that the scattering cross section of a sphere, as determined from the field equations, indeed approaches the result of geometrical optics in the limit of very short wavelengths; and White succeeded in finding the first correction to the geometrical-optics cross section of a sphere. In the last two papers listed, the result of White was essentially rediscovered by different methods, perhaps with a little addition in mathematical rigor. Accordingly, when the present investigation was initiated, the state of knowledge was limited to the assertion that at very high frequencies the difference between the exact scattering cross section of a circular cylinder or sphere and the approximation of geometrical optics is asymptotically given by some known constant multiplying the two-thirds power of the wavelength. Furthermore, neither the method of White nor that of Rubinow and Wu can be generalized readily to give higher-order corrections. The work of Kear actually was not available until after this investigation was completed; it does not seem feasible to get these corrections of higher order through his method either.

In the present treatment, a practical procedure is developed to find any number of terms of the asymp-

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[1] G. Mie, Ann. Physik 25, 377 (1908). P. Debye, Münchener dissertation, 1908 (unpublished); Ann. Physik 30, 57 (1909). T. T. Wu and S. I. Rubinow, Cruft Laboratory Scientific Report 3, Harvard University, 1955 (unpublished); J. Appl. Phys. (to be published). G. Kear, New York University Research Report EM-86, 1955 (unpublished).

totic series for the cross sections of a circular cylinder or a sphere with simple boundary conditions. This is accomplished through a new technique, which is by no means restricted to the calculation of the scattering cross sections. However, in this paper the discussion is limited to the scattering cross sections, except for a passing remark in the last section about the surface current. Five terms in the asymptotic series are evaluated explicitly in addition to the leading term provided by geometrical optics.

The method to be used here is briefly as follows. First, the much-discussed concept of the creeping wave around an obstacle is reconsidered from the standpoint of physics, in the sense that it is formulated in accordance with the intuitive concept and then asymptotically evaluated for high frequencies. In view of the questionable nature of the usual identification of creeping waves with the terms of a series, this is interesting for two reasons. The concept of the creeping wave is extended to all wavelengths, not necessarily small as compared with the size of the obstacle. And, even more important, the asymptotic formula for the creeping wave thus obtained is essentially different from the usual expression given, for example, by Franz.[2] At least in the case of the circular cylinder, the asymptotic formula for the far-zone field is much easier to obtain for the creeping waves separately. This is probably true for an arbitrary obstacle. By taking an appropriate sum, it is then easy to get the asymptotic formula for the forward far-zone field, and hence that for the total scattering cross section.

Because of the lack of physical content in some of the rather tedious mathematical manipulations that are necessary, the detailed development is not presented here. Readers interested in the mathematical manipulations are referred to the author's thesis[3] or to a technical report,[3] which is available for distribution on request.

2. PRELIMINARIES

In this section, the plan of attack on the problem of creeping waves is outlined and the simplest problem of this sort is solved explicitly for later use. In the study of the diffraction of a plane wave by a conducting half-plane, Sommerfeld[4] used the fictitious two-sheeted Riemann space[5] to simulate the effect of the screen. More recently, Friedlander,[6] in studying the diffraction of a pulse by a circular cylinder, made important use of the infinite-sheeted Riemann space covering the exterior of the cylinder. Since the solution for the harmonic time-dependent problem can be obtained by integration once the pulse solution is known, the total field due to the diffraction of a cylindrical harmonic wave by a circular cylinder also has a definite meaning on this artificial Riemann space. Alternatively, this total field may be found more directly by solving the reduced wave equation $(\Delta+k^2)\psi=0$ on the Riemann space instead of the conventional Euclidean space.

This consideration naturally suggests the following procedure, at least for two-dimensional problems. Consider the diffraction of a known field by a set of nonintersecting impenetrable domains (closed sets). The appropriate linear field equation—in particular, the reduced wave equation—is solved not on E, the region outside of all these impenetrable domains, but on the universal covering space[7] of E. This solution then gives complete information about the creeping wave structure of the total field. The term "creeping wave" is used in this paper to denote a wave creeping around the obstacle. This is intuitively clear but the precise formulation is not entirely trivial and is given in the Appendix. Loosely, the universal covering space of E may be thought of as the simply-connected many-sheeted space with the same local structure as E. If the creeping wave structure is known, that is, the solution is known on this universal covering space, then the required solution on E may be found by adding up the values of the solution on the universal covering space at the corresponding points. The situation may be easily visualized in the simplest case where the space E is doubly connected. In this case, consider a stack of identical copies of E. Let each of them be slit along some line and then glue each edge to that of the next copy in the simplest manner. The resulting simply-connected space is the required universal covering space. To get the final answer on E, it is only necessary to add up the values of the solution at the corresponding points of the various copies of E. If the space E is of higher connectivity, then the process of slitting and gluing will be more complicated, although not fundamentally different. Of course the diffracted field on E may be found by subtracting the incident field from the total field on E. The diffracted field in general has no creeping-wave structure by itself, because the incident field may not be defined on the universal covering space.

A serious complication arises from the fact that no radiation condition is known for a space that is not Euclidean. A good rule to follow is to use Hankel functions of the first kind, even if the index is not integral. In the remainder of this section the simplest nontrivial creeping-wave problem is solved, namely, the problem of the diffraction of a scalar plane wave by an infinite straight line or cylinder of zero radius. In this case, the total field is verified to be actually causal;

[2] W. Franz and K. Depperman, Ann. Physik **10**, 361 (1952).

[3] T. T. Wu, thesis, Harvard University, 1956 (unpublished); Cruft Laboratory Technical Report 232, Harvard University, 1956 (unpublished).

[4] A. Sommerfeld, *Optics* (Academic Press, Inc., New York, 1954), p. 249 ff.

[5] Not to be confused with Riemannian space in general relativity. See, for example, B. B. Baker and E. T. Copson, *Huygen's Principle* (Clarendon Press, Oxford, 1950), p. 129.

[6] F. G. Friedlander, New York University Report EM-64, 1954 (unpublished).

[7] This deviates somewhat from the standard terminology.

this indicates that it is admissible to follow the above rule. This problem is interesting since in the proposed line of investigation a knowledge of its solution is a prerequisite to the study of the creeping waves around a circular cylinder of finite radius. For this purpose, let a polar coordinate system be set up as in Fig. 1. The set E consists of the entire Euclidean space except the origin. This is a doubly-connected region and it is clear what its universal covering space R is. Let the polar coordinate be extended onto R in the obvious manner. The fundamental sheet of R, defined by $-\pi<\theta\leqslant\pi$, $0<r<\infty$, is called R_0 and has a natural one-to-one correspondence with E.

The resulting total field is constructed by taking the limit of the Green's function for R. For that purpose, let a unit source-point be at $(r_0,0)$, and the induced field be $G(r,r_0;\theta)$. Let $r_>$ be the larger one of r and r_0, both positive, and $r_<$ the smaller one. Since for $\nu\geqslant 0$

$$(\Delta+k^2)J_\nu(kr_<)H_\nu^{(1)}(kr_>)e^{\pm i\nu\theta}=\frac{2i}{\pi r_0}\delta(r-r_0)e^{\pm i\nu\theta},$$

it is clear that

$$G(r,r_0;\theta)=\frac{i}{4}\int_\infty^\infty J_{|\nu|}(kr_<)H_{|\nu|}^{(1)}(kr_>)e^{i\nu\theta}d\nu. \tag{2.1}$$

And hence the total field due to the plane wave e^{-ikx} is given by

$$\psi^{\text{tot}}(r,\theta)=\lim_{r_0\to\infty}G(r,r_0;\theta)\Big/\frac{i}{4}H_0^{(1)}(kr_0)$$

$$=2\int_0^\infty J_\nu(kr)e^{-i\nu\pi/2}\cos\nu\theta d\nu. \tag{2.2}$$

This is the desired expression for the creeping waves. As an application, it may be remarked that the solution of the problem of diffraction by a wedge of arbitrary angle may be obtained from (2.2) by summation.

A few things may be done with this result. First, the various creeping waves may be added together. Thus, by the Poisson summation formula,

$$\sum_{n=-\infty}^{\infty}\psi^{\text{tot}}(r,\theta+2n\pi)=e^{-ikr\cos\theta}=\psi^{\text{inc}}(r,\theta). \tag{2.3}$$

Note here that since the left- and the right-hand sides are defined on different spaces, they can only be compared through the correspondence between R_0 and E or through the polar coordinate system. Equation (2.3) is a well-known result.

Secondly, the total field induced by the scattering of a plane pulse by a line may be found. A subscript k may be used to indicate the wave number. When k is replaced by $-k$, it follows from $\psi_k^{\text{inc}}(r,\theta)=\psi_{-k}^{\text{inc}*}(r,\theta)$ that $\psi_k^{\text{tot}}(r,\theta)=\psi_{-k}^{\text{tot}*}(r,\theta)$ where * means complex conjugate. This is satisfied formally for (2.2) provided the value of J_ν is taken on the principal Riemann

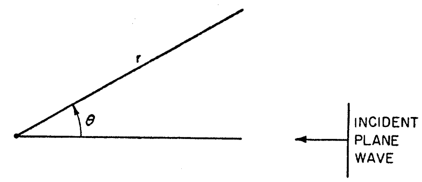

FIG. 1. Coordinate system for the scattering by a cylinder of zero radius.

sheet. For the plane pulse, the field is represented by $\psi_\delta^{\text{inc}}(r,\theta;t)=\delta(r\cos\theta+ct)$. Therefore

$$\psi_\delta^{\text{tot}}(r,\theta;t)=\frac{1}{2\pi}\int_{-\infty}^\infty\psi_k^{\text{tot}}(r,\theta)e^{-ikct}dk$$

$$=\int_0^\infty F(r,\nu;t)\cos\nu\theta d\nu,$$

where

$$F(r,\nu;t)=\frac{1}{\pi}e^{-i\nu\pi/2}\int_{-\infty}^\infty J_\nu(kr)e^{-ikct}dk.$$

The function F may be found by integration as follows:

$$F(r,\nu;t)$$

$$=\begin{cases}\dfrac{2}{\pi}(r^2-c^2t^2)^{-\frac{1}{2}}\cos\nu\left(\dfrac{\pi}{2}+\sin^{-1}\dfrac{ct}{r}\right) & \text{for}\quad r>ct,\\[2ex] -\dfrac{2}{\pi}(c^2t^2-r^2)^{-\frac{1}{2}}\left(\dfrac{r}{ct+(c^2t^2-r^2)^{\frac{1}{2}}}\right)^\nu\sin\nu\pi & \text{for}\quad r<ct.\end{cases}$$

Another integration gives, for $r>ct$,

$$\psi_\delta^{\text{tot}}(r,\theta;t)=\begin{cases}\delta(r\cos\theta+ct) & \text{for}\quad |\theta|<\pi,\\ 0 & \text{otherwise.}\end{cases}$$

Correspondingly, for $r<ct$,

$$\psi_\delta^{\text{tot}}(r,\theta;t)=\frac{1}{\pi}(c^2t^2-r^2)^{-\frac{1}{2}}\left(\frac{\theta-\pi}{[\cosh^{-1}(ct/r)]^2+(\theta-\pi)^2}-\frac{\theta+\pi}{[\cosh^{-1}(ct/r)]^2+(\theta+\pi)^2}\right).$$

These two formulas verify that the field is indeed causal by definition.

Thirdly, asymptotic formulas for the total field are found, to be used later. If $r\to\infty$ with θ fixed, there are three distinct cases depending on the relative magnitude of θ and π. First let $|\theta|>\pi$. It is found that

$$\frac{\partial}{\partial\nu}\arg H_\nu^{(2)}(kr)>0,\quad\frac{\partial^2}{\partial\nu^2}\arg H_\nu^{(2)}(kr)<0,$$

$$\frac{\partial}{\partial\nu}\arg H^{(2)}(kr)\,|_{\nu=0}=\frac{\pi}{2}\quad\text{for}\quad\nu\geqslant 0.$$

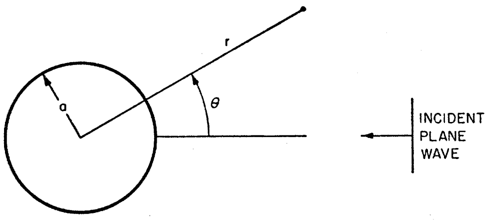

FIG. 2. Coordinate system for the scattering by a circular cylinder.

Hence if J_ν is replaced by $\frac{1}{2}(H_\nu^{(1)}+H_\nu^{(2)})$ in (2.2), the resulting integrals have no point of stationary phase, and an integration by parts gives

$$\psi^{\text{tot}}(r,\theta)\sim\frac{\pi i}{\theta^2-\pi^2}H_0^{(1)}(kr) \quad \text{for} \quad |\theta|>\pi \tag{2.4}$$

as $kr\to\infty$. Equation (2.3) furnishes the required information for all the other cases. Thus, as $kr\to\infty$,

$$\psi^{\text{tot}}(r,\theta)\sim e^{-ikr\cos\theta}+\frac{\pi i}{\theta^2-\pi^2}H_0^{(1)}(kr) \quad \text{for} \quad |\theta|<\pi, \tag{2.5}$$

and

$$\psi^{\text{tot}}(r,\pm\pi)\sim\tfrac{1}{2}e^{+ikr}-\frac{i}{4\pi}H_0^{(1)}(kr). \tag{2.6}$$

3. CROSS SECTION OF AN INFINITE CIRCULAR CYLINDER

In this section, the radiation field caused by the scattering of a plane wave by an infinite circular cylinder is studied. The polarization is first assumed to be such that the electric field vector is parallel to the axis of the cylinder. The other polarization is considered in the next section. Let a polar coordinate system be set up as in Fig. 2, and the boundary condition to be used is that $\psi^{\text{tot}}(a,\theta)=0$, where ψ^{tot} is defined on R as before. The simplest procedure here is to use the total field, given by (2.2), as the incident field ψ^{inc}. It then follows directly from the boundary condition that

$$\begin{aligned}\psi^{\text{sc}}(r,\theta)&=\psi^{\text{tot}}(r,\theta)-\psi^{\text{inc}}(r,\theta)\\&=-2\int_0^\infty J_\nu(ka)\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}e^{-i\nu\pi/2}\cos\nu\theta d\nu.\end{aligned} \tag{3.1}$$

As $kr\to\infty$, this gives the radiation field

$$\psi^{\text{sc}}(r,\theta)\to-2H_0^{(1)}(kr)\int_0^\infty\frac{J_\nu(ka)}{H_\nu^{(1)}(ka)}e^{-i\nu\pi}\cos\nu\theta d\nu. \tag{3.2}$$

Therefore, the far-zone field is completely specified by the function

$$f(\theta)=f_1(\theta)+2\int_0^\infty\frac{J_\nu(ka)}{H_\nu^{(1)}(ka)}e^{-i\nu\pi}\cos\nu\theta d\nu, \tag{3.3}$$

where, according to (2.4–6),

$$f_1(\theta)=\begin{cases}-\pi i/(\theta^2-\pi^2) & \text{for} \quad |\theta|\neq\pi,\\ i/4\pi & \text{for} \quad |\theta|=\pi.\end{cases}$$

As a shorthand, let

$$\sum f(\theta)=\sum_{n=-\infty}^{\infty}f(\theta+2n\pi);$$

then the scattering cross section $\sigma_D^{(c)}$ is given by

$$\sigma_D^{(c)}=(4/k)\operatorname{Re}\sum f(\pi), \tag{3.4}$$

where "Re" means "the real part of." Note that $f(\theta)=f(-\theta)$.

Let $\kappa=ka$ for simplicity; then

$$f(\theta)=f_1(\theta)+\lim_{A\to\infty}\left[\int_0^A e^{-i\nu\pi}\cos\nu\theta d\nu +\frac{1}{2}\int_{-A}^{A}\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}e^{-i\nu(\pi-\theta)}d\nu\right]. \tag{3.5}$$

The second integral in this expression diverges as $A\to\infty$. In this case and a number of other cases, it is very desirable to give a meaning to divergent integrals of this variety in order to save the trouble of complicated limiting processes, which tend to obscure the real issue at hand. For the problems treated here, it is simplest perhaps to use the Abel summability, i.e.,

$$I=\int_{0,-\infty}^{0,\infty}f(x)dx$$

in the Abel sense if

$$I=\lim_{\epsilon\to0+}\int_{0,-\infty}^{0,\infty}f(x)e^{-\epsilon|x|}dx.$$

In this sense, (3.5) may be rewritten as

$$f(\theta)=\begin{cases}\dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty}\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}e^{-i\nu(\pi-\theta)}d\nu & \text{for } \theta\geqslant0 \text{ and } \theta\neq\pi,\\ \dfrac{1}{2}\displaystyle\lim_{A\to\infty}\left[A+\int_{-\infty}^{A}\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}d\nu\right] & \text{for } \theta=\pi.\end{cases} \tag{3.6}$$

In case $\theta>\pi$, the integral may be reduced to a residue sum by closing the contour of integration by a large semicircle in the upper half-plane. The result is that

$$f(\theta)=\pi i\sum_j e^{-i\nu_j(\pi-\theta)}\lim_{\nu\to\nu_j}(\nu-\nu_j)\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}, \tag{3.7}$$

where ν_j is the jth zero of $H_\nu^{(1)}(\kappa)$. This equation is exactly true. However, as $\kappa\to\infty$, each term in the series is exponentially small compared with the previous term. Hence, in the sense of Poincaré,[8] the asymptotic

[8] See, for example, E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (The Macmillan Company, New York, 1943), p. 151.

expansion for $f(\theta)$ is completely identical with that of the first term. The leading terms in the asymptotic series for ν_1 are

$$\nu_1 \sim \kappa\left\{1+\tfrac{1}{2}e^{i\pi/3}\left(\frac{3x}{\kappa}\right)^{2/3}\right\}, \tag{3.8}$$

where x is the first zero of $J_{1/3}+J_{-1/3}$; numerically, $x=2.3834466$. Hence the asymptotic expansion of $f(\theta)$ is of the form

$$-\sqrt{3}\pi e^{i\pi/3}\frac{J_{1/3}(x)}{J_{2/3}(x)-J_{-2/3}(x)}\left(\frac{3x}{\kappa}\right)^{-1/3} \times\exp[-\tfrac{1}{4}3^{5/6}x^{2/3}\kappa^{1/3}(\theta-\pi)] \times\exp\{i\kappa(\theta-\pi)[1+\tfrac{1}{4}(3x/\kappa)^{2/3}]\} \times\{1+\text{power series in } \kappa^{-1/3}\}. \tag{3.9}$$

The existence of the exponentially decaying factor in this formula is the basic reason why the concept of creeping waves is particularly useful at high frequencies. From geometrical optics, it is known that $\kappa=O(\sum f(\pi))$ as $\kappa\rightarrow\infty$. Hence it can be shown that, in the sense of Poincaré, the functions $\sum f(\pi)$ and $2f(\pi)$ must have exactly the same asymptotic series in inverse powers of κ as $\kappa\rightarrow\infty$. This observation is most important in the determination of the asymptotic series of the scattering cross section. Physically, it means that, for a circular cylinder in the high-frequency limit, the scattering cross section is determined entirely by the grazing rays in the terminology of geometrical optics; all of the more complicated creeping waves around the cylinder contribute nothing. It may be conjectured that this is true for any impenetrable convex obstacle provided its boundary curve is infinitely differentiable.

The remaining part of this section is devoted to the determination of the asymptotic expansion for $f(\pi)$. From (3.6),

$$f(\pi)=\frac{\nu_0}{2}+\frac{1}{2}\int_{-\infty}^{\nu_0}\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}d\nu+\int_{\nu_0}^{\infty}\frac{J_\nu(\kappa)}{H_\nu^{(1)}(\kappa)}d\nu. \tag{3.10}$$

This is true for any ν_0, which will be chosen later. For the asymptotic evaluation of (3.10), the formulas of Cherry[9] may be used. Without worrying about routine rigor, it can be seen that $f(\pi)$ may be obtained correctly with appropriate order of magnitude for the error, when the Cherry formulas are directly used in (3.10). For this purpose, it is clear that the variable of integration should be changed from ν to the new variable $v=\nu^{2/3}\zeta$, using Cherry's notation. It is thus necessary to express ν in a series involving v. The first six terms in the expansion for ν has been found, the leading term being κ. The number six is a compromise, for, on the one hand, the required amount of labor increases rapidly with the number of terms, and on the other hand, six terms seem to be sufficient for all reasonable purposes at the present time. The main idea is to perform a perturbation calculation on the series for ζ. To get six terms, one perturbation is enough. The result is

$$\frac{\nu}{\kappa}=1+\frac{1}{2}v\left(\frac{\kappa}{2}\right)^{-2/3}+\frac{1}{120}v^2\left(\frac{\kappa}{2}\right)^{-4/3}-\left(\frac{1}{2800}v^3+\frac{1}{280}\right)\left(\frac{\kappa}{2}\right)^{-2}+\left(\frac{281}{9072000}v^4+\frac{29}{25200}v\right)\left(\frac{\kappa}{2}\right)^{-8/3} -\left(\frac{73769}{20956320000}v^5+\frac{7361}{23284800}v^2\right)\left(\frac{\kappa}{2}\right)^{-10/3}+\cdots. \tag{3.11}$$

This can be used in (3.10), with ν_0 chosen such that $v(\nu_0)=0$, or

$$\nu_0=\kappa[1-(1/70)\kappa^{-2}+\cdots]. \tag{3.12}$$

Consequently, $f(\pi)$ is given by

$$f(\pi)=\frac{1}{2}\left\{\nu_0+\int_{-\infty}^{0}e^{i2\pi/3}\frac{A_i(e^{-i2\pi/3}v)}{A_i(e^{i2\pi/3}v)}\frac{d\nu}{dv}dv-\int_0^{\infty}e^{-i2\pi/3}\frac{A_i(v)}{A_i(e^{i2\pi/3}v)}\frac{d\nu}{dv}dv\right\}\{1+O(\kappa^{-4})\}, \tag{3.13}$$

where $d\nu/dv$ is a polynomial in v. If

$$M_n=\int_{-\infty}^{0}e^{i2\pi/3}\frac{A_i(e^{-i2\pi/3}v)}{A_i(e^{i2\pi/3}v)}v^n dv-\int_0^{\infty}e^{-i2\pi/3}\frac{A_i(v)}{A_i(e^{i2\pi/3}v)}v^n dv, \tag{3.14}$$

then from (3.11–13)

$$\frac{2f(\pi)}{\kappa}=1+\frac{1}{2}M_0\left(\frac{\kappa}{2}\right)^{-2/3}+\frac{1}{60}M_1\left(\frac{\kappa}{2}\right)^{-4/3}-\left(\frac{3}{2800}M_2+\frac{1}{280}\right)\left(\frac{\kappa}{2}\right)^{-2} +\left(\frac{281}{2268000}M_3+\frac{29}{25200}M_0\right)\left(\frac{\kappa}{2}\right)^{-8/3}-\left(\frac{73769}{4191264000}M_4+\frac{7361}{11642400}M_1\right)\left(\frac{\kappa}{2}\right)^{-10/3}+\cdots. \tag{3.15}$$

This gives the desired answer when the electric field is parallel to the axis of the cylinder.

[9] T. M. Cherry, Trans. Am. Math. Soc. 68, 224 (1950).

4. ALTERNATIVE PROCEDURE AND THE SOLUTION FOR THE OTHER POLARIZATION

Although the procedure presented in the last section is a natural one, it has the serious limitation of depending critically on Cherry's formula. If the incident field is polarized such that the magnetic field vector is parallel to the axis of the cylinder, it is necessary to have a corresponding formula for the first derivative of the Bessel function in order to carry through the analogous calculation. Such a formula probably does not exist, since the differential equation for $Z_\nu'(\kappa)$ has an apparent singularity[10] at $\kappa=\nu$. In any case, it is very desirable to have an alternative approach and for this purpose the formulas of Schöbe[11] may be used. With the formulas of Schöbe, the investigation of the case of the other polarization can be made along the same line, although it is somewhat more complicated. On the other hand, there is some uncertainty about the region in the ν,κ plane where the Schöbe formulas are valid. The check of the answers in the first case shows the validity of the procedure, although it may be necessary to use the uniform asymptotic series of Olver[12] in the second case in order to be mathematically rigorous.

To apply the Schöbe formula, integrals of the following kind have to be evaluated:

$$\int_0^{\pm\infty}\frac{P(\xi)\rho_1(\xi)+Q(\xi)\rho_1'(\xi)}{P(\xi)\rho_2(\xi)+Q(\xi)\rho_2'(\xi)}d\xi,$$

where ρ_1 and ρ_2 are two solutions of the Airy differential equation

$$(d^2/d\xi^2-\xi)\rho_1(\xi)=0, \tag{4.1}$$

$\rho'(\xi)=(d/d\xi)\rho(\xi)$, and $Q(\xi)$ is assumed to be small compared with $P(\xi)$ in some sense. Let

$$R(\xi)=Q(\xi)/P(\xi), \tag{4.2}$$

and

$$\rho(\xi)=\rho_1(\xi)/\rho_2(\xi); \tag{4.3}$$

then the integral under consideration is

$$\int_0^{\pm\infty}\rho(\xi)d\xi+\int_0^{\pm\infty}\rho(\xi)R(\xi)\frac{\rho_1'(\xi)/\rho_1(\xi)-\rho_2'(\xi)/\rho_2(\xi)}{1+R(\xi)\rho_2'(\xi)/\rho_2(\xi)}d\xi.$$

The second integral here may be written as

$$\int_0^{\pm\infty}\left[\left(R+\frac{1}{3}R^3\xi+\frac{1}{12}R^4\right)+\left(\frac{1}{2}R^2+\frac{1}{3}R^4\xi\right)\frac{d}{d\xi}+\frac{1}{6}R^3\frac{d^2}{d\xi^2}+\frac{1}{24}R^4\frac{d^3}{d\xi^3}+\cdots\right]\rho' d\xi.$$

If $P(\xi)$ and $Q(\xi)$ are the series given by Schöbe, then a direct algebraic calculation together with an integration by parts yields the following result:

$$\begin{aligned}\int_0^{\pm\infty}&\frac{P(\xi)\rho_1(\xi)+Q(\xi)\rho_1'(\xi)}{P(\xi)\rho_2(\xi)+Q(\xi)\rho_2'(\xi)}d\xi\\ &=\int_0^{\pm\infty}\rho(\xi)\left[1+\frac{1}{30}\xi\left(\frac{\kappa}{2}\right)^{-2/3}-\frac{3}{1400}\xi^2\left(\frac{\kappa}{2}\right)^{-4/3}\right.\\ &\quad+\left(\frac{281}{1134000}\xi^3+\frac{29}{12600}\right)\left(\frac{\kappa}{2}\right)^{-2}\\ &\quad\left.-\left(\frac{73769}{2095632000}\xi^4+\frac{7361}{5821200}\xi\right)\left(\frac{\kappa}{2}\right)^{-8/3}+\cdots\right]d\xi\\ &\quad-\frac{1}{140}\left(\frac{\kappa}{2}\right)^{-2}\rho(0)-\frac{1}{39200}\rho'(0).\end{aligned} \tag{4.4}$$

Equation (3.15) then follows directly from (3.10), the formulas of Schöbe, and (4.4), with the terms involving $\rho'(0)$ cancelled out.

Attention is now directed to the problem of the other polarization, namely, the determination of the asymptotic expansion for the scattering cross section of a perfectly conducting cylinder with the incident plane wave polarized such that the magnetic field vector is parallel to the axis of the cylinder. Let a bar be used to designate quantities pertaining to this problem as distinguished from the corresponding quantities for the problem in the last section. Thus, for example, $\bar{\psi}^{\text{inc}}(r,\theta)=\psi^{\text{inc}}(r,\theta)$. Since the boundary condition is $(\partial/\partial r)\bar{\psi}^{\text{tot}}(a,\theta)=0$, it corresponds to (3.3) that

$$f(\theta)=f_1(\theta)+2\int_0^{\infty}\frac{J_\nu'(ka)}{H_\nu^{(1)\prime}(ka)}e^{-i\nu\pi}\cos\nu\theta d\nu. \tag{4.5}$$

In particular, analogously to (3.6),

$$f(\pi)=\frac{1}{2}\lim_{A\to\infty}\left[A+\int_{-\infty}^{A}\frac{H^{(2)\prime}(\kappa)}{H^{(1)\prime}(\kappa)}d\nu\right]. \tag{4.6}$$

By reasoning completely analogous to that in the beginning of Sec. 3, it is found that, in the sense of Poincaré, the functions $\sum\bar{f}(\pi)$ and $2\bar{f}(\pi)$ have exactly the same asymptotic expansion in inverse powers of κ as $\kappa\to\infty$.

The determination of the asymptotic series for $\bar{f}(\pi)$ is even more laborious than for $f(\pi)$. When Schöbe's formula is used an evaluation of the following kind of integral is necessary:

$$\int_0^{\pm\infty}\frac{\bar{P}(\xi)\rho_1(\xi)+\bar{Q}(\xi)\rho_1'(\xi)}{\bar{P}(\xi)\rho_2(\xi)+\bar{Q}(\xi)\rho_2'(\xi)}d\xi,$$

where $\bar{P}(\xi)$ and $\bar{Q}(\xi)$ are again the series given by Schöbe in connection with the formulas for $Z_\nu'(\kappa)$. This is formally the same as the previous expression, except

[10] See, for example, E. L. Ince, *Ordinary Differential Equations* (Longmans Green and Company, London, 1927), p. 406.
[11] W. Schöbe, Acta Math. **92**, 265 (1954).
[12] F. W. J. Olver, Trans. Roy. Soc. (London) **A247**, 328 (1954).

that $\bar{P}(\xi)$ is assumed to be small compared with $\bar{Q}(\xi)$. Thus let

$$\bar{\rho}(\xi)=\rho_1'(\xi)/\rho_2'(\xi)\,; \tag{4.7}$$

then a similar calculation gives, with the lower limit of integration replaced by a small positive ϵ to avoid divergence,

$$\begin{aligned}\int_\epsilon^{\pm\infty}\frac{\bar{P}(\xi)\rho_1(\xi)+\bar{Q}(\xi)\rho_1'(\xi)}{\bar{P}(\xi)\rho_2(\xi)+\bar{Q}(\xi)\rho_2'(\xi)}d\xi=&\int_\epsilon^{\pm\infty}\bar{\rho}(\xi)d\xi+\int_\epsilon^{\pm\infty}\Bigg[-\left(\frac{1}{60}\xi^3-\frac{1}{10}\right)\left(\frac{\kappa}{2}\right)^{-2/3}+\left(\frac{1}{1400}\xi^4-\frac{1}{50}\xi+\frac{1}{200}\xi^{-2}\right)\left(\frac{\kappa}{2}\right)^{-4/3}\\ &-\left(\frac{281}{4536000}\xi^5-\frac{611}{126000}\xi^2-\frac{1}{3000}\xi^{-1}-\frac{1}{2000}\xi^{-4}\right)\left(\frac{\kappa}{2}\right)^{-2}+\Bigg(\frac{73769}{10478160000}\xi^6-\frac{56299}{24948000}\xi^3-\frac{1679}{1108800}\\ &+\frac{7}{120000}\xi^{-3}+\frac{1}{16000}\xi^{-6}\Bigg)\left(\frac{\kappa}{2}\right)^{-8/3}+\cdots\Bigg]\frac{1}{\xi}\bar{\rho}'(\xi)d\xi-\Bigg[\frac{1}{200}\epsilon^{-1}\left(\frac{\kappa}{2}\right)^{-4/3}-\left(\frac{9}{4000}-\frac{1}{6000}\epsilon^{-3}\right)\left(\frac{\kappa}{2}\right)^{-2}\\ &+\left(\frac{1}{30000}\epsilon^{-2}+\frac{1}{80000}\epsilon^{-5}\right)\left(\frac{\kappa}{2}\right)^{-8/3}\Bigg]\frac{\bar{\rho}'(\epsilon)}{\epsilon}-\left[\frac{1}{6000}\epsilon^{-2}\left(\frac{\kappa}{2}\right)^{-2}-\frac{13}{120000}\epsilon^{-1}\left(\frac{\kappa}{2}\right)^{-8/3}\right]\left(\frac{\rho'(\xi)}{\xi}\right)'\Bigg|_{\xi=\epsilon}\\ &+\left(\frac{1}{360000}-\frac{1}{240000}\epsilon^{-3}\right)\left(\frac{\kappa}{2}\right)^{-8/3}\left(\frac{\bar{\rho}'(\xi)}{\xi}\right)''\Bigg|_{\xi=\epsilon}+\cdots.\end{aligned}\tag{4.8}$$

The long integral is not further integrated by parts. The reason will be clear later. If (4.8) is used to evaluate (4.6), it may be seen that for $\xi<0$, $\bar{\rho}(\xi)$ has the form

$$e^{i2\pi/3}\frac{d}{d\xi}[\mathrm{Ai}(e^{-i2\pi/3}\xi)]\Big/\frac{d}{d\xi}[\mathrm{Ai}(e^{i2\pi/3}\xi)],$$

and for $\xi>0$, it has the form

$$-e^{-i2\pi/3}\frac{d}{d\xi}[\mathrm{Ai}(\xi)]\Big/\frac{d}{d\xi}[\mathrm{Ai}(e^{i2\pi/3}\xi)].$$

Hence $\bar{\rho}'(\xi)$ has the same form for $\xi<0$ and $\xi>0$. Because of the complexity of (4.8), let the symbol $\lim_{\epsilon\to 0}F$ be introduced to mean the finite part in the limit $\epsilon\to 0$ in the sense of a Laurent series expansion. For example,

$$\lim_{\epsilon\to 0}F\left(\frac{1}{\epsilon\sin\epsilon}\right)=\frac{1}{6}.$$

As an analog to (3.14), let

$$\bar{M}_{n-1}=\frac{1}{n}e^{-i2\pi/3}\int_{-\infty}^{\infty}\xi^n\frac{d}{d\xi}\left[\frac{\frac{d}{d\xi}[\mathrm{Ai}(\xi)]}{\frac{d}{d\xi}[\mathrm{Ai}(e^{i2\pi/3}\xi)]}\right]d\xi \quad \text{for } n>0, \tag{4.9}$$

and

$$\bar{M}_{n-1}=\lim_{\epsilon\to 0}F\left[\frac{1}{n}e^{-i2\pi/3}\left(\int_{-\infty}^{-\epsilon}+\int_\epsilon^{\infty}\right)\xi^n\frac{d}{d\xi}\left[\frac{\frac{d}{d\xi}[\mathrm{Ai}(\xi)]}{\frac{d}{d\xi}[\mathrm{Ai}(e^{i2\pi/3}\xi)]}\right]d\xi\right] \quad \text{for } n<0. \tag{4.10}$$

Note that in particular

$$\bar{M}_0=\int_{-\infty}^{0}e^{i2\pi/3}\frac{\frac{d}{d\xi}[\mathrm{Ai}(e^{-i2\pi/3}\xi)]}{\frac{d}{d\xi}[\mathrm{Ai}(e^{i2\pi/3}\xi)]}d\xi-\int_0^{\infty}e^{-i2\pi/3}\frac{\frac{d}{d\xi}[\mathrm{Ai}(\xi)]}{\frac{d}{d\xi}[\mathrm{Ai}(e^{i2\pi/3}\xi)]}d\xi. \tag{4.11}$$

Finally, substitution of (4.8)–(4.11) in (4.6) gives

$$\begin{aligned}\frac{2\bar{f}(\pi)}{\kappa}=&1+\frac{1}{2}\bar{M}_0\left(\frac{\kappa}{2}\right)^{-2/3}+\left(\frac{1}{60}\bar{M}_1+\frac{1}{20}\bar{M}_{-2}\right)\left(\frac{\kappa}{2}\right)^{-4/3}\\ &-\left(\frac{3}{2800}\bar{M}_2+\frac{1}{100}-\frac{3}{400}\bar{M}_{-4}\right)\left(\frac{\kappa}{2}\right)^{-2}+\Bigg(\frac{281}{2268000}\bar{M}_3\\ &-\frac{611}{252000}\bar{M}_0+\frac{1}{3000}\bar{M}_{-3}+\frac{1}{800}\bar{M}_{-6}\Bigg)\left(\frac{\kappa}{2}\right)^{-8/3}\\ &-\Bigg(\frac{73769}{4191264000}\bar{M}_4-\frac{56299}{24948000}\bar{M}_1+\frac{1679}{2217600}\bar{M}_{-2}\\ &-\frac{7}{60000}\bar{M}_{-5}-\frac{7}{32000}\bar{M}_{-8}\Bigg)\left(\frac{\kappa}{2}\right)^{-10/3}+\cdots.\end{aligned}\tag{4.12}$$

This is the desired answer when the magnetic field is parallel to the axis of the cylinder.

5. SCATTERING BY A SPHERE

In the last two sections, the scattering cross section of an infinitely long circular cylinder has been obtained in two cases of physical interest. It is natural, then, to try to do the same for the sphere. Unfortunately, for reasons to be given in the Appendix, the present method of solving creeping wave problems does not work satisfactorily for three-dimensional obstacles of finite size. An alternative, and admittedly less satisfactory, procedure is given in this section for the determination of the scattering cross section of a sphere. The main idea is to compare the problem of a sphere with that of a cylinder. This is another reason why the scattering cross section of a cylinder has been considered first.

There are essentially three different problems of scattering from an impenetrable sphere, depending on the nature of the incident plane wave. Of the three, the most important is the quantum-mechanical scattering of a free particle by an impenetrable spherical core. This approximates some nucleon-nucleon scattering at very high energy. Mathematically, the solution of this kind of problem is the determination of a function satisfying the reduced wave equation with the Dirichlet boundary condition $\psi(a,\theta)=0$. Secondly, there is the problem of the acoustic scattering by a rigid sphere. In this case, the velocity potential satisfies the Neumann boundary condition $(\partial/\partial r)\psi(r,\theta)=0$ for $r=a$. Thirdly, there is the problem of the scattering of a plane electromagnetic wave by a perfectly conducting sphere. These three problems are now studied in turn.

In the first problem the function of interest is

$$F_1=\sum_{n=0}^{\infty}(2n+1)\frac{j_n(\kappa)}{h_n^{(1)}(\kappa)},\tag{5.1}$$

where $\kappa=ka$ as before, and j and $h^{(1)}$ are spherical Bessel functions of the first and third kinds, respectively. The quantity F_1 is actually the forward scattering amplitude. The scattering cross section is related to F_1 by

$$\sigma_D^{(s)}=(4\pi/k^2)\,\mathrm{Re}F_1.\tag{5.2}$$

The analogous expression for the cylinder is

$$\sum_{n=0}^{\infty}\frac{\epsilon_n}{2}\frac{J_n(\kappa)}{H_n^{(1)}(\kappa)}.$$

By comparison with (3.3), it can be seen that the Poisson summation formula should be applied to (5.1) to yield

$$F_1=\sum_{n=-\infty}^{\infty}\int_0^{\infty}2\nu\frac{J_\nu(\kappa)}{H_\nu^{(1)}(\kappa)}e^{i2\pi(\nu-\frac{1}{2})n}d\nu.\tag{5.3}$$

If Abel summability is used again, then, for $n\neq 0$, let

$$F_{1,n}=\int_0^{\infty}\nu\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}e^{i2\pi(\nu-\frac{1}{2})n}d\nu.\tag{5.4}$$

Then,

$$\int_{-\infty}^{0}\nu\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}e^{i2\pi(\nu-\frac{1}{2})n}d\nu =\int_0^{\infty}\nu\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}e^{-i2\pi(\nu-\frac{1}{2})(n+1)}d\nu=F_{1,-(n+1)}.\tag{5.5}$$

Accordingly, from (5.3), insofar as the asymptotic series in the sense of Poincaré is concerned,

$$F_1\sim\int_{-\infty}^{0}\nu\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}d\nu+2\int_0^{\infty}\nu\frac{J_\nu(\kappa)}{H_\nu^{(1)}(\kappa)}d\nu +\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\int_0^{\infty}\nu e^{i2\pi(\nu-\frac{1}{2})n}d\nu.\tag{5.6}$$

The last term may be evaluated as follows:

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\int_0^{\infty}\nu e^{i2\pi(\nu-\frac{1}{2})n}d\nu=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^{n+1}\int_0^{\infty}\nu e^{-2\pi\nu n}d\nu =\sum_1^{\infty}\frac{(-1)^{n+1}}{2\pi^2n^2}=\frac{1}{24}.\tag{5.7}$$

This answer may also be obtained from $\lim[\sum(n+\frac{1}{2})-\int\nu d\nu]$, if the limiting process is understood to be in the sense of second-order Cesaro summability. Substitution of (5.7) into (5.6) yields

$$F_1\sim G_1,\tag{5.8}$$

where

$$G_1=\lim_{A\to\infty}\left[\int_{-\infty}^{A}\nu\frac{H_\nu^{(2)}(\kappa)}{H_\nu^{(1)}(\kappa)}d\nu+\frac{A^2}{2}\right]+\frac{1}{24}.\tag{5.9}$$

Equation (5.9) is very similar to (3.6).

If Cherry's formula is used to evaluate G_1, the procedure is straightforward and the result is

$$\frac{2G_1}{\kappa^2}=1+M_0\left(\frac{\kappa}{2}\right)^{-2/3}+\frac{8}{15}M_1\left(\frac{\kappa}{2}\right)^{-4/3}+\left(\frac{4}{175}M_2+\frac{23}{420}\right)\left(\frac{\kappa}{2}\right)^{-2}-\left(\frac{64}{70875}M_3+\frac{2}{1575}M_0\right)\left(\frac{\kappa}{2}\right)^{-8/3} +\left(\frac{2944}{3274425}M_4+\frac{334}{363825}M_1\right)\left(\frac{\kappa}{2}\right)^{-10/3}+\cdots.\tag{5.10}$$

This is the desired answer for the scattering cross section of an impenetrable spherical well in quantum mechanics.

The other two cases will now be treated together. Referring to (5.1), the corresponding function for the acoustic or Neumann case is

$$F_2=\sum_{n=0}^{\infty}(2n+1)\frac{j_n'(\kappa)}{h_n^{(1)\prime}(\kappa)}, \tag{5.11}$$

and that for the electromagnetic case is

$$F_3=\tfrac{1}{2}(F_1+F_{31})-1, \tag{5.12}$$

where

$$F_{31}=\sum_{n=0}^{\infty}(2n+1)\frac{\dfrac{\partial}{\partial\kappa}[\kappa j_n(\kappa)]}{\dfrac{\partial}{\partial\kappa}[\kappa h_n^{(1)}(\kappa)]}. \tag{5.13}$$

The corresponding G functions are

$$G_2=\lim_{A\to\infty}\left\{\int_{-\infty}^{A}\nu\frac{\dfrac{\partial}{\partial\kappa}[\kappa^{-\frac{1}{2}}H_\nu^{(2)}(\kappa)]}{\dfrac{\partial}{\partial\kappa}[\kappa^{-\frac{1}{2}}H_\nu^{(1)}(\kappa)]}d\nu+\frac{A^2}{2}\right\}+\frac{1}{24}, \tag{5.14}$$

and

$$G_{31}=\lim_{A\to\infty}\left\{\int_{-\infty}^{A}\nu\frac{\dfrac{\partial}{\partial\kappa}[\kappa^{\frac{1}{2}}H_\nu^{(2)}(\kappa)]}{\dfrac{\partial}{\partial\kappa}[\kappa^{\frac{1}{2}}H_\nu^{(1)}(\kappa)]}d\nu+\frac{A^2}{2}\right\}+\frac{1}{24}. \tag{5.15}$$

By comparison with the case of the circular cylinder, let G_0 be defined by

$$G_0=\lim_{A\to\infty}\left[\int_{-\infty}^{A}\nu\frac{H_\nu^{(2)\prime}(\kappa)}{H_\nu^{(1)\prime}(\kappa)}d\nu+\frac{A^2}{2}\right]+\frac{1}{24}. \tag{5.16}$$

Within the accuracy required,

$$\frac{\dfrac{\partial}{\partial\kappa}[\kappa^{-\frac{1}{2}}H_\nu^{(2)}(\kappa)]}{\dfrac{\partial}{\partial\kappa}[\kappa^{-\frac{1}{2}}H_\nu^{(1)}(\kappa)]}-\frac{\dfrac{\partial}{\partial\kappa}[\kappa^{\frac{1}{2}}H_\nu^{(2)}(\kappa)]}{\dfrac{\partial}{\partial\kappa}[\kappa^{\frac{1}{2}}H_\nu^{(1)}(\kappa)]}\sim-\frac{4i}{\pi}\kappa^{-2}\frac{1}{[H_\nu^{(1)\prime}(\kappa)]^2}\left\{1+\frac{1}{4}\kappa^{-2}\left[\frac{H_\nu^{(1)}(\kappa)}{H_\nu^{(1)\prime}(\kappa)}\right]^2\right\}, \tag{5.17}$$

and

$$\frac{\dfrac{\partial}{\partial\kappa}[\kappa^{-\frac{1}{2}}H_\nu^{(2)}(\kappa)]}{\dfrac{\partial}{\partial\kappa}[\kappa^{-\frac{1}{2}}H_\nu^{(1)}(\kappa)]}+\frac{\dfrac{\partial}{\partial\kappa}[\kappa^{\frac{1}{2}}H_\nu^{(2)}(\kappa)]}{\dfrac{\partial}{\partial\kappa}[\kappa^{\frac{1}{2}}H_\nu^{(1)}(\kappa)]}-2\frac{\dfrac{\partial}{\partial\kappa}H_\nu^{(2)}(\kappa)}{\dfrac{\partial}{\partial\kappa}H_\nu^{(1)}(\kappa)}\sim-\frac{2i}{\pi}\kappa^{-3}\frac{H_\nu^{(1)}(\kappa)}{[H_\nu^{(1)\prime}(\kappa)]^3}\left\{1+\frac{1}{4}\kappa^{-2}\left[\frac{H_\nu^{(1)}(\kappa)}{H_\nu^{(1)\prime}(\kappa)}\right]^2\right\}. \tag{5.18}$$

It is now required to consider integrals of the following kinds:

$$I_n=\int_0^{\pm\infty}\left[\kappa+\xi\left(\frac{\kappa}{2}\right)^{\frac{1}{3}}\right]\frac{1}{[\bar{P}(\xi)\rho_2(\xi)+\bar{Q}(\xi)\rho_2'(\xi)]^2}\left[\frac{P(\xi)\rho_2(\xi)+Q(\xi)\rho_2'(\xi)}{\bar{P}(\xi)\rho_2(\xi)+\bar{Q}(\xi)\rho_2'(\xi)}\right]^n\left(\frac{\kappa}{2}\right)^{\frac{1}{3}}d\xi, \tag{5.19}$$

with $0\leqslant n\leqslant 3$. Integrals of this type may be evaluated by the procedure given in the last section. A few more pages of calculation yield the following results:

$$\frac{2G_0}{\kappa^2}=-1+\frac{4\bar{f}(\pi)}{\kappa}+\bar{M}_1\left(\frac{\kappa}{2}\right)^{-4/3}+\left(\frac{1}{20}\bar{M}_2+\frac{29}{240}\right)\left(\frac{\kappa}{2}\right)^{-2}-\left(\frac{29}{12600}\bar{M}_3-\frac{11}{600}\bar{M}_0\right)\left(\frac{\kappa}{2}\right)^{-8/3}$$
$$+\left(\frac{227}{907200}\bar{M}_4-\frac{2}{225}\bar{M}_1+\frac{29}{12000}\bar{M}_{-2}\right)\left(\frac{\kappa}{2}\right)^{-10/3}+\cdots, \tag{5.20}$$

$$\frac{G_2-G_{31}}{\kappa^2}=\frac{1}{4}\bar{M}_{-2}\left(\frac{\kappa}{2}\right)^{-4/3}+\left(\frac{1}{10}+\frac{3}{40}\bar{M}_{-4}\right)\left(\frac{\kappa}{2}\right)^{-2}+\left(\frac{2}{215}\bar{M}_0+\frac{49}{2400}\bar{M}_{-3}+\frac{37}{640}\bar{M}_{-6}\right)\left(\frac{\kappa}{2}\right)^{-8/3}$$
$$+\left(-\frac{4}{2025}\bar{M}_1+\frac{283}{252000}\bar{M}_{-2}+\frac{259}{12000}\bar{M}_{-5}+\frac{203}{6400}\bar{M}_{-8}\right)\left(\frac{\kappa}{2}\right)^{-10/3}+\cdots, \tag{5.21}$$

and

$$\frac{G_2+G_{31}-2G_0}{\kappa^2}=-\frac{3}{32}\bar{M}_{-4}\left(\frac{\kappa}{2}\right)^{-2}-\left(\frac{1}{80}\bar{M}_{-3}-\frac{3}{64}\bar{M}_{-6}\right)\left(\frac{\kappa}{2}\right)^{-8/3}$$
$$-\left(\frac{19}{9600}\bar{M}_1-\frac{1067}{806400}\bar{M}_{-2}+\frac{367}{19200}\bar{M}_{-5}+\frac{343}{10240}\bar{M}_{-8}\right)\left(\frac{\kappa}{2}\right)^{-10/3}+\cdots. \quad (5.22)$$

These formulas give all the desired information about the scattering cross sections for the acoustic scattering by a rigid sphere and the electromagnetic scattering by a perfectly conducting sphere.

6. CONCLUSION AND DISCUSSION

With the procedure outlined, it is only necessary to evaluate the integrals M_n and $\bar{M}_n$, in order to get the desired numerical answers for the scattering cross sections. The main idea of the evaluation is to convert the integrals into residue series. It is rather complex to carry out the conversion because of the necessity of interpreting the integrals in a special manner. After the numerical summation of the residue series, the results are

$$M_0=1.25507437e^{i\pi/3}, \qquad M_1=0.53225036e^{i2\pi/3},$$
$$M_2=0.0935216, \qquad M_3=0.772793e^{i\pi/3},$$
$$M_4=1.0992e^{i2\pi/3}, \qquad \bar{M}_0=-1.088874119e^{i\pi/3},$$
$$\bar{M}_1=-0.93486491e^{i2\pi/3}, \qquad \bar{M}_2=-0.1070199,$$
$$\bar{M}_3=-0.757663e^{i\pi/3}, \qquad \bar{M}_4=-1.1574e^{i2\pi/3},$$
$$\bar{M}_{-2}=-3.70409389e^{-i\pi/3}, \qquad \bar{M}_{-3}=0.41682138e^{-i2\pi/3},$$
$$\bar{M}_{-4}=3.17579652, \qquad \bar{M}_{-5}=2.55965945+3.12247506e^{-i\pi/3},$$
$$\bar{M}_{-6}=2.06575721e^{-i2\pi/3}, \qquad \bar{M}_{-8}=-1.36515171-2.94764528e^{-i\pi/3}.$$

The expressions for the total scattering cross sections are then

$$\sigma_D^{(c)}/4a=1+0.49807659(ka)^{-2/3}-0.01117656(ka)^{-4/3}-0.01468652(ka)^{-2}$$
$$+0.00488945(ka)^{-8/3}+0.00179345(ka)^{-10/3}+\cdots,$$
$$\sigma_N^{(c)}/4a=1-0.43211998(ka)^{-2/3}-0.21371236(ka)^{-4/3}+0.05573255(ka)^{-2}$$
$$-0.00055534(ka)^{-8/3}+0.02324932(ka)^{-10/3}+\cdots,$$
$$\sigma_D^{(s)}/2\pi a^2=1+0.99615319(ka)^{-2/3}-0.35764983(ka)^{-4/3}+0.2275982(ka)^{-2}$$
$$-0.0072753(ka)^{-8/3}-0.007443(ka)^{-10/3}+\cdots,$$
$$\sigma_N^{(s)}/2\pi a^2=1-0.86423996(ka)^{-2/3}-0.4162852(ka)^{-4/3}+0.7352097(ka)^{-2}$$
$$-0.0298539(ka)^{-8/3}+0.058616(ka)^{-10/3}+\cdots,$$
$$\sigma_E^{(s)}/2\pi a^2=1+0.06595661(ka)^{-2/3}+0.7797489(ka)^{-4/3}-2.8713350(ka)^{-2}$$
$$-0.3385447(ka)^{-8/3}+0.058460(ka)^{-10/3}+\cdots.$$

These series are asymptotic expressions for $ka\to\infty$. In each of the numbers given above, the last place may be inaccurate. The values of the Airy integrals are taken from a table by Miller.[13]

Since exact computations exist only for intermediate values of ka, a comparison is made for $1\leqslant ka\leqslant 20$. This is given in Fig. 3 for the first four of the above cases. For $ka\leqslant 10$, the exact curves are computed from known values of the phase shifts.[14] Additional points for $ka=15$, 20 are obtained by direct summation of the series expansions. The agreement is excellent as shown in Fig. 3. The exact curve for the fifth case has not been computed.

Perhaps a most interesting fact is that, for a circular cylinder, the contribution to the asymptotic expansion for the forward field comes entirely from the grazing rays. In a talk presented at the URSI Symposium, Michigan, 1955, J. B. Keller has conjectured this to be true for the first term of the asymptotic expansion. If this is considered to be a general property of high-frequency scattering phenomena, then the following conjecture may be stated. Consider the two-dimensional scattering problem of a scalar wave by an obstacle with either Dirichlet or Neumann boundary condition.

[13] J. C. P. Miller, *British Association Tables* (Cambridge University Press, Cambridge, 1946), Part-Volume B.

[14] A. N. Lowan *et al.*, National Defense Research Council Report NDRC 62. 1R, 1945 (unpublished).

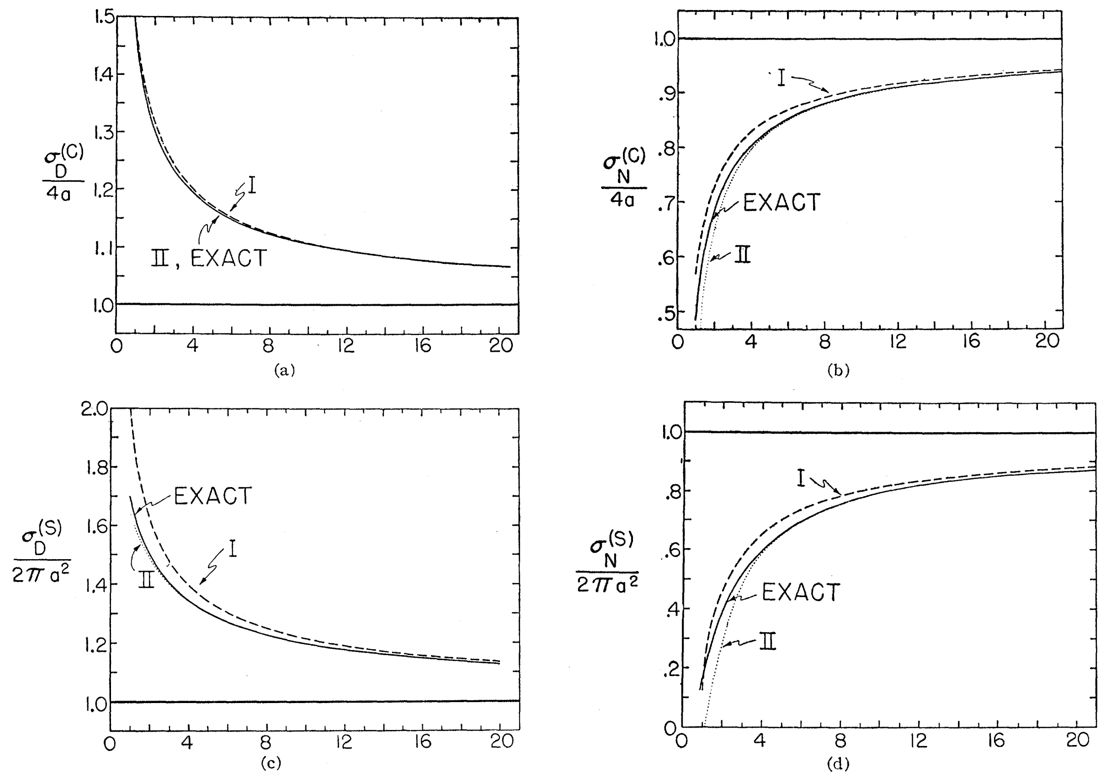

FIG. 3. Comparison of scattering cross sections. (a) The scattering cross section of a circular cylinder with Dirichlet boundary condition; (b) the scattering cross section of a circular cylinder with Neumann boundary condition; (c) the scattering cross section of a sphere with Dirichlet boundary condition; (d) the scattering cross section of a sphere with Neumann boundary condition. In these diagrams, curve I gives the result obtained by geometrical optics with one correction term, and curve II gives the result obtained by geometrical optics with two correction terms.

If the obstacle is strictly convex and the boundary curve is infinitely differentiable, then

$$\sigma(\lambda)=\sigma(0)+\sum_{n=0}^{\infty} a_n\lambda^{(2+n)/3},$$

where λ is the wavelength. Furthermore, a_n depends only on the first $n+2$ derivatives of the boundary curve at the shadow boundary points. In particular

$$a_0=C(r_1^{\frac{1}{3}}+r_2^{\frac{1}{3}}),$$

where r_1 and r_2 are the radii of curvature at the two shadow boundary points, and C is to be determined by comparison with the case of the circular cylinder If the obstacle is symmetrical about the shadow boundary points, $a_n=0$ for all odd n.

As an application, consider the case of an elliptic cylinder as shown in Fig. 4. Then the foregoing conjecture leads to

$$\sigma_D{}^e/4b=1+0.498076595(kb^2/a)^{-\frac{2}{3}}+\cdots,$$

and

$$\sigma_N{}^e/4b=1-0.43211998(kb^2/a)^{-\frac{2}{3}}+\cdots,$$

for all positive a and b. Similar statements may be made for three-dimensional problems with an axis of symmetry.

Finally, the possible development from procedures of this type may be anticipated. For the case of a circular cylinder or sphere, the machinery as developed here is in no way restricted to the computation of the total cross section. For example, it may be used to study the structure of the creeping wave at finite points. Or else it may be used to obtain the asymptotic expansion of the current near the shadow boundary, in the cases of electromagnetic scattering. For the cylinder, the result is as follows. The first approximation to the

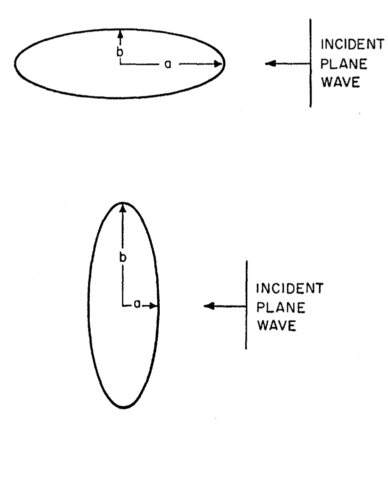

FIG. 4. Scattering by an elliptic cylinder.

current is the Fock current,[15] and higher order approximations are furnished by appropriate combinations of the derivatives and integrals of the Fock current. When the near-zone calculation is carried out, it is found that the general structure is rather independent of the fact that the obstacle is of the particularly simple shape. At least for convex obstacles with analytic surfaces and bounded radii of curvature, this yields an extension of the eiconal solution[16] into the shadow region. This kind of consideration may also decide the validity of the above conjecture about the scattering cross section.

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APPENDIX. CONCEPT OF CREEPING WAVES

In this Appendix, the relation between creeping waves and the spaces E and R is discussed for rather general situations. In Fig. 5 are drawn three curves

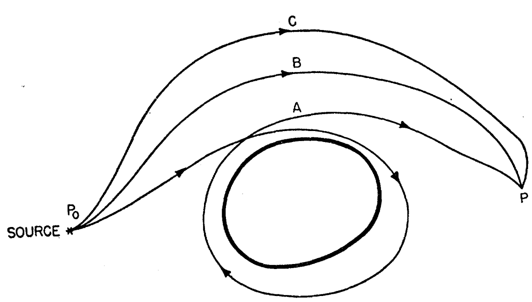

FIG. 5. Wave paths.

connecting the points P_0 and P. Each curve remains in E. Loosely, a wave originating at P_0 may arrive at P through any of these curves. In the intuitive sense, and also suggested by the work of Friedlander, it is expected that it is impossible to distinguish between waves arriving at P through curve B or curve C, but the wave through the curve A is of a different nature. This intuitive feeling is the basis for the concept of creeping waves, and may be restated as follows. In the scattering by impenetrable obstacles, the path of the wave should be regarded as the homotopy class of curves joining the initial point and the end point. Topological concepts are very useful in this discussion, although they are avoided in the main text.

Next, how is this idea of a wave path related to the space R? To answer this question, it is necessary to give a brief definition of the universal covering space. Let E be a connected, locally arcwise connected, metric space. Let P_0 be a point in E. Define the metric space F to be the class of all continuous functions on the closed unit interval $[0,1]$ into E such that 0 goes over to P_0. The metric is given by

$$\bar{\rho}(f_1,f_2) = \underset{0\leq t\leq 1}{\text{l.u.b.}}\ \rho(f_1(t),f_2(t)),$$

where ρ is the metric in E. Let R be the set of all arc components of sets of functions satisfying $f(1)=P$ with P in E. Then there are unique mappings M from F to R and N from R to E such that f is in $M(f)$ and $MNf=f(1)$. The space R has a natural topology, and in fact has a local metric structure similar to that of E because every little disk D in E has a one-to-one correspondence with any component of $N^{-1}(D)$. The space R is here called the universal covering space of E. With this definition, the correlation with the physical situation is simple. The space F is just the class of all continuous curves (parametrized) in the physical space E, and the space R is just the set of all wave paths mentioned before. Since the various creeping waves must satisfy the same field equations as the total field, they may be found by solving the field equations in R. This makes sense when the field equations are differential equations (of finite order), because R has the same local metric structure as E. In the simple two-dimensional problem where the obstacle is a point, E is the punched plane, and R is the familiar Riemann surface associated with the punched plane.

It should be noted that this idea works only in the case of two-dimensional problems. In the case of three-dimensional problems, the space E is in most cases simply connected, and hence R is homeomorphic to E. Yet somehow from intuition, there should still be a nontrivial creeping wave structure. There is still a certain amount of work to be done in this direction.

[15] V. A. Fock, J. Phys. (U.S.S.R.) **10**, 130 (1946).
[16] Keller, Lewis, and Seckler, New York University Report EM-81, 1955 (unpublished).