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Pion-Nucleon Coupling Constant and Scattering Phase Shifts*

URI HABER-SCHAIM†
Department of Physics, University of Illinois, Urbana, Illinois

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Goldberger's relations for the forward scattering of pions are used in the following way. Two linear functions of ν^2, where ν is the total pion energy in the laboratory system, are constructed from quantities taken from experiment, i.e., forward amplitudes and integrals over total cross sections. The extrapolation of one of these functions to $\nu=0$ gives $2f^2$, where f is the renormalized pion-nucleon coupling constant. Various sets of phase shifts are compared as to their compatibility with the above functions.

I. DISPERSION RELATIONS

In a recent paper Goldberger, Miyazawa, and Oehme[1] have written down dispersion relations for the forward amplitude for pion-nucleon scattering. They split the forward scattering amplitude of a pion from an isotopic spin state β to an isotopic spin state α into two parts, corresponding to no isotopic spin flip and isotopic spin flip, respectively:

$$T_{\alpha\beta}(\nu)=\delta_{\alpha\beta}T^{(1)}(\nu)+\tfrac{1}{2}[\tau_\alpha\tau_\beta]T^{(2)}(\nu), \tag{1}$$

where ν is the total energy in the laboratory system. $T^{(2)}(\nu)$, which is the amplitude we shall discuss first, can be expressed in terms of the coherent π^- and π^+ scattering amplitudes or the isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ amplitudes.

$$T^{(2)}(\nu)=\tfrac{1}{2}(T_-(\nu)-T_+(\nu))=\tfrac{1}{3}(T^{\frac{1}{2}}(\nu)-T^{\frac{3}{2}}(\nu)). \tag{2}$$

Using the relation between the imaginary part of the coherent scattering amplitude and the total cross section we then find the following equation:

$$\mathrm{Re}T^{(2)}(\nu)=\frac{2f^2}{\nu^2-\nu_B{}^2}+\frac{\nu}{2\pi^2}\times\int_\mu^\infty\left(\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'^2-\nu^2}. \tag{3}$$

f is the renormalized pion nucleon coupling constant; $\nu_B=\mu^2/2M$ where μ and M are the pion and nucleon masses, respectively. σ_- and σ_+ are the total cross sections for negative (positive) pions on protons, and $q=(\nu^2-\mu^2)^{\frac{1}{2}}$. All quantities are in the laboratory system. Now we make use under the integral of the following identity:

$$\frac{1}{\nu'^2-\nu^2}=\frac{1}{\nu'^2}+\frac{\nu^2}{\nu'^2(\nu'^2-\nu^2)},$$

and multiply both sides by $(\nu^2-\nu_B{}^2)/\nu$ to obtain

$$\left(\frac{\nu^2-\nu_B{}^2}{\nu}\right)\mathrm{Re}T^{(2)}(\nu)-\frac{\nu^2(\nu^2-\nu_B{}^2)}{2\pi^2}\times\int_\mu^\infty\left(\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'^2(\nu'^2-\nu^2)}$$
$$=2f^2+\frac{\nu^2-\nu_B{}^2}{2\pi^2}\int_\mu^\infty\left(\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'^2}. \tag{4}$$

Since $\nu_B{}^2=0.55\times10^{-2}\mu^2$, we may neglect it and obtain a simplified expression:

$$\nu\,\mathrm{Re}T^{(2)}(\nu)-\frac{\nu^4}{2\pi^2}\int_\mu^\infty\left(\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'^2(\nu'^2-\nu^2)}$$
$$=2f^2+\frac{\nu^2}{2\pi^2}\int_\mu^\infty\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\frac{q'd\nu'}{\nu'^2}. \tag{4'}$$

From (2) one finds for $\mathrm{Re}T^{(2)}(\nu)$:

$$\mathrm{Re}T^{(2)}(\nu)=\frac{1}{6q}\left(1+\frac{2\nu}{M}+\frac{\mu^2}{M^2}\right)\times[\sin2\alpha_1+\sin2\alpha_{11}+2\sin2\alpha_{13}+\cdots-\sin2\alpha_3-\sin2\alpha_{31}-2\sin2\alpha_{33}-\cdots]. \tag{5}$$

The α's are the phase shifts in their usual notation.

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† Present address: Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts.

[1] Goldberg, Miyazawa, and Oehme, Phys. Rev. 99, 986 (1955).

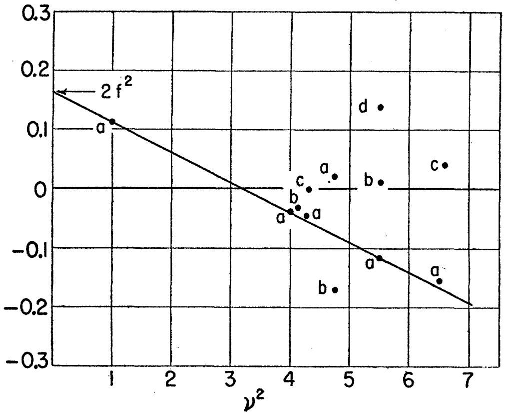

FIG. 1. The left-hand side of Eq. (4′) plotted against ν^2. The experimental points are taken from the following papers: (a) J. Orear, Phys. Rev. **100**, 288 (1955); (b) H. L. Anderson *et al.*, Phys. Rev. **100**, 268–279 (1955), 165 and 189 Mev, respectively; (c) J. Ashkin *et al.*, Phys. Rev. **101**, 1149 (1956), 150 Mev; M. O. Stern *et al.*, Bull. Am. Phys. Soc. Ser. II, **1**, 72 (1956), 220 Mev; (d) W. Rarita (private communication to Professor Bernardini), 217 Mev.

The integral on the left-hand side of (4′) is certainly rapidly convergent and should be well approximated by replacing the upper limit by 1.9 Bev. The integral on the right-hand side will converge only if

$$\mathrm{Lim}_{\nu\to\infty}[\sigma_-(\nu)-\sigma_+(\nu)]=0.$$

Let us assume that this is the case. Then the apparently complicated ν dependence indicated by the left-hand side of (4′) boils down to a linear function of ν^2. Its value at $\nu=0$ is equal to $2f^2$ and its slope is given by the integral. If we do not want to assume the convergence of the integral on the right-hand side of (4′), we may write down a dispersion relation for $T(\nu)/\nu^2$.[2] This will result in the same equation as in (4′), with the divergent integral replaced by an unknown constant. Actually, as will be seen later, the value of the coefficient of ν^2 as determined by the slope is very close to that found by performing the integration up to 1.9 Bev. This may be taken as an indication that the integral converges.[1]

II. COUPLING CONSTANT

The results of the calculation are shown in Fig. 1.[3] The energy scale is taken in units of μ^2. For the determination of the straight line the following two points were used: (1) $\nu^2=1$, i.e., zero kinetic energy. This point is practically determined by the difference between the scattering lengths, $a_1-a_3=0.27$. The contribution of the integral on the left-hand side of (4′) is only about 10%. (2) $\nu^2=4$ or 140 Mev kinetic energy. Here $\alpha_{33}\approx 45°$ and therefore $2\sin 2\alpha_{33}=2$, which means that the most important contribution to $\mathrm{Re}T^{(2)}(\nu)$ at this point comes from α_{33} and yet it is very insensitive to the exact value of α_{33}. It will be noted that only around 140 Mev are the results of the various sets of phase shifts in agreement with each other. The extrapolation to $\nu=0$ yields $2f^2=0.164$. Considering the experimental error for a_1-a_3 we find $f^2=0.082\pm 0.015$. The quoted error is meant only as an indication of the uncertainties involved.

This value of the coupling constant is in close agreement with the one found by Chew and Low[4] by using quite a different method of extrapolation. Whereas their procedure involves a linear function of ν and depends on the $(\frac{3}{2},\frac{3}{2})$ state only, our method depends both on s and p states, and requires a smaller extrapolation due to the linearity in ν^2.

The value of the right-hand integral in (4′) as given by the slope is $I_{\mathrm{slope}}=-5.1\times 10^{-2}$ as compared to the calculated value $I_{\mathrm{calc}}=-5.15\times 10^{-2}$. They are certainly equal within experimental errors even when allowance is made for possible small contributions from higher energies. We can, therefore, also define the straight line by using only the point at $\nu=1$ and the slope as given by the integral, thus using only total cross sections.

III. PHASE SHIFTS

Continuing our straight line towards higher energies (Fig. 1), we notice that the points calculated with Orear's prescription for the phase shifts[5] fall pretty well on it. The exception at $\nu^2=4.75$ (165 Mev) is not disturbing since that point is very sensitive to the exact shape of $\frac{1}{2}[\sigma_-(\nu)-\sigma_+(\nu)]$ in the vicinity of the

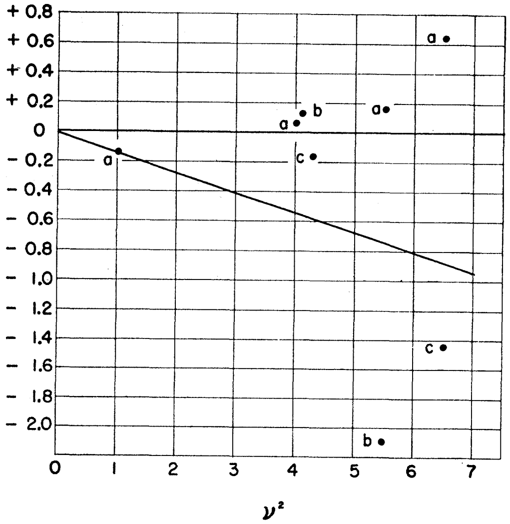

FIG. 2. The left-hand side of Eq. (7) plotted against ν^2. The experimental points are taken from the same papers as in Fig. 1.

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[2] This has to be performed with some care. See Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954). An alternative procedure is to write down Eq. (3) for some particular value of ν, say $\nu=\mu$, and subtract it from (3) itself. This was done by Goldberger *et al.*[1]

[3] The total cross section were taken from Anderson, Davidon, and Kruse, Phys. Rev. **100**, 337 (1955), R. L. Cool and O. Piccioni, Bull. Am. Phys. Soc. Ser. II, **1**, 173 (1956).

[4] G. E. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[5] J. Orear, Phys. Rev. **100**, 288 (1955).

maximum. This is not the case for the other points, i.e., those at 189 and 220 Mev. One may wonder if the good fit of the Orear phase shifts (with $\alpha_{11}=\alpha_{13}=\alpha_{31}=0$) has some deep meaning or is merely the result of an accidental cancellation. To investigate this question, we return to the non-isotopic-spin-flip amplitude $T^{(1)}(\nu)$. Like $T^{(2)}(\nu)$, it can be expressed in the following form:

$$T^{(1)}(\nu)=\tfrac{1}{2}(T_-(\nu)+T_+(\nu))=\tfrac{1}{3}(T^{\frac{1}{2}}(\nu)+2T^{\frac{3}{2}}(\nu)). \quad (6)$$

It is readily seen that the dispersion integral for $T^{(1)}$ will diverge and we therefore write it from the start for $T^{(1)}(\nu)/\nu^2$ and obtain

$$(\nu^2-\nu_B{}^2)\,\mathrm{Re}T^{(1)}(\nu)-\frac{\nu^2(\nu^2-\nu_B{}^2)}{2\pi^2}\times\int_\mu^\infty\left(\frac{\sigma_-(\nu')+\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'(\nu'^2-\nu^2)} \quad (7)$$
$$=-\nu_B{}^2T^{(1)}(0)+\nu^2\left[T^{(1)}(0)-\frac{g^2}{M}\right],$$

where $g^2=(2M/\mu)^2f^2$. $T^{(1)}(0)$ is the amplitude at zero total energy and is an unknown real number. $\mathrm{Re}T^{(1)}(\nu)$ will again be expressed in terms of phase shifts:

$$\mathrm{Re}T^{(1)}(\nu)=\frac{1}{6q}\left(1+\frac{2\nu}{M}+\frac{\mu^2}{M^2}\right)\times[\sin 2\alpha_1+\sin 2\alpha_{11}+2\sin 2\alpha_{13}+\cdots$$
$$+2\sin 2\alpha_3+2\sin 2\alpha_{31}+4\sin 2\alpha_{33}+\cdots]. \quad (8)$$

Evaluating (7) for $\nu=1$, we obtain an equation for $T^{(1)}(0)$ which yields $T^{(1)}(0)=2.07$. It should be noticed that the main contribution to the left-hand side of (7) for $\nu=1$ comes from the integral since $a_1+2a_3=-0.06$ is small. Using the above value for $T^{(1)}(0)$, we find that the straight line defined by (7) starts out very close to the origin and goes down with a slope $T^{(1)}(0)-g^2/M=-0.13$. The integral has been evaluated numerically up to 1.9 Bev and analytically from there on, assuming a constant value for the sum of the cross sections $\frac{1}{2}(\sigma_-+\sigma_+)=28.7$ mb.

We can now use the dispersion relation for $T^{(1)}$ as an additional test for the various sets of phase shifts used previously in the dispersion relation for $T^{(2)}$. The results are shown in Fig. 2. We notice that the points obtained with Orear's precription for the phase shifts lead in the wrong direction as the energy increases. This may indicate that some contributions of the small *p*-phase shifts and perhaps a nonlinear behavior of the *s*-phase shifts canceled out in (4′) but added up in (7). Also the other points in Fig. 2 are quite far from the line.

One has, of course, to bear in mind that these tests are fairly severe. The functional form of the right hand side of (4) and (7) results from the cancellations of two large terms. The distance of the points in Fig. 1 and 2 from the straight lines is therefore a "second-order" effect as compared to the general behavior of the scattering amplitudes found by Anderson *et al.*[3] In some cases, though, the effect is quite large. This need not always be due to the incorrectness of the set of phase shifts in question. The total cross sections are not known accurately enough to permit the drawing of unique curves for σ_- and σ_+ as functions of energy. The uncertainties may be important for principal-value integrals evaluated in the vicinity of the resonance. The straight line in (7), however, depends on the value of the integral at $\nu=1$ and is therefore quite reliable.

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