Boundary Value Treatment of Nucleon-Nucleon Phase Shifts*

A. M. SAPERSTEIN† AND LOYAL DURAND III‡
Yale University, New Haven, Connecticut
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The recent phase-shift fits of Feshbach and Lomon to the nucleon-nucleon scattering data in the energy range $0 \leq E \leq 274$ Mev are examined in detail. An attempt is made to improve the agreement with experiment, and the limitations of the Feshbach-Lomon fits are brought out. It is not found possible to obtain agreement with the data at all energies, even if all previously omitted phase shifts for $L \leq 3$ are included. An independent fit to the proton-proton data is then developed by a boundary value method similar to that used by Breit and Bouricius, and a set of phase shifts for $L \leq 4$ is found which fits the proton-proton cross section and polarization throughout the energy range $0 \leq E \leq 310$ Mev. There seems to be no major obstacle to using the proposed p-p fit as the basis for a charge-independent fit to all the nucleon-nucleon scattering data.

I. INTRODUCTION AND NOTATION

A BOUNDARY-VALUE approach to the analysis of proton-proton scattering at low energies was shown to be possible by Breit and Bouricius.[1] These authors pointed out that such an approach has more generality than a treatment by means of a potential energy. In particular, they stress the fact that changes in the kinetic energy of the incident nucleon can be partially masked in the interaction region, provided that many mesons are present and that the incident energy is shared with them. The extreme situation of many mesons and strong sharing of the energy leads to the possibility of approximate energy independence of the logarithmic derivative of the nucleon-nucleon wave function near the boundary of the interaction region. Breit and Hull[2] pointed out the relationship of an energy-independent boundary condition at such a fixed internucleon distance to the limiting case of interaction through a very deep, very short-ranged potential acting outside an infinitely repulsive core. The connection of the boundary value approach to the phenomenological potential of Jastrow[3] and the meson-theoretic potential of Lévy[4] and the subsequently proposed modifications of Blatt and Kalos[5] and others was thus made clear. The extension to energy-dependent values of the logarithmic derivative and the boundary radius allows, furthermore, a possible description of the nucleon-nucleon interaction even when potential energy descriptions become velocity dependent or fail entirely.

The following is a list of the notations used in this paper.

θ	nucleon scattering angle in the center-of-mass system.
Θ	nucleon scattering angle in the laboratory.
E	laboratory energy of the incident nucleon.
v	laboratory velocity of the incident nucleon.
$\eta = e^2/\hbar v$,	where v is the relativistic relative velocity of the colliding nucleons.
$k = 2\pi$	times the relativistic wave number of relative motion of the nucleons.
${}^3\delta^L{}_J$	triplet phase shift for the state with total angular momentum J, and which would have orbital angular momentum L in the limit of zero coupling.
ϵ_J	the coupling parameter between triplet states of the total angular momentum J, orbital angular momenta $J-1$, $J+1$.
K_L	singlet phase shift for angular momentum L.
$\sigma(\theta)$	differential scattering cross section as a function of θ in the center of mass.
$P\sigma(\theta)$	polarization times the differential cross section as a function of θ in the center of mass.
$I(\theta)$	Coulomb interference terms in the p-p cross section $\sigma_{p\text{-}p}(\theta)$.

II. COMPARISON OF FESHBACH-LOMON P–P FITS WITH EXPERIMENT

Feshbach and Lomon[6] have recently attempted a charge-independent boundary value fit to the nucleon-nucleon scattering, spanning the energy range from zero to 274 Mev. In all states except the ${}^1S_0$ state, the radius at which boundary values were applied was kept constant; in the ${}^1S_0$ state, the boundary radius was allowed to decrease with increasing energy. In all states, the value of the logarithmic derivative of the wave function at the boundary was taken to be independent of the energy. The data fitted were the proton-proton and neutron-proton differential scattering cross sections and the magnitude of the proton-proton polarization at a scattering angle of $\Theta = 20°$ in the laboratory. Two sets of proton-proton phase shifts were determined, sets A and B. These have the same bound-

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[1] G. Breit and W. G. Bouricius, Phys. Rev. **75**, 1029 (1949).
[2] G. Breit and M. H. Hull, Jr., Am. J. Phys. **21**, 184 (1953).
[3] R. Jastrow, Phys. Rev. **79**, 389 (1950); **81**, 165 (1951).
[4] M. M. Lévy, Phys. Rev. **88**, 725 (1952).
[5] J. M. Blatt and M. H. Kalos, Phys. Rev. **92**, 1563 (1953).
[6] H. Feshbach and E. Lomon, Phys. Rev. **102**, 891 (1956). The considerations in the introduction to this reference regarding the bearing on meson theory and relationsip to hard core potentials are essentially the same as in references 1 and 2.

ary condition in the $^3P_0$ state, hence, the same phase shifts $^3\delta^P{}_0$ for that state, but differ otherwise. Fit A contains nonvanishing values of the singlet phase shifts K_0, K_2 and the triplet phase shifts $^3\delta^P{}_0$, $^3\delta^P{}_1$, $^3\delta^P{}_2$ only. The parameters $^3\delta^S{}_1$, ϵ_1, and $^3\delta^D{}_1$ are then added to fit the neutron-proton data. Fit B is similarly confined to K_0, K_2, $^3\delta^P{}_0$, $^3\delta^P{}_2$, $^3\delta^F{}_2$, $^3\delta^F{}_3$, $^3\delta^S{}_1$, ϵ_1, $^3\delta^D{}_1$. This work does not consider the neutron-proton polarization and the angular distribution of the proton-proton polarization, nor does it consider Coulomb interference effects in the proton-proton polarization and cross section. Consequently, the fits have been examined here with regard to these features.

Coulomb interference in the small-angle cross section can rule out an otherwise acceptable p-p cross section fit,[7] as can the angular distribution of the polarization.[8] The Feshbach-Lomon p-p fits have been examined in these particulars, and to allow greater leeway in fitting the additional data, the possible existence of small phase shifts in those states with $L\leq 3$ left undetermined by the Feshbach-Lomon fits has been assumed. Thus, small $^3\delta^F{}_2$, $^3\delta^F{}_3$, $^3\delta^F{}_4$ were added to p-p fit A; and small $^3\delta^P{}_1$, $^3\delta^F{}_4$ to fit B. The additions were kept small so as not to depart from the approximately angle-independent p-p cross section obtained at larger angles from the Feshbach-Lomon fits. The $J=2$ coupling parameter, ϵ_2, was not used, since this would necessitate changes in the $J=2$ boundary values; furthermore, numerical computations have shown that $\sigma_{p\text{-}p}(\theta)$ and $P\sigma_{p\text{-}p}(\theta)$ are not too sensitive to small values of ϵ_2.

The polarization was first computed using those parts of Eq. (1) of Hull and Saperstein[9] not involving Coulomb interference effects:

$$k^2 P\sigma_{pp} = \sin\theta\,\cos\theta(a+b\cos^2\theta+c\cos^4\theta). \tag{1}$$

The coefficients a and b were obtained from the least-squares analysis of Fischer and Baldwin,[10] while c was kept small. The values used are given in Table I. The

TABLE I. Coefficients for $(P\sigma)_{pp}$, in Eq. (1). The experimental values at 130 and 170 Mev are from Fischer and Baldwin.[10] The 274-Mev data are interpolated from Fig. 5 of Fischer and Baldwin.

	Experiment		F-L fit A		F-L fit B	
E_{Mev}	a	b	a	b	a	b
130	0.13 ± 0.06	0.25 ± 0.12	0.356	0.000	0.462	0.020
170	0.23 ± 0.07	0.23 ± 0.11	0.595	0.000	0.652	0.042
274	0.60 ± 0.27	0.53 ± 0.26	0.836	0.000	0.783	0.054

[7] Breit, Condon and Present, Phys. Rev. **50**, 825 (1936); Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939); Breit, Kittel, and Thaxton, Phys. Rev. **57**, 155 (1940); Thaler, Bengston, and Breit, Phys. Rev. **94**, 683 (1954); R. M. Thaler and J. Bengston, Phys. Rev. **94**, 679 (1954); H. P. Stapp, University of California Radiation Laboratory Report UCRL 3098, 1955 (unpublished); C. A. Klein, Nuovo cimento **1** (Series 10), 581 (1955); S. Ohnuma and D. Feldman, Phys. Rev. **102**, 1641 (1956).

[8] Breit, Ehrman, Saperstein, and Hull, Phys. Rev. **96**, 807 (1954).

[9] M. H. Hull, Jr., and A. M. Saperstein, Phys. Rev. **96**, 806 (1954).

[10] D. Fischer and J. Baldwin, Phys. Rev. **100**, 1445 (1955).

TABLE II. Feshbach-Lomon p-p phase shifts for their cases A and B.

Phase	98 Mev		130 Mev		170 Mev		274 Mev	
shift	A	B	A	B	A	B	A	B
K_0	25°	20°	16°	9°	11°	0	24.72°	12.60°
K_2	1.1	2	1.6	3	2.2	4.1	0.75	4.68
$^3\delta^P{}_0$	−31	−31	−40	−40	−52	−52	−77.93	−77.93
$^3\delta^P{}_1$	2	···	2.3	···	2.1	···	0	···
$^3\delta^P{}_2$	6	7.5	7.3	8.7	8.4	7.9	8.22	1.95
$^3\delta^F{}_2$	···	~ 0	···	0.8	···	− 1.6	···	− 6.03
$^3\delta^F{}_3$	···	~ 0	···	0.17	···	0.37	···	1.2
$^3\delta^F{}_4$								

Coulomb interference term $I(\theta)$ in the cross section was computed using a form[11] linear in η:

$$I(\theta)\simeq -\frac{2\eta}{k^2}\csc^2\theta\{S^S{}_0+5S^D{}_2P_2(\cos\theta)$$
$$+[5S^P{}_2+3S^P{}_1+S^P{}_0]\cos^2\theta$$
$$+[9S^F{}_4+7S^F{}_3+5S^F{}_2](\cos\theta)P_3(\cos\theta)\}. \tag{2}$$

Here $S^L{}_J=\frac{1}{2}\sin 2\delta^L{}_J$; the relativistic value[12] of $\eta=e^2/\hbar v$ should be used, and $k/2\pi$ is the wave number of relative motion: $1/k^2\simeq 830/E_{\text{Mev}}$ mb. The calculations were later repeated exactly, using the α_i form of the scattering matrix due to Breit, Ehrman, and Hull.[13]

Feshbach-Lomon p-p Fit at High Energies

The phase shifts given by Feshbach and Lomon for fits A and B at the energies $E=130$, 170, and 274 Mev are shown in Table II. The values of a, b computed from these phase shifts are given in Table I, along with the experimental values. Comparison of the theoretical and experimental values shows that a is always considerably too large and b, too small. With some study of the magnitudes and signs of the Feshbach-Lomon phase shifts, and the formulas for a, b of Hull and Saperstein,[9] it becomes apparent that the effects of adding small F phase shifts to the fit will appear primarily in the interference terms between $^3\delta^F{}_J$ and $^3\delta^P{}_0$. The large size of $|^3\delta^P{}_0|$ relative to $|^3\delta^P{}_1|$ and $|^3\delta^P{}_2|$ makes it by far the dominant P wave. One wishes to decrease a while increasing b and is restricted to small $^3\delta^F{}_J$ so as not to destroy the isotropy of $\sigma_{p\text{-}p}(\theta)$ at large angles. There is no interference between the $^3F_3-{}^3P_0$ states, so a small $^3\delta^F{}_3$ will have very little effect on the polarization. The $^3F_2-{}^3P_0$ interference appears only in a, but the $^3F_4-{}^3P_0$ interference appears in both a and b, and with the correct sign, i.e., positive in b, negative in a if $^3\delta^F{}_4>0$. Thus, the fitting procedure was to fit a, b as well as possible using the phase shift $^3\delta^F{}_4>0$. For set A, the $^3\delta^F{}_2$ phase shift was free, and was used to adjust a further, while $^3\delta^F{}_3$ was used to adjust the isotropy of the cross section or the Coulomb interference. For set B,

[11] M. H. Hull, Jr. (private communication).

[12] G. Breit, Phys. Rev. **99**, 1581 (1955) has shown that relativistic effects can be accounted for, to first order, by using the relativistic velocity in η.

[13] Breit, Ehrman, and Hull, Phys. Rev. **97**, 1051 (1955).

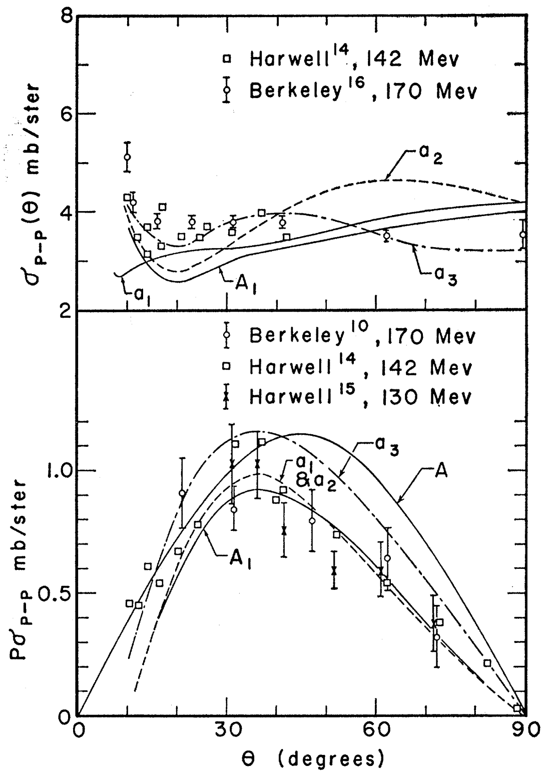

FIG. 1. The upper curves give the p-p cross sections at 130 Mev and 170 Mev derived from the modified Feshbach-Lomon phase shift sets. The curves are explained in the text. The lower curves give the corresponding polarizations at 130 and 170 Mev. Curve A is the unmodified Feshbach-Lomon A fit at 130 Mev.

the ${}^3\delta^F{}_4$ phase shift was again free, but ${}^3\delta^F{}_3$ and ${}^3\delta^F{}_2$ were already specified. The remaining phase shift was ${}^3\delta^P{}_1$. Small values of ${}^3\delta^P{}_1$ produce negligible effects on a, b, but may be important in $I(\theta)$ and $\sigma_{p\text{-}p}(\theta)$.

At 130 Mev, fit A without additions yields $a=0.356$, $b=c=0$. The addition to fit A of ${}^3\delta^F{}_2=2°$, ${}^3\delta^F{}_4=1°$, makes $a=0.16$, $b=0.26$, an acceptable fit to the large-angle polarization. There are no small-angle cross section data at this energy, but fit A with and without the above additions shows a large dip in the small-angle cross section caused by destructive Coulomb interference. There is no such pronounced dip at 170 Mev. The recent 142-Mev data from Harwell[14] do show considerable interference in the small angle region but not as much as required by Feshbach and Lomon. The 130-Mev data[15] and fit are given in Fig. 1, curve $A1$.

Using fit B without additions, $a=0.462$, $b=0.020$. Small ${}^3\delta^P{}_1$ does not help a or b appreciably, and ${}^3\delta^F{}_4$ must be used to improve the fit. Assuming ${}^3\delta^F{}_4=2°$ makes $a=0.238$, but $b=0.562$. Further increase in ${}^3\delta^F{}_4$ will reduce a to within the experimental limits, but b is too large already and will continue to increase. Thus, fit B cannot be made to fit $P\sigma_{p\text{-}p}$ at $E=130$ Mev through the addition of phase shifts for states $L\leq 3$.

At 170 Mev,[10,16] fit A yields $a=0.595$, $b=c=0$, again making $P\sigma$ much too large. The ${}^3F_4-{}^3P_0$ interference is used to reduce a and increase b; with ${}^3\delta^F{}_4=1°$, one obtains $a=0.430$, $b=0.396$, so that both coefficients are too large. Decreasing ${}^3\delta^F{}_4$ to 0.65° makes $b=0.255$, an acceptable value, and $a=0.489$. The ${}^3F_2-{}^3P_0$ interference is now used with ${}^3\delta^F{}_2=4°$ to reduce a to 0.222. This is a fit to the least-squares polarization coefficients: $a=0.23\pm0.07$, $b=0.23\pm0.11$; but the new fit makes the Coulomb interference in the cross section assume a value of -4.96 mb at $\theta=10°$, as shown in Fig. 1, curve a_1. The experimental data show a gradual rise in $\sigma_{p\text{-}p}(\theta)$ from $\theta=90°$ to $\theta=10°$, followed by the steep Coulomb rise for $\theta\leq10°$. Thus, the Coulomb interference term $I(\theta)$ must just cancel the pure Coulomb rise in $\sigma_{p\text{-}p}(\theta)$ in the range $10°\leq\theta\leq30°$. The nuclear cross section is assumed to be relatively flat, even at small angles. Computation from the data then implies $I(10°)\simeq-2.6$ mb. The discrepancy can be corrected at $\theta=10°$ by making ${}^3\delta^F{}_3=-3°$; the polarization coefficients are not changed appreciably ($a=0.217$, $b=0.253$) but the resultant cross section then has a large dip at $\theta=20°$ and is too big at larger angles as is shown by curve a_2 of Fig. 1. Thus, the addition of fairly large F phase shifts, necessary to fit the additional requirements of small angle cross section and polarization, has destroyed the large-angle cross section fit originally given by case A.

Case B at 170 Mev gives $a=0.652$, $b=0.042$, and $c=0.0$. Since the results are insensitive to small values of ${}^3\delta^P{}_1$, the phase shift ${}^3\delta^F{}_4$ is picked to give a correct value of b; for ${}^3\delta^F{}_4=0.6°$, one obtains $b=0.272$ and $a=0.555$. As ${}^3\delta^F{}_4$ is increased, b will increase at a rate greater than that at which a decreases; hence further change of ${}^3\delta^F{}_4$ will not improve the B fit at 170 Mev.

The highest energy for which Feshbach and Lomon give phase shifts is 274 Mev. The nearest experimental differential cross section is at 260 Mev, the nearest polarization angular distribution, at 310 Mev. For convenience in first calculations, values of a and b are obtained at 274 Mev by linear interpolation between 170 and 310 Mev in Fig. 5 of Fischer and Baldwin[10]; one obtains $a=0.60\pm0.27$ and $b=0.53\pm0.26$. Without additions, fit A yields $a=0.836$ and $b=0$. By adding a ${}^3\delta^F{}_4=1°$, the following fit to the polarization coefficients is obtained: $a=0.589$, $b=0.582$. To duplicate the differential cross section at $\theta=10°$, the magnitude of the destructive Coulomb interference must be reduced; from Eq. (2) this can be done by adding negative ${}^3\delta^F{}_2$ and ${}^3\delta^F{}_3$ phase shifts, $P_3(\cos\theta)$ being positive in this region. A comparison of the desired amount of Coulomb interference with that actually obtained without the ${}^3\delta^F{}_2$ and ${}^3\delta^F{}_3$ additions, similar to the comparison illustrated at 170 Mev, provides the upper limits on the

[14] A. Taylor (private communication of Harwell data to Professor G. Breit).
[15] J. M. Dickson and D. C. Salter, Nature 173, 946 (1954).
[16] O. Chamberlain and J. D. Garrison, Phys. Rev. 95, 1349 (1954).

phase shifts. Rather than taking either phase shift individually very negative, it was found desirable for $\sigma_{p\text{-}p}(\theta)$ to have $|{}^3\delta^F{}_J|$ as small as possible: thus, ${}^3\delta^F{}_2={}^3\delta^F{}_3=-2.5°$. Taking these upper limits in the additions gives the just acceptable coefficients $a=0.80$, $b=0.65$; the addition of such large F phase shifts produces a dubious, but perhaps allowable, hump in $\sigma_{p\text{-}p}(\theta)$ in the region around $\theta=40°$. The quantities $P\sigma_{p\text{-}p}$ and $\sigma_{p\text{-}p}(\theta)$ are shown in Fig. 2 with the 250-Mev, 260-Mev, and 310-Mev data.[16–19] Comparing $P\sigma_{p\text{-}p}$ with the polarization data at 310 Mev, fit A, as modified above, produces too much polarization at large angles and too little at small angles; correcting the Coulomb interference term in the cross section has still left too much destructive interference in the polarization.

Case B implies $a=0.783$, $b=0.0535$ at 274 Mev. When a ${}^3\delta^F{}_4=1°$ is added, $a=0.544$, $b=0.557$. The additional F phase shift is not large enough to destroy the isotropy of the large-angle cross section; furthermore the negative K_0 diminishes the destructive Coulomb interference so as to provide a good small-angle cross section as shown in curve B of the upper part of Fig. 2. However, there appears to be too much destructive Coulomb interference in the polarization: when compared with the 310-Mev data, the polarization decreases too rapidly at small angles.

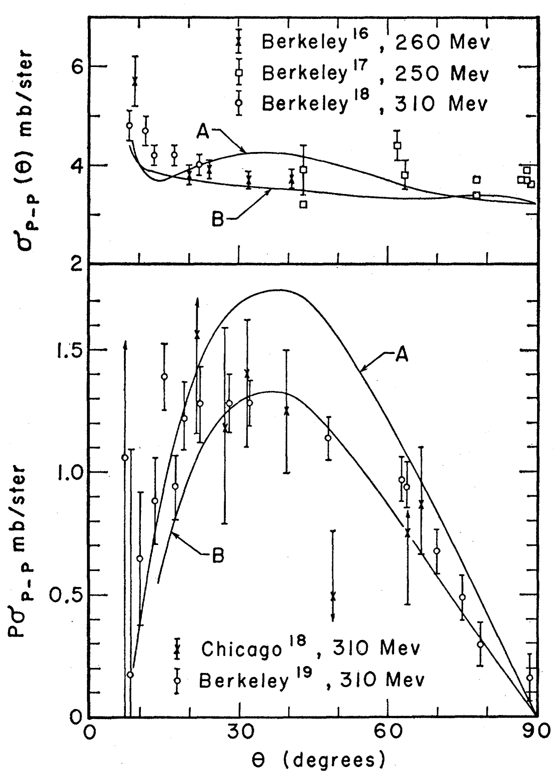

FIG. 2. The curves show the p-p cross sections and polarizations at 274 Mev calculated from Feshbach-Lomon fits A and B with F wave phase shifts added. The curves are explained in the text.

[17] Chamberlain, Segrè, and Wiegand, Phys. Rev. **83**, 923 (1951).
[18] Marshall, Marshall, and Carvalho, Phys. Rev. **93**, 1431 (1954).
[19] Chamberlain, Pettengill, Segrè, and Wiegand, Phys. Rev. **95**, 1348 (1954).

Attempts at Modification of the Feshbach-Lomon Fits

The Feshbach-Lomon fits are hardest to justify at 170 Mev. This energy is at about the middle of the interval over which their model is supposed to be valid, so failure at 170 Mev can hardly be disregarded. In the hope that the Feshbach-Lomon fits are in the neighborhood of a correct fit at 170 Mev, they were re-examined at this energy, allowing for the possibility of a readjustment.

Feshbach and Lomon assumed a pure $\sin\theta\cos\theta$ variation of $P\sigma$. They obtained the consequently large value of a by using a ${}^3\delta^P{}_0$ of very large absolute value. The coefficient b then arises through interference between ${}^3\delta^F{}_4$ and ${}^3\delta^P{}_0$, but $|{}^3\delta^P{}_0|$ has been taken so large that b becomes extremely sensitive to ${}^3\delta^F{}_4$; a very small value of ${}^3\delta^F{}_4$ thus gives a suitable value for b but is too small to decrease a sufficiently. The fits were modified therefore by reducing $|{}^3\delta^P{}_0|$ by 10°: *i.e.*, taking ${}^3\delta^P{}_0=-42°$ at 170 Mev. In order to maintain the value of the cross section at $\theta=90°$ and simultaneously keep the large angle cross section angle independent, $|K_0|$ was increased, keeping $\sin^2K_0+\sin^2({}^3\delta^P{}_0)$ constant. This gave $K_0=27.2°$ for case A and $K_0=24.6°$ for case B.

With these changes, fit A gives $a=0.43$, $b=c=0$. Adding ${}^3\delta^F{}_4=1°$ makes $a=0.31$ and $b=0.28$, which is the best that can be done by changing ${}^3\delta^F{}_4$ alone, since a further increase in ${}^3\delta^F{}_4$ will increase b more rapidly than a is decreased. A reasonable value of $I(10°)$ can be obtained by taking ${}^3\delta^F{}_3=-1.7°$, ${}^3\delta^F{}_2=-1°$; these additions change the polarization only very slightly. The resultant plot of the cross section against scattering angle is too low at $\theta=20°$ and is not sufficiently flat, as seen in curve a_3 of the upper part of Fig. 1.

With reduced ${}^3\delta^P{}_0$, case B gives, at 170 Mev, $a=0.455$ and $b=0.042$; the best fit to $P\sigma$ is obtained with ${}^3\delta^F{}_4=1°$, the polarization again being relatively insensitive to ${}^3\delta^P{}_1$, the other free parameter. The polarization coefficients are still too large: $a=0.33$, $b=0.32$. To fit the Coulomb interference in the cross section at $\theta=10°$ requires the addition of ${}^3\delta^P{}_1\leq-7.2°$. The addition ${}^3\delta^P{}_1=-7.2°$ increases the polarization coefficients slightly to $a=0.37$, $b=0.34$. These values are considerably outside the rather liberal limit of experimental error for a, and just reach the limit for b. The relatively large size of $|{}^3\delta^P{}_1|$ also adversely affects $\sigma_{p\text{-}p}(\theta)$ for $\theta\geq30°$.

Feshbach-Lomon p-p Fits at Low Energies

Feshbach and Lomon's low-energy fits are also not very satisfactory. When $\sigma(90°)$ is calculated, cases A and B are acceptable at 38.5 Mev incident energy, but at 80 Mev, A is 0.2 mb and B is 0.4 mb below the lower

TABLE III. The boundary values given by Feshbach and Lomon[a] were used to compute f for their cases. The boundary values for the present work were taken as: $Y_0=0.131-0.00615E$ at $r_b=1.32\times10^{-13}$ cm for $E<50$ Mev. The phase shift calculated from Y_0 was supplemented by that caused by the potential tail discussed in the text.

E_{Mev}	f, F-L set A	f, F-L set B	f, present work	f experimental[b]
1	7.8	7.8	9.1	8.72±0.1
2	8.8	8.7	9.5	9.64±0.1
6	11.7	11.4	12.6	13.3 ±0.5
10	15.4	15.6	16.1	16.7 ±0.5
20	25.3	26.2	24.5	25 ±1.0
30	35.8	37.6	32.6	32.5 ±1.0

[a] See reference 6.
[b] See reference 21.

experimental limits.[20] Both cases produce a differential cross section which increases from $\theta=30°$ to $\theta=90°$ whereas the 95-Mev data of Kruse, Teem, and Ramsey[20] show just the opposite slope in this angular range.

At energies below 30 Mev the f function of Breit, Condon, and Present[7] can be computed for cases A and B and compared with the experimental results as summarized in Yovits, Smith, Hull, Bengston, and Breit.[21] Some results are tabulated in Table III. Considering the precision of the low-energy experiments, it is seen that the Feshbach-Lomon fits are not entirely satisfactory.

Summary of Examination of Feshbach-Lomon Proton-Proton Fits

Neither of the Feshbach-Lomon p-p fits is completely satisfactory over the entire high-energy range 100 Mev$\leq E\leq$300 Mev. Unmodified, neither fit A nor fit B gives an adequate representation of the angular distribution of the polarization at any energy. Thus, the parameters for $L\leq 3$ left free by Feshbach and Lomon were added in an attempt to improve the fitting. At 130 Mev, fit A with the additions ${}^3\delta^{F}{}_2=2°$ and ${}^3\delta^{F}{}_4=1°$ gives an acceptable fit to both $P\sigma_{p\text{-}p}$ and $\sigma_{p\text{-}p}(\theta)$. At 170 Mev, the addition of ${}^3\delta^{F}{}_2=4°$ and ${}^3\delta^{F}{}_4=1°$ gives a barely acceptable fit to $P\sigma_{p\text{-}p}$, but causes excessive Coulomb interference in the cross section at $\theta=10°$. This may be corrected with ${}^3\delta^{F}{}_3=-3°$, but this change in turn destroys the agreement of $\sigma_{p\text{-}p}(\theta)$ with experiment near 20° and at large angles. The alteration of the Feshbach-Lomon fit at 170 Mev with respect to the very large value of $|{}^3\delta^{P}{}_0|$ was considered. For $\delta^{P}{}_0$ changed to $-42°$, and with the phase shifts ${}^3\delta^{F}{}_2=-1°$, ${}^3\delta^{F}{}_3=-1.7°$, and ${}^3\delta^{F}{}_4=1°$ added, some improvement in the fit to $\sigma_{p\text{-}p}(\theta)$ was found; but $P\sigma$ was made slightly too big at large angles by these changes, and further adjustment of the Feshbach-Lomon phases seemed necessary. At 274 Mev, with added phase shifts ${}^3\delta^{F}{}_2=-2.5°$, ${}^3\delta^{F}{}_3=-2.5°$, and ${}^3\delta^{F}{}_4=1°$, case A fits both $P\sigma_{p\text{-}p}$ and $\sigma_{p\text{-}p}(\theta)$, but not well. The quantities $P\sigma$ and $\sigma_{p\text{-}p}(\theta)$ are both too large in the region about $\theta=40°$.

Fit B fares less well; at 130 Mev there seems no chance of a fit starting with the Feshbach-Lomon phase shifts and adding only small ${}^3\delta^{P}{}_1$ and ${}^3\delta^{F}{}_4$ phase shifts. Case B also fails to fit the angular distribution of the polarization at 170 Mev; the fit can be improved somewhat by modifying ${}^3\delta^{P}{}_0$, but an increase by 10° is not sufficient to bring a and b within the experimental limits. The Coulomb interference and the large-angle isotropy of $\sigma_{p\text{-}p}(\theta)$ cannot be fitted simultaneously in the modified case. However, at 274 Mev, case B, with added ${}^3\delta^{F}{}_4$, does fit $\sigma_{p\text{-}p}(\theta)$ and $P\sigma_{p\text{-}p}$ reasonably well. A possible exception is the small-angle polarization: there the effect of too much destructive Coulomb interference is apparent.

It would appear that major modifications of the Feshbach-Lomon fits are necessary in order to fit satisfactorily the high-energy proton-proton scattering data. The fits have been improved somewhat by the addition of further phase shifts, but no attempt has been made to add these in a manner consistent with constant boundary values for the logarithmic derivative. This arbitrariness in the introduction of additional phase shifts, contrary to the spirit of the boundary-value method, can be taken as an argument against interpreting the modified phase shift sets too hopefully. Coupled with the lack of a consistently good high-energy fit, is the failure of the Feshbach-Lomon phase shifts to fit the low-energy p-p scattering to within the accuracy of the experiments. It seems, therefore, difficult to accept the Feshbach-Lomon proton-proton fits without allowing for major modifications.

III. FESHBACH-LOMON n-p FITS AT HIGH ENERGY

The Feshbach-Lomon charge-independent fits to the n-p scattering data are not, as given, satisfactory. Some defects in the n-p cross sections have been discussed by the authors themselves,[6] but the n-p polarization has not been examined by them. Neutron-proton polarization data are available only at 98 Mev,[14,22] 310 Mev,[23] and at 350 Mev.[24] The 310-Mev data are at an energy close enough to the Feshbach-Lomon fit at 274-Mev to make comparison of $P\sigma_{n\text{-}p}$ at the two energies possess some validity.

At 98 Mev the Harwell group[14,22] has measured the angular distribution of $P\sigma_{n\text{-}p}$ in the angular range $20°<\theta<180°$. The experimental data are reproduced in Fig. 3. To the Feshbach-Lomon p-p phase shifts at 98 Mev given in Table II, the n-p fit adds the following phase shifts: $K_1=-13°$, ${}^3\delta^{S}{}_1=46°$, $\epsilon_1=27°$ and ${}^3\delta^{D}{}_1=-12°$. The n-p polarization and cross section computed using these sets of phase shifts are shown in

[20] Kruse, Teem, and Ramsey, Phys. Rev. **94**, 1795 (1954).
[21] Yovits, Smith, Hull, Bengston, and Breit, Phys. Rev. **85**, 540 (1952).
[22] P. Hillman and G. H. Stafford, Harwell (private communication of data to Professor G. Breit).
[23] Chamberlain, Donaldson, Segrè, Tripp, Wiegand, and Ypsilantis, Phys. Rev. **95**, 850 (1954).
[24] Siegel, Hartzler, and Love, Phys. Rev. **101**, 838 (1956).

Fig. 3, curves A and B. The cross section calculated at 98 Mev was normalized to 90 Mev using the approximate E^{-1} dependence of the n-p total cross section. Both sets A and B yield cross sections which are too low by several millibarns near $\theta=0°$ and $\theta=180°$. The cross section for set A is slightly too asymmetric about $\theta=90°$, $\sigma_{n\text{-}p}(180°)$ being about 1.5 mb higher than $\sigma_{n\text{-}p}(0°)$. The cross section at $\theta=90°$ is slightly low for both cases compared to the 90-Mev experimental data,[25] but this discrepancy may not be significant due to the uncertainty in the absolute normalization of the data. The n-p polarization given by sets A and B is in somewhat better agreement with the experiments. The agreement is good for small angles, but $P\sigma_{n\text{-}p}$ is systematically too large for $\theta>60°$, and fails to become negative at very large scattering angles. Set A gives a somewhat better polarization than does set B.

The isotopic singlet phase shifts ${}^3\delta^D{}_2$, ${}^3\delta^D{}_3$, and K_3 are available to improve the agreement of the Feshbach-Lomon n-p fits with experiment. The isotopic triplet phase shifts added in the foregoing p-p analysis are also available, but since these are, to some extent, determined by the adjustment of the p-p fits, these phase shifts should be used only if absolutely necessary.

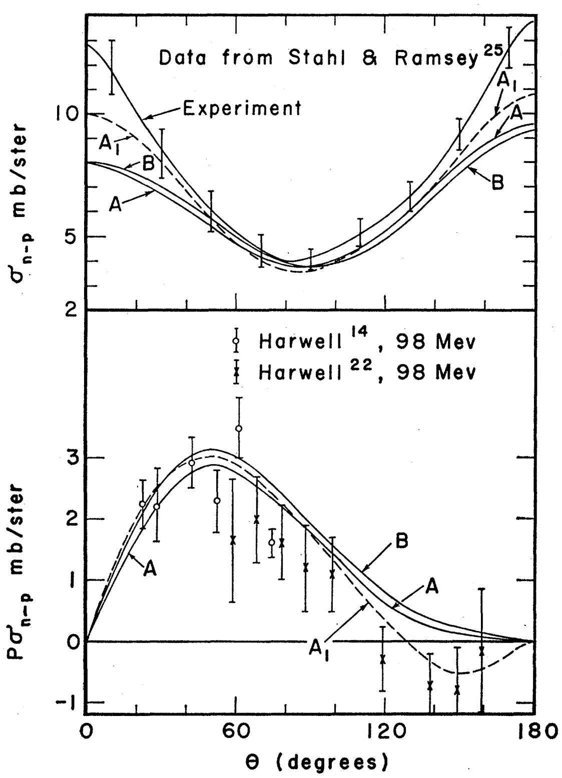

FIG. 3. The n-p cross section at 90 Mev and polarization at 98 Mev from the unmodified and the modified n-p fits of Feshbach and Lomon. The cross bars on the cross section data give the approximate experimental limits, taken from the curves of Stahl and Ramsey.[25]

[25] R. H. Stahl and N. F. Ramsey, Phys. Rev. **96**, 1310 (1954). The 90-Mev n-p experimental data are summarized in this paper.

Since the expansions of $\sigma_{n\text{-}p}$ and $P\sigma_{n\text{-}p}$ in terms of trigonometric functions of the phase shifts are rather unwieldy, the n-p calculations were carried out using the α_i formulation of the scattering matrix, due to Breit, Ehrman, and Hull,[13] computing each α_i separately. Examination of the amplitudes and the Feshbach-Lomon phase shifts at 98 Mev indicates that $P\sigma_{n\text{-}p}$ is a sensitive function of ${}^3\delta^D{}_3$. A small ${}^3\delta^D{}_3$ phase shift may produce quite large changes in the polarization, mainly through its interference effects with the large ${}^3S_1+{}^3D_1$ phase shifts. For ${}^3\delta^D{}_3>0$, the changes will increase $P\sigma_{n\text{-}p}$ at small angles and decrease it at large angles. The interference effects with the isotopic triplet phase shifts are small, increasing $P\sigma_{n\text{-}p}$ at both large and small angles. The net change arising from ${}^3\delta^D{}_3>0$ is in the desired direction. On the other hand, ${}^3\delta^D{}_3>0$ decreases the already low cross section at $\theta=90°$. This effect in $\sigma_{n\text{-}p}$ may be canceled by the addition of ${}^3\delta^D{}_2>0$, an addition which tends to decrease $P\sigma_{n\text{-}p}$ at small angles and increase it at large angles. The polarization is not as sensitive to ${}^3\delta^D{}_2$ as it is to ${}^3\delta^D{}_3$, so a compromise may be reached improving both $\sigma_{n\text{-}p}$ and $P\sigma_{n\text{-}p}$. The inclusion of ${}^3\delta^D{}_3=3°$ with set A improved the 98-Mev n-p polarization somewhat, especially in the region $\theta>140°$, where $P\sigma_{n\text{-}p}$ became negative in agreement with experiment. However, $\sigma_{n\text{-}p}(90°)$ was reduced by 0.4 millibarn to considerably below the experimental limits. Although the 0° and 180° cross sections were both increased by adding ${}^3\delta^D{}_3$, the asymmetry in the cross section was also increased. The ${}^3\delta^D{}_2$ phase shift was added next. After some experimentation, the combination ${}^3\delta^D{}_2={}^3\delta^D{}_3=4°$ was settled upon. The polarization could not be improved by further adjustment of these D-wave phase shifts without making the fit to $\sigma_{n\text{-}p}$ worse. The defects in $\sigma_{n\text{-}p}$ could not be corrected using K_3 alone, so the F-wave phase shifts were examined. From the adjustment of the p-p fits required at higher energies, one would expect the addition of ${}^3\delta^F{}_4>0$ to the Feshbach-Lomon fits to be necessary to correct the angular distribution of $P\sigma_{p\text{-}p}$ at 98 Mev. However, the addition of ${}^3\delta^F{}_4=1°$ under the assumption of charge independence of the nucleon-nucleon interaction led to undesirable effects on $P\sigma_{n\text{-}p}$ and $\sigma_{n\text{-}p}$. Changing the sign of the phase shift gave some improvement in the fit, but, as one would like to keep the same sign of ${}^3\delta^F{}_4$ at 98 Mev as that required by the p-p work at the nearby energy of 130 Mev, this phase shift was dropped from consideration. The n-p polarization is quite insensitive to ${}^3\delta^F{}_3$, but ${}^3\delta^F{}_3=-1°$ made the cross section more symmetric about $\theta=90°$. The use of ${}^3\delta^F{}_3$ was not ultimately necessary, as both $P\sigma_{n\text{-}p}$ and $\sigma_{n\text{-}p}$ were sufficiently improved by the addition to set A of ${}^3\delta^F{}_2=2°$. This phase shift has the same sign as that required by the p-p analysis. The cross section was further improved by the addition of $K_3=-1°$. Increasing both ${}^3\delta^F{}_2$ and $|K_3|$ slightly will help $\sigma_{n\text{-}p}$ at all angles, but larger ${}^3\delta^F{}_2$ will increase the already too large value of $P\sigma_{n\text{-}p}$ at 90°. The two phase shifts must be adjusted together so as not to destroy the symmetry characteristics of the cross section. The final modification of the Feshbach-Lomon set A contained the additions: ${}^3\delta^D{}_2={}^3\delta^D{}_3=4°$, ${}^3\delta^F{}_2=2°$, and $K_3=-1°$. The final fits are shown in Fig. 3, curves $A1$. The cross section is much improved over that for the unmodified set A, and the polarization is improved at large angles, remaining too large in the region $60°<\theta<130°$. The symmetry of $\sigma_{n\text{-}p}$ about 90° is fitted quite well: $\sigma_{n\text{-}p}(180°)$ is slightly larger than $\sigma_{n\text{-}p}(0°)$, and the minimum in the cross section occurs for $\theta<90°$. The magnitude of the cross section is not fitted quite so well, especially near 0° and 180°. Despite the improvements, the fit to the data is considerably outside the supposed limits of experimental error for $\sigma_{n\text{-}p}$ at low and high angles. Its agreement with measurements of $P\ \sigma_{n\text{-}p}$ is slightly outside the limits of error. It is also questionable whether the adjusted fit is meaningful, as it relies heavily on the F-wave phase shift ${}^3\delta^F{}_2$, a phase shift to which the p-p polarization and, especially, the p-p cross section are sensitive. It is conceivable therefore that a simultaneous fit to $\sigma_{p\text{-}p}$ and $(P\sigma)_{p\text{-}p}$ at 95 Mev will be impossible for the ${}^3\delta^F{}_2$ needed to satisfy the n-p data. Data on $(P\sigma)_{p\text{-}p}$ at this energy might help to clarify this point.

Case B at 98 Mev behaved quite similarly to case A, but it was not possible to obtain as good a fit to the experimental data as with case A, the ${}^3\delta^F{}_2$ and ${}^3\delta^F{}_3$ phase shifts already being specified. Aside from the D-wave phase shifts, the remaining parameters were ${}^3\delta^P{}_1$, to which $P\sigma_{n\text{-}p}$ is not very sensitive, and ${}^3\delta^F{}_4$, which did not help much. The best fits were obtained with ${}^3\delta^D{}_2$ and ${}^3\delta^D{}_3$ positive and located in the general range $3°\lesssim\delta\lesssim5°$. The polarization for the unmodified set B was too large, and any such additions, designed to improve $P\sigma_{n\text{-}p}$ in the large-angle region where agreement with the data was worst, caused the agreement to become worse at small angles. As this case could not be improved nearly so much as case A, final curves of $P\sigma_{n\text{-}p}$ and $\sigma_{n\text{-}p}$ are not given.

The main difficulty with the Feshbach-Lomon fits at 98 Mev arises from the large value of ${}^3\delta^P{}_0$ relative to ${}^3\delta^P{}_2$, a condition resulting from the method used for fitting the p-p polarization. The interference of this ${}^3\delta^P{}_0-{}^3\delta^P{}_2$ combination with the ${}^3S_1+{}^3D_1$ and the ${}^3\delta^D{}_3$ phase shifts produces a large asymmetric term in the cross section which builds up $\sigma_{n\text{-}p}(180°)$ and reduces $\sigma_{n\text{-}p}(0°)$. Since the ${}^3S_1+{}^3D_1$ phase shifts are fixed, and since the asymmetry in $\sigma_{n\text{-}p}$ cannot be corrected using K_3 alone, the allowable asymmetry provides an upper limit on the size of ${}^3\delta^D{}_3$. The size of ${}^3\delta^D{}_2$ is fixed by ${}^3\delta^D{}_3$ and the 90° cross section; the possibility of improving $P\sigma_{n\text{-}p}$ and $\sigma_{n\text{-}p}$ simultaneously using D-wave phase shifts and K_3 alone is thus virtually nonexistent. With a smaller $|{}^3\delta^P{}_0|$ and a larger ${}^3\delta^P{}_2$, these limitations would not be so severe.

At 274 Mev, the Feshbach-Lomon n-p fits are formed by supplementing the p-p phase shifts at this energy, as given in Table II, by the isotopic singlet parameters ${}^3\delta^S{}_1=16.5°$, $\epsilon_1=55.3°$, and ${}^3\delta^D{}_1=-54.3°$. Using the Feshbach-Lomon sets A and B, $\sigma_{n\text{-}p}$ and $P\sigma_{n\text{-}p}$ were calculated, and the results are shown in Fig. 4, curves A and B. The experimental data used for comparison with $\sigma_{n\text{-}p}$ are at 310 Mev[26] and at 260 Mev,[27] effectively bracketing the energy of the calculation. Neither set A nor set B yields a cross section in complete agreement with the data. The cross section calculated from fit A does not have the proper shape near $\theta=90°$, the minimum in $\sigma_{n\text{-}p}(\theta)$ falling near $\theta=80°$, while the experimental minimum appears to lie above 90°. The cross section is also somewhat too large in the angular regions $\theta<40°$ and $100°<\theta<160°$ as shown in Fig. 4. The cross section corresponding to fit B is much better for $\theta>60°$, but is too high at smaller angles. This excessive size of the small-angle cross section cannot be reduced through the addition of the available singlet phase shift K_3 alone, without simultaneously destroying the match to the data at large angles. The 310-Mev n-p polarization data[23] are not fitted at all well, as shown in Fig. 4. Some discrepancy between the theoretical polarization and the data might well be permissible since the calculated polarization is at 274 Mev, but the predicted angular distributions and magnitudes of

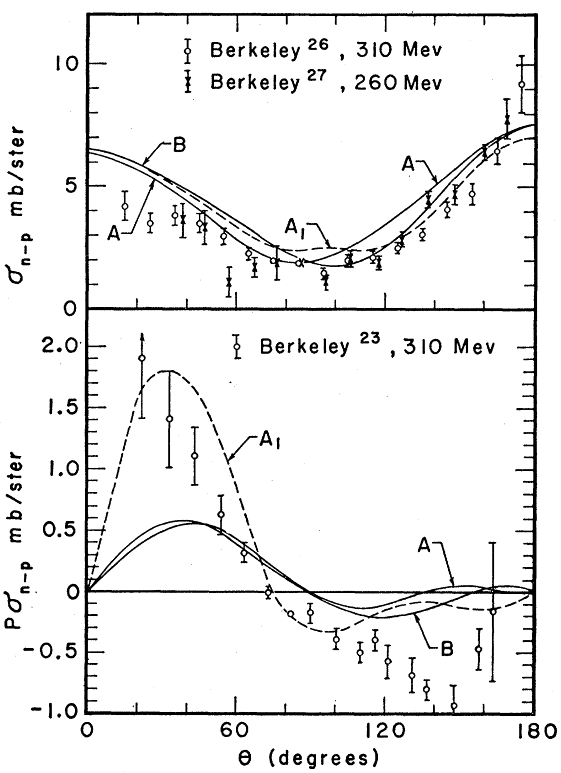

FIG. 4. The n-p cross section and polarizations at 274 Mev from the unmodified and the modified Feshbach-Lomon fit. The modifications are explained in the text.

[26] J. de Pangher, Phys. Rev. 99, 1446 (1955).
[27] Kelly, Leith, Segrè, and Wiegand, Phys. Rev. 79, 96 (1950).

$P\sigma_{n\text{-}p}$ are quite different from those observed thus far at any energy. The lack of agreement with experiment is especially striking near $\theta=140°$.

Since the Feshbach-Lomon fits at 274 Mev to the n-p polarization are definitely unsatisfactory, the main efforts to improve the fits centered on $P\sigma_{n\text{-}p}$. The effects of adding positive ${}^3\delta^D{}_3$ and ${}^3\delta^D{}_2$ phase shifts to sets A and B were to improve the small-angle polarization considerably, without significantly altering the results at large angles, even for phase shifts as large as 8°. The positive D-wave phase shifts added to $P\sigma_{n\text{-}p}$ the desired odd term, negative for $\theta>90°$, mainly through interference between the added phase shifts ${}^3\delta^D{}_2$ and ${}^3\delta^D{}_3$ and the large ${}^3S_1+{}^3D_1$ phase shifts. This term was, however, swamped near 0° and 180° by a large positive, even interference term between the added phase shifts and the isotopic triplet phase shifts. The net result was a rapid increase in $P\sigma_{n\text{-}p}$ at small angles, some improvement at $\theta=90°$, and virtually no change at large angles. The size of the added D-wave phase shifts was limited by the rapid buildup of $P\sigma_{n\text{-}p}$ for $\theta<60°$. The positive D-wave phase shifts also affected $\sigma_{n\text{-}p}$ adversely. The cross section was lowered at 180°, raised slightly at 0°, and raised significantly at 90°. This reversal of the desired asymmetry in $\sigma_{n\text{-}p}$ was not amenable to correction through the addition of K_3. These effects are seen on curves A and A_1 of Fig. 4. The behavior of sets A and B was in this respect quite similar. As it did not seem possible to improve the Feshbach-Lomon fits substantially through the addition of D-wave phase shifts alone, the effect of adding the F-wave phase shifts and, for set B, of adding ${}^3\delta^P{}_1$, was examined, with improvements in $P\sigma_{n\text{-}p}$ at large angles again essentially nil. This work was carried out using semianalytic formulas for the changes in $P\sigma_{n\text{-}p}$ at various angles caused by the addition of one or two phase shifts, and it seems fairly certain that it will not be possible to fit the n-p polarization data by adding only phase shifts for $L<4$. The upper size limits on the phase shifts tried were determined by the point at which the cross section or the small angle polarization began to go bad. The best fit obtained is shown in Fig. 4, curve A_1. This is set A with the following additions: ${}^3\delta^D{}_2=8°$, ${}^3\delta^D{}_3=8°$, ${}^3\delta^F{}_2=1°$, and ${}^3\delta^F{}_3=-6°$. A more thorough search might improve this fit somewhat, but the attainment of a really satisfactory fit to both $\sigma_{n\text{-}p}$ and $P\sigma_{n\text{-}p}$ appears very unlikely. Set B was not extensively examined, as the situation was very similar to that with set A, except that fewer parameters were available for making adjustments.

The conclusion seems apparent that the sets of phase shifts proposed by Feshbach and Lomon as a fit to the high-energy n-p cross section data are not adequate if the cross sections are examined in detail and if the n-p polarization data are utilized. It is not obvious to what extent the failure of the n-p fits is due to the inadequacy of the isotopic singlet phase shifts alone, as the considerations at 98 Mev and the foregoing examination of the p-p fits suggest that the p-p phase shifts may be largely to blame. This should be especially true at 98 Mev, where the ${}^3S_1+{}^3D_1$ phase shifts are still fairly well determined by the low-energy data,[6] and at which energy the remaining D-wave phase shifts would be expected to be small.

IV. INDEPENDENT BOUNDARY-VALUE FIT TO THE p-p DATA

An attempt has been made to obtain a better fit to the p-p data than that given by Feshbach and Lomon, using techniques of calculation similar to those employed there[6] and in Breit and Bouricius.[1] The failure of the Feshbach-Lomon fits seemed to eliminate the possibility of fitting the data with both the boundary radii and the boundary values of the logarithmic derivatives held energy-independent in the $L>0$ states. In the ${}^1S_0$ state Feshbach and Lomon themselves allowed the boundary radius to change with the energy of the incident nucleon. To allow a more general type of description, the boundary values of the logarithmic derivatives were assumed to depend on E, the energy of the incident nucleon, but the boundary radii were kept fixed. In this matter the present treatment follows closely that of Breit and Bouricius[1] who have considered both constant and energy dependent values of the homogeneous logarithmic derivative at a constant radius. The general formalism is otherwise quite similar to that of Feshbach and Lomon.

The following formalism was employed. Let ψ_J be the nucleon-nucleon wave function corresponding to total angular momentum J, considered in the center-of-mass system. Then for values of $r>r_b$, where r_b is the boundary radius, $r\psi J$ can be expressed for the general case in which coupling is present in the form:

$$r\psi_J=\mathcal{Y}_{J-1,\,J}\mathfrak{F}_{J-1,\,J}+\mathcal{Y}_{J+1,\,J}\mathfrak{F}_{J+1,\,J}, \tag{3}$$

where $\mathcal{Y}_{L,\,J}$ is the standard angular spin function as in Breit, Ehrman, and Hull.[13] The functions $\mathfrak{F}_{L,\,J}$ are linear combinations of the regular and irregular Coulomb functions F_L, G_L. The boundary values Y are then defined at $r=r_b$ by the condition:

$$\begin{pmatrix} rd\mathfrak{F}_{J-1,\,J}/dr \\ rd\mathfrak{F}_{J+1,\,J}/dr \end{pmatrix}=\begin{pmatrix} Y_{J-1,\,J} & Y_J{}^c \\ Y_J{}^c & Y_{J+1,\,J} \end{pmatrix}\begin{pmatrix} \mathfrak{F}_{J-1,\,J} \\ \mathfrak{F}_{J+1,\,J} \end{pmatrix}. \tag{3.1}$$

Here superscript c indicates coupling. The second subscript refers to the value of the total angular momentum. In the figures and tables the specification of the orbital angular momentum quantum number L is usually made by using the letters $S, P, D\cdots$ for $L=0, 1, 2, \cdots$. For uncoupled states

$$rd\mathfrak{F}_{L,\,J}/dr=Y_{L,\,J}\mathfrak{F}_{L,\,J}, \tag{3.2}$$

as in the work of Breit and Bouricius.[1] Equation (3.1) is equivalent to the procedure of Feshbach and Lomon. The quantity called by them $f_{J,\,L}$ is related to the $Y_{L,\,J}$

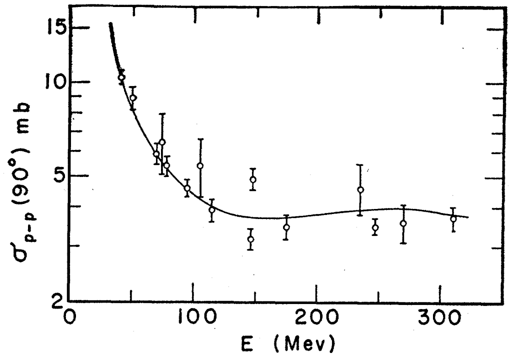

FIG. 5. The p-p cross section at $\theta=90°$ as a function of the incident proton energy, from the data of Kruse, Teem, and Ramsey.[20] The theoretical curve is for the new fit to the p-p data proposed in this paper.

used here by

$$f_{J,L}=Y_{L,J}-1 \tag{3.3}$$

for the coupled and uncoupled cases and by

$$f_J^{(t)}=Y_J^c \tag{3.4}$$

for the off-diagonal element of the boundary value matrix. The employment of the homogeneous logarithmic derivative is convenient as a safeguard regarding errors in changes of units of length. Specification of the $Y_{L,J}$ and Y_J^c and r_b at any energy determines the scattering matrix at that energy. The boundary radius r_b may depend on the state considered.

As a starting point for the numerical work of fitting the data, some hypothesis regarding the energy variation of the Y's was necessary. The assumption was made that the state of the two-nucleon system was everywhere describable by a many-body Schrödinger equation containing only energy-independent interaction terms, and that no free mesons are present. Under this assumption, and assuming no nucleon-nucleon interaction for internucleon separations $r>r_b$, it is necessary for the uncoupled case that the logarithmic derivative of the wave function for given L, J, evaluated at a fixed boundary radius, satisfy the inequality[7]

$$\frac{dY_{L,J}(E)}{dE}\leq 0. \tag{4}$$

The meaning of this condition has been discussed by Breit and Bouricius.[1] This condition was imposed upon the Y's as a tentative hypothesis, and was met by the final fit. A generalization of Eq. (4) was used for the coupled case. This generalization applies if d/dr enters only as the usual d^2/dr^2 in the diagonal terms of the coupled differential equations. A sufficient condition that the p-p system possess no bound state is that[6]

$$Y_{L,J}(0)>-L \tag{5}$$

provided no coupling is present. This condition was assumed in all cases. Further information regarding the energy dependence of the Y's was deduced from the experimental data.

Below 30 Mev, the p-p data can be fitted fairly well with almost pure $^1S_0$—Coulomb scattering, all other states contributing only slightly. The $Y_{L,J}$ for $L>0$ cannot, however, be determined in the low-energy region with available experimental information. Assuming a nucleon-nucleon interaction of the type discussed in the introduction, it would be reasonable to assume energy independence of the boundary values for low enough incident nucleon energies. At higher energies, a decrease in the boundary values with energy would be expected. At about 100 Mev, there is a marked change in the p-p scattering. Below this energy, the p-p cross section for $\theta=90°$ falls off approximately as E^{-1}, as shown in Fig. 5. At 95 Mev, there is still a definite angular dependence of the differential scattering cross section in the large angle region.[20] However, for energies between 100 Mev and 400 Mev, the 90° cross section remains approximately constant at 3.7 mb, and the differential cross section is practically isotropic for $\theta>30°$. The p-p polarization has virtually the same shape from 130 Mev to 440 Mev, but it increases slowly in magnitude. The change in the behavior of the proton-proton data near 100 Mev should perhaps be reflected in a change in the energy variation of the $Y_{L,J}$ near this energy. If, furthermore, the expansion of the cross section in Legendre polynomials as given by Feshbach and Lomon,[6] and the expansion of the polarization in powers of $\cos\theta$, as given by Hull and Saperstein,[9] are examined, it is seen that the high-energy characteristics of the data can be approximately represented by taking the sines of all phase shifts proportional to k for 100 Mev$\leq E\leq$400 Mev. The 90° cross section will then remain roughly constant, and the polarization will retain the same general shape, but its magnitude will be nearly proportional to k. The desired isotropy of $\sigma_{p-p}(\theta)$ will be obtained if K_0 and

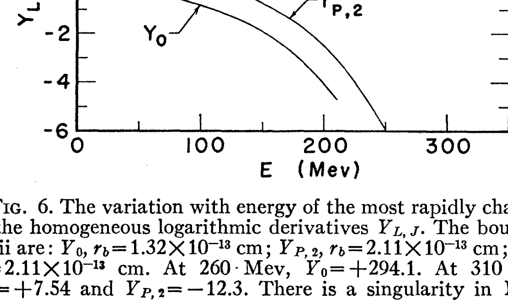

FIG. 6. The variation with energy of the most rapidly changing of the homogeneous logarithmic derivatives $Y_{L,J}$. The boundary radii are: Y_0, $r_b=1.32\times10^{-13}$ cm; $Y_{P,2}$, $r_b=2.11\times10^{-13}$ cm; $Y_{F,J}$, $r_b=2.11\times10^{-13}$ cm. At 260 Mev, $Y_0=+294.1$. At 310 Mev, $Y_0=+7.54$ and $Y_{P,2}=-12.3$. There is a singularity in Y_0 between 170 Mev and 260 Mev corresponding to a zero of the wave function at the boundary.

$\delta^P{}_0$ are the dominant phase shifts. Given one set of phase shifts in the high-energy region, this simple hypothesis for the energy dependence of $\sin\delta^L{}_J$ gives a method for finding fits at other energies. Having an approximate energy dependence for the phase shifts, the boundary radii r_b for each state can be determined by requiring the Y for that state to satisfy Eq. (4), to have a minimum variation with energy at high energies, and to extrapolate smoothly into a constant value at low energies.

The low- and high-energy regions were treated interdependently. The low-energy data do not determine the $L>0$ phase shifts, but only limits on their size. The polarization depends critically on the P and F wave phase shifts, so possible sets of these parameters may be found in the high-energy region, and from them the corresponding Y's and r_b's. With boundary radii so determined, the preliminary boundary values were changed so as to simultaneously improve the agreement with the data and to smooth out their variation with energy.

The low-energy p-p scattering data were fitted in terms of the f function of Breit, Condon, and Present.[7] Using the f function values of YSHBB[21] to compute Y_0, it was found that Y_0 decreased nearly linearly with energy in the range 14 Mev$<E<$50 Mev, while the boundary value work of Breit and Bouricius[1] showed that the experimental values of the f function for $E<14$ Mev could be satisfied with constant $r_b=1.32\times10^{-13}$ cm and constant Y_0. In order to extend the approximately linear variation of Y_0 with energy over the entire range $0\leq E\leq 50$ Mev, a small attractive potential,

$$V=V_0\exp(-r/r_0) \tag{6}$$

was assumed to exist outside the radius r_b at which the boundary value Y_0 was applied. The potential had a range $r_0=2.83\times10^{-13}$ cm; and a depth $V_0=-0.454$ Mev, chosen so that the total phase shift from the boundary condition and the potential roughly matched the experimental phase shift at $E=2$ Mev and at 10 Mev. The added potential was important only for $E<50$ Mev; the effects produced at higher energies were negligible. The final values of the f function are given in Table II, along with the experimental values and the values of Feshbach and Lomon.

It was pointed out to the authors by Professor G. Breit at the beginning of this work, that the potential energy type of nucleon-nucleon interaction may be believed more literally for larger internucleon distances, since the phenomena leading one to expect a breakdown of the potential energy description are weakest under these conditions. A treatment of nucleon-nucleon interactions which substitutes a boundary condition for the strong interactions at small distances, but still retains a potential energy tail for large separations, would appear to be more plausible than either a pure boundary-value treatment or a pure potential-energy type of treatment and was part of the suggestion just referred to. Feshbach and Lomon[6] have subsequently expounded a similar view. It is not as yet clear to what degree such potential tails in other states would affect the present energy dependence of the boundary values.

The final adjustments of the fit were made using the Coulomb interference effects in the high-energy data, and the 310-Mev triple scattering data. No exhaustive search for a "best fit" was attempted, nor were the possible effects of phase shifts for $L>4$ considered. Some further adjustment of the p-p fit may be necessary, particularly if H waves are required by the 310-Mev data. The final values of the Y's are shown in

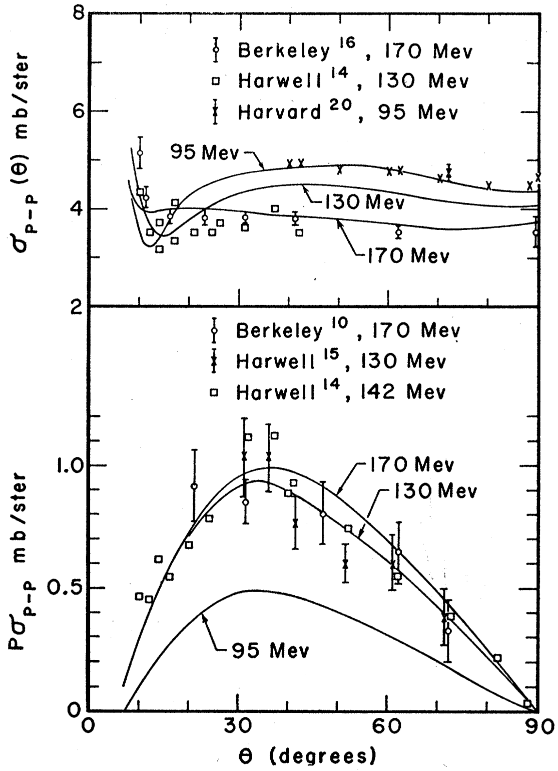

FIG. 7. The cross section and polarization at 95 Mev, 130 Mev, and 170 Mev given by new fit to the p-p scattering data proposed in this paper. The phase shifts at these energies are given in Table V. The Harnwell cross-section data are at 142 Mev rather than 130 Mev as marked on the figure.

TABLE IV. Values at the energy independent boundary radii r_b of the homogeneous logarithmic derivatives $Y_{L,J}(E)$.

$Y_{L,J}$ / r_b in. cm $\times10^{-13}$	Y_2 1.32	Y_4 2.38	$Y_{P,0}$ 2.11	$Y_{P,1}$ 1.32	Y_{2^c} 2.11
0 Mev	1.97	2.60	1.55	1.55	−0.10
100 Mev	1.97	2.60	1.52	1.41	−0.10
150 Mev	1.97	2.60	1.18	1.26	−0.10
200 Mev	1.97	2.60	0.33	1.02	−0.10
250 Mev	1.97	2.60	−0.57	0.79	−0.18
300 Mev	1.67	2.60	−1.49	0.49	−0.64

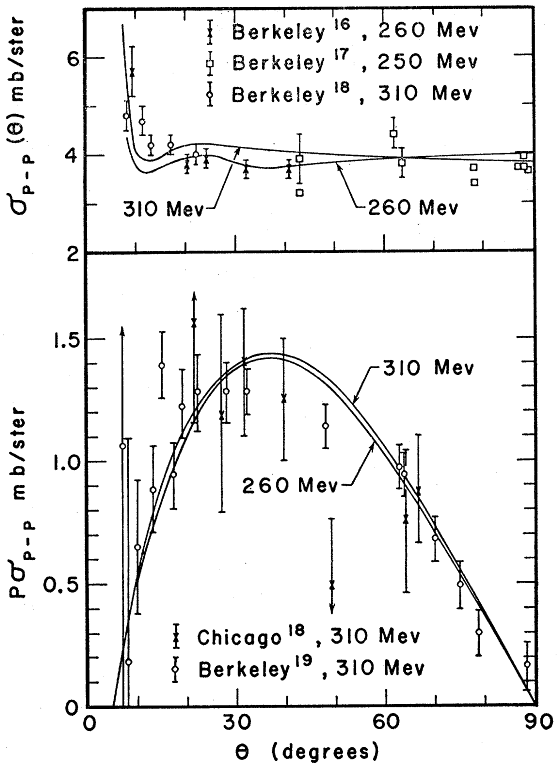

FIG. 8. The cross section and polarization at 260 Mev and 310 Mev given by the new fit to the p-p scattering data proposed in this paper. The phase shifts at these energies are given in Table V.

Fig. 6 and Table IV for the entire energy range $0 \leq E \leq 310$ Mev. The final fits to the p-p cross section and polarization data are shown in Figs. 7 and 8, and the phase shifts at the corresponding energies are given in Table V. The cross section and polarization are fitted reasonably well at all energies through 310 Mev. The Coulomb interference in the cross section is not always quite correct, but the deviation from experiment is not large. The fit to the triple-scattering experiments is not as good as the fit to the cross section and polarization experiments: the 310-Mev depolarization fit is good up to $\theta = 75°$ but fails at larger angles, and the 310-Mev rotation data are fitted rather poorly. It may well be possible to improve the 310-Mev fits through the addition of very small H waves, but this has not yet been investigated. On the whole, this new fit to the proton-proton scattering data is considerably more satisfactory than either of the Feshbach-Lomon fits. The p-p cross section is fitted at least as well as by their phase shift sets, and the angular distribution of the polarization is fitted much better. The new fit also shows good agreement with the low-energy data, while the Feshbach-Lomon fits do not.

The recent Harwell data[14] on the proton-proton polarization and cross section at 142 Mev have not been used in obtaining this new fit, but the polarization and cross section calculated from it agree fairly well with these experimental results. The new p-p phase shifts have not been extensively tested regarding their suitability as a basis for a charge independent fit to the n-p data. Some preliminary work has produced a good fit to the n-p cross section and polarization at 95 Mev, and there appears to be no major obstacle to obtaining charge independent fits at other energies as well.

V. CONCLUSION

The fits to the nucleon-nucleon scattering data obtained through a boundary value method by Feshbach and Lomon have been examined in some detail. The failure of these fits with respect to the p-p and n-p polarization experiments was brought out, and improvements were attempted. It was not found possible to bring the Feshbach-Lomon fits into agreement with both the polarization and cross-section data through the addition of those phase shifts for $L<4$ which were neglected in the original work. An independent fit to the proton-proton scattering data was then developed, using another boundary-value approach. It is not claimed that the new fit is the only one consistent with the data used. The p-p cross section and polarization are fitted quite well, but the requirements of the triple-scattering experiments are satisfied only crudely. Furthermore, as work proceeds on the extension of the p-p fit into a charge-independent n-p fit, it may become necessary to alter the p-p phase shifts and the boundary conditions somewhat, perhaps by adding supplementary potentials similar to that used for the ${}^1S_0$ state. The boundary-value method as such is completely general, being equivalent to the general phase shift method.[1] In this paper the variation of the boundary values has been restricted by conditions which can be interpreted in terms of physical pictures. It may be of interest that these restrictions still allow a fit to the data.

TABLE V. Phase shifts for the new p-p fit presented in this paper.

Phase shift \ T	30 Mev	95 Mev	130 Mev	170 Mev	260 Mev	310 Mev
K_0	50.3°	39.0°	37.0°	36.5°	47.3°	53.3°
K_2	1.3	0.93	1.51	2.00	1.41	3.00
K_4	···	0.35	0.98	2.00	2.74	0.65
δ_0^P	1.2	−14.8	−27.0	−29.0	−36.8	−43.0
$\delta^P{}_1$	1.0	2.22	1.71	1.00	−0.47	−2.00
$\delta^P{}_2$	1.5	8.8	10.0	11.5	14.2	15.6
ϵ_2	···	0.50	0.88	1.50	3.50	5.00
$\delta^F{}_2$	···	−0.27	−1.00	−1.15	−1.42	−1.53
$\delta^F{}_3$	···	−0.99	−2.60	−3.00	−3.70	−4.05
$\delta^F{}_4$	···	0.75	1.50	1.10	1.36	1.50

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Pion-Nucleon Coupling Constant and Scattering Phase Shifts*

URI HABER-SCHAIM†
Department of Physics, University of Illinois, Urbana, Illinois
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Goldberger's relations for the forward scattering of pions are used in the following way. Two linear functions of ν^2, where ν is the total pion energy in the laboratory system, are constructed from quantities taken from experiment, i.e., forward amplitudes and integrals over total cross sections. The extrapolation of one of these functions to $\nu=0$ gives $2f^2$, where f is the renormalized pion-nucleon coupling constant. Various sets of phase shifts are compared as to their compatibility with the above functions.

I. DISPERSION RELATIONS

In a recent paper Goldberger, Miyazawa, and Oehme[1] have written down dispersion relations for the forward amplitude for pion-nucleon scattering. They split the forward scattering amplitude of a pion from an isotopic spin state β to an isotopic spin state α into two parts, corresponding to no isotopic spin flip and isotopic spin flip, respectively:

$$T_{\alpha\beta}(\nu)=\delta_{\alpha\beta}T^{(1)}(\nu)+\tfrac{1}{2}[\tau_\alpha\tau_\beta]T^{(2)}(\nu), \tag{1}$$

where ν is the total energy in the laboratory system. $T^{(2)}(\nu)$, which is the amplitude we shall discuss first, can be expressed in terms of the coherent π^- and π^+ scattering amplitudes or the isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ amplitudes.

$$T^{(2)}(\nu)=\tfrac{1}{2}(T_-(\nu)-T_+(\nu))=\tfrac{1}{3}(T^{\frac{1}{2}}(\nu)-T^{\frac{3}{2}}(\nu)). \tag{2}$$

Using the relation between the imaginary part of the coherent scattering amplitude and the total cross section we then find the following equation:

$$\mathrm{Re}T^{(2)}(\nu)=\frac{2f^2}{\nu^2-\nu_B{}^2}+\frac{\nu}{2\pi^2}\times\int_\mu^\infty\left(\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'^2-\nu^2}. \tag{3}$$

f is the renormalized pion nucleon coupling constant; $\nu_B=\mu^2/2M$ where μ and M are the pion and nucleon masses, respectively. σ_- and σ_+ are the total cross sections for negative (positive) pions on protons, and $q=(\nu^2-\mu^2)^{\frac{1}{2}}$. All quantities are in the laboratory system. Now we make use under the integral of the following identity:

$$\frac{1}{\nu'^2-\nu^2}=\frac{1}{\nu'^2}+\frac{\nu^2}{\nu'^2(\nu'^2-\nu^2)},$$

and multiply both sides by $(\nu^2-\nu_B{}^2)/\nu$ to obtain

$$\left(\frac{\nu^2-\nu_B{}^2}{\nu}\right)\mathrm{Re}T^{(2)}(\nu)-\frac{\nu^2(\nu^2-\nu_B{}^2)}{2\pi^2}\times\int_\mu^\infty\left(\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'^2(\nu'^2-\nu^2)}=2f^2+\frac{\nu^2-\nu_B{}^2}{2\pi^2}\int_\mu^\infty\left(\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'^2}. \tag{4}$$

Since $\nu_B{}^2=0.55\times10^{-2}\mu^2$, we may neglect it and obtain a simplified expression:

$$\nu\,\mathrm{Re}T^{(2)}(\nu)-\frac{\nu^4}{2\pi^2}\int_\mu^\infty\left(\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\right)\frac{q'd\nu'}{\nu'^2(\nu'^2-\nu^2)}=2f^2+\frac{\nu^2}{2\pi^2}\int_\mu^\infty\frac{\sigma_-(\nu')-\sigma_+(\nu')}{2}\frac{q'd\nu'}{\nu'^2}. \tag{4'}$$

From (2) one finds for $\mathrm{Re}T^{(2)}(\nu)$:

$$\mathrm{Re}T^{(2)}(\nu)=\frac{1}{6q}\left(1+\frac{2\nu}{M}+\frac{\mu^2}{M^2}\right)\times[\sin2\alpha_1+\sin2\alpha_{11}+2\sin2\alpha_{13}+\cdots-\sin2\alpha_3-\sin2\alpha_{31}-2\sin2\alpha_{33}-\cdots]. \tag{5}$$

The α's are the phase shifts in their usual notation.

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† Present address: Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts.

[1] Goldberg, Miyazawa, and Oehme, Phys. Rev. 99, 986 (1955).