Application of an Artificial Satellite to the Measurement of the General Relativistic "Red Shift"*

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In view of the paucity of experimental tests for the general theory of relativity, it is desirable to consider the uses to which a satellite vehicle could be put. The advance of the perigee is calculated similarly to the perihelion advance of mercury; it amounts to only 15 seconds of arc per year. However, the effect on a satellite clock is large and could be measured. With respect to an earth clock it is calculated to be a "red shift" for low-altitude orbits, zero shift for an orbit at one-half the earth's radius, and a "violet shift" for higher altitudes, where it approaches 7×10^{-10}.

Some experimental schemes for the measurement of the clock shift are discussed; a counting technique seems to be best suited since it is capable of higher ultimate accuracy and avoids signaling problems during intercomparison arising from the motion of the satellite.

INTRODUCTION

ONE of the important tasks of observational astronomy is the verification of predictions of physical theories. The theory of general relativity, because of its very nature, can at the present time only be verified by certain astronomical measurements, the most important of these being (i) the advance of the perihelion of the planet Mercury, (ii) the gravitational deflection of light rays, and (iii) the gravitational displacement of spectral lines. These are the tests which apply to Einstein's theory of *local* gravitation. The consequences of the theory of general relativity to cosmology would lead into a completely different range of observations, into which we will not enter here. It is important, however, to point out that the observational discovery of the extragalactic red shifts which has led to a consideration of nonstatic world models is a Doppler red shift giving evidence of radial velocities, and is not related to the red shift with which we will be concerned in the main body of this paper.

The perihelion and light deflection effects of general relativity are not entirely crucial tests; G. J. Whitrow in particular has called attention to the fact that other theories lead to the same results, for example, Whitehead's theory, which differs from Einstein's in major respects. Unfortunately, however, the tests which Einstein's theory has received so far are not all completely convincing. We will therefore first review the present status of the verification of the three tests.

A planet may be considered a free particle and therefore describes a space-time trajectory given by the equation for a geodesic. If one solves this equation using the Schwarzschild line element, one arrives at a force law which differs from the inverse square law by a small term. It is well known that this will lead to a nonreentrant orbit and therefore to a rotation of the line of apsides of the ellipse. The fraction of a revolution through which the perihelion rotates per revolution can be derived; it is largest for the planet Mercury and amounts to 43.03 ± 0.93 seconds of arc per century. The observations of the planet are extremely difficult; in addition, its orbit is subject to many perturbations by other planets, and to other classical effects. Clemence has summarized the current status and gives the observed perihelion advance as 5599.74 seconds per century; of this amount 42.56 ± 0.94 cannot be ascribed to classical effects.[1]

We have calculated the advance of perigee of a satellite in an orbit in the earth's gravitational field near the earth.[2] While it is of the order of 30 times greater than the Mercury perihelion advance (see Appendix II), it would be difficult to eliminate the various classical effects which enter. The main effect is caused by the nonspherical shape of the earth.[3] In addition there are many extraneous effects producing minute perturbations of the satellite's orbit. It is conceivable, however, that means may be found for measuring the general relativistic effect on the satellite orbit, although this seems very unlikely for a minimum satellite close to the earth.

The second effect, the deflection of light rays, should lead to a bending near the limb of the sun of 1.75 seconds of arc as calculated by Einstein, compared to half that amount as calculated by the Newtonian theory of gravitation. The observations of the deflection seems to indicate somewhat higher values although a recent observation by Van Biesbroeck gives a value which agrees with Einstein's figure. It is considered that the verification is not yet fully made.

The theory of relativity gives a gravitational displacement of spectral lines by the amount $\Delta\lambda/\lambda=GM/rc^2$. This is perhaps not such a crucial test; the gravitational red shift is also derived if one assumes that the photon has mass according to the relation $m=h/\lambda c^2$; and also $V=-GM/r$, the ordinary Newtonian gravitational

* Presented at the Tenth Anniversary Meeting of the Upper Atmosphere Rocket Research Panel, Ann Arbor, Michigan (January, 1956).

[1] G. M. Clemence, Revs. Modern Phys. **19**, 361 (1947).
[2] See also L. LaPaz, Publs. Astron. Soc. Pacific **66**, 13 (1954); J. J. Gilvarry, Phys. Rev. **89**, 1046 (1953).
[3] Indeed the observed perigee motion may be used to determine the figure of the earth.

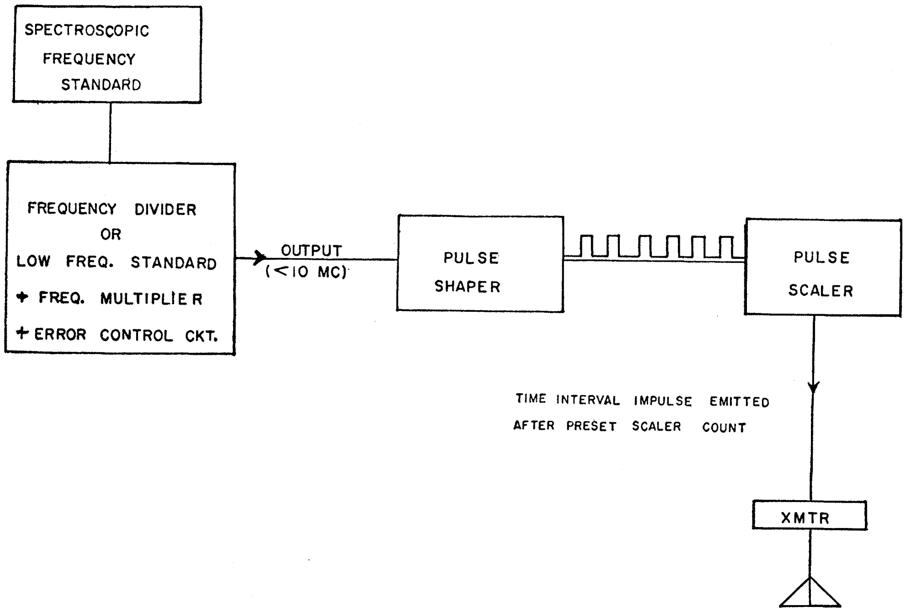

FIG. 1. Experimental scheme for measuring time shift.

potential. The reddening of spectral lines can then be viewed as the loss of energy by the photons as they leave the region of high gravitational potential. For the sun the shift would be 2.12×10^{-6}. Observationally, however, the predicted value is observed only at the extreme limb of the sun while the value over most of the disk, in contradiction to the theory, is considerably less.[4] It may be, as has been suggested by St. John, that the red shift is masked by violet shifts due to upward radial currents of extremely high velocities. The red shift is also too small for the white dwarf Sirius *B*. The result there hinges on our knowledge of its radius. Recent data of Kuiper indicate that this radius may be less than 1% of the sun's radius; the observed red shift is then too small by about a factor 3.[4]

SATELLITE TESTS

All of these considerations speak very strongly for additional tests of the general theory of relativity. An artificial satellite provides the possibility for a test of the gravitational red shift. Before discussing the experiment, we must first of all derive the shift experienced by a clock located in the satellite, relative to a clock located on the earth's surface. Details of this derivation are given in Appendix I. The result is as follows:

$$\Delta\equiv\frac{dt_{\text{sat}}-dt_{\text{earth}}}{dt_{\text{earth}}}\sim7\times10^{-10}\left(\frac{1.5}{1+h/R_E}-1\right).$$

It might be thought off-hand that since the satellite is freely falling and therefore forms an inertial system, no red shift will be experienced by its clock, which would therefore run at a faster rate (or be more violet) than an earth clock. Contrary to this expectation, the satellite clock runs slower than the earth clock until the satellite altitude h reaches the value $\frac{1}{2}R_E$. Beyond this, the satellite clock should run faster than the earth clock. It should be noted that the effects are quite large and therefore conceivably subject to observation.

EXPERIMENTAL SCHEME FOR TIME SHIFT MEASUREMENTS

The method used in astronomy employs simultaneous comparison of two spectral lines. We cannot use this method here because it has limited accuracy and because the effect here is much smaller. Instead we employ a digital or counting method which is capable of much greater accuracy since long integration times can be used; this demands that the clock have a frequency in the radio region and not in the optical region. The digital method eliminates also the problem of the Doppler effects of the signals used to exchange the information. Instead, we will consider that the satellite clock has associated with it a scaler which counts its ticks. After a certain predetermined number of ticks have accumulated, the scaler sends a short signal to the ground. If the accumulation time or running time of the clock is very long, then the detailed means of comparison between the satellite scaler and the ground become quite unimportant.

In order to achieve a high degree of frequency stability, it is probably necessary to use an atomic clock. Such clocks have been constructed based on the absorption of ammonia, with a stability of 1 part in 10^8.[5] Atomic beams would be even more stable; they can

[4] O. Struve, Sky and Telescope **13**, 225 (1954). See also C. E. Moore, in *The Sun*, edited by G. P. Kuiper (University of Chicago, Chicago, 1953), p. 186.

[5] H. Lyons, Ann. N. Y. Acad. Sci. **55**, 831 (1952).

give even higher Q's since they are not subject to collision and Doppler broadening. Cesium and thallium clocks[6] are said to be capable of accuracies of better than 1 part in 10^{12}.

Whatever technique is employed, the problem is one of distinguishing a random error from the systematic relativistic drift. If the errors are truly random, then it is only a matter of time before the real drift can be seen above the noise. For the sake of illustration, consider the cesium line which has a frequency of 9192.632 Mc/sec or $\sim 10^{10}$ cps, with an error of 1 part in 10^9, i.e., ± 10 cps. The relativistic drift is to be measured to ~ 1 cps. After 10^4 seconds of running time, the drift is 10 000 cycles, while the standard error, which increases as the square root of time, is ± 1000. If the atomic clock is frequency-divided down to a 10-Mc oscillator whose pulses are then scaled (see Fig. 1), our count will be out by 10 pulses because of the relativistic drift. In order to transmit this information accurately, we must use a 10^{-7}-sec pulse. The problem becomes much easier if we are willing to wait 10^5 seconds or even longer; we then have 1 μsec or longer to transmit the timing interval impulse.†

ACKNOWLEDGMENTS

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APPENDIX I. RELATIVISTIC EFFECTS OF SATELLITE CLOCKS

We will treat this problem by considering an observer far away from the earth viewing two physically identical clocks, one located on the earth's surface, the other in the satellite.

The element for interval in covariant form is

$$ds^2 = g_{\mu\nu} d\xi^\mu d\xi^\nu. \tag{1}$$

For our observer far from the earth, $g_{44}=1$, and $ds = dt_{\rm obs}$, i.e., the proper time interval ds of the clock at his location is exactly equal to his coordinate time interval dt. We now wish to calculate the relation between ds and $dt_{\rm sat}$, as well as ds and dt_E.

To do this we use the expression for ds of the Schwarzschild line element (see Appendix II), and obtain

$$\left(\frac{ds}{dt}\right)^2 = 1 - \left[\frac{1}{1-2m/r}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2 \sin^2\theta\left(\frac{d\phi}{dt}\right)^2 + \frac{2m}{r}\right] \tag{2}$$

as the relation between proper time (measured by a clock on, e.g., the satellite) and the coordinate time dt. We now assume a circular orbit for the satellite, with $r=R_E+h$, and located in the $\theta=\pi/2$ plane, so that $\sin\theta=1$ and $d\theta^2/dt^2=0$; then

$$\left(\frac{ds}{dt_{\rm sat}}\right)^2 = 1 - (R_E+h)^2\left(\frac{d\phi}{dt}\right)^2 - \frac{2m}{R_E+h}. \tag{3}$$

For the earth clock at sea level, $r=R_E$, and

$$(ds/dt_E)^2 = 1 - 2m/R_E. \tag{4}$$

Therefore

$$(dt_{\rm sat}/dt_E)^2 = (1-2m/R_E) \times [1-(R_E+h)^2(d\phi/dt)^2 - 2m/(R_E+h)]^{-1}. \tag{5}$$

Now since $GM_E m/r^2 = mv^2/r = m\omega^2 r$, we have $r^2\omega^2 = GM_E/r$; hence $(R_E+h)^2(d\phi/dt)^2 = m/(R_E+h)$. Equation (5) now becomes

$$(dt_{\rm sat}/dt_E)^2 = (1-2m/R_E)[1-3m/(R_E+h)]^{-1}. \tag{6}$$

We can write approximately

$$dt_{\rm sat}/dt_E = (1-m/R_E)[1-\tfrac{3}{2}m/(R_E+h)]^{-1}. \tag{7}$$

We can now define

$$\Delta \equiv (dt_{\rm sat}-dt_E)/dt_E = (dt_{\rm sat}/dt_E) - 1.$$

Then from Eq. (7)

$$\Delta = m\{[1.5/(R_E+h)] - 1/R_E\}[1-1.5m/(R_E+h)]^{-1} \approx m\{[1.5/(R_E+h)] - 1/R_E\}. \tag{8}$$

Numerically,

$$\Delta = 5.98\times 10^{27}\,\text{g}\cdot(6.37\times 10^8\,\text{cm})^{-1} \times (1.35\times 10^{28})^{-1}[1.5(1+h/R_E)^{-1}-1] \sim 7\times 10^{-10}[1.5(1+h/R_E)^{-1}-1].$$

It is plotted as a function of h/R_E in Fig. 2. We see that $\Delta = +3.5\times 10^{-10}$ for a satellite in an orbit near sea level; $\Delta=0$ when $h=\frac{1}{2}R_E$. For higher altitudes, Δ is negative and it can become as large as -7×10^{-10}. In other words: with respect to the earth clock the satellite ticks more slowly for low-altitude orbits, at the same rate when the orbit altitude is one-half the earth's radius ($h=3135$ km, about 2000 miles), and faster at higher altitudes. This result is seen to be identical to the one obtained from the usual gravitational red-shift, to which is added the relativistic (second-order) Doppler shift.

[6] J. Zacharias (private communication).

† The uncertainty in the determination of the satellite's position corresponds to a timing error of $10^{-8}-10^{-1}$ sec.

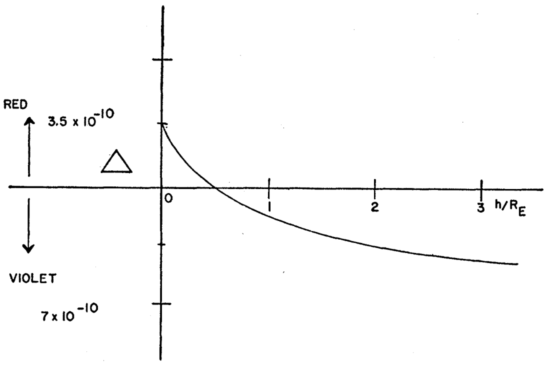

FIG. 2. Satellite clock shift *vs* satellite altitude.

The calculation for a highly elliptical orbit follows the general scheme above but is somewhat more complicated since $d\phi/dt$ and r are no longer constant. A qualitative picture can show the advantages of a highly elongated elliptical orbit:

(1) Since it spends very little of its time in the vicinity of the earth, it will show a large "violet" shift.

(2) Only a small amount of additional velocity applied at a low altitude can produce such an orbit.

(3) The satellite returns to a sufficiently low perigee altitude to make data transfer easy.

APPENDIX II. RATE OF ROTATION OF THE LINE OF APSIDES OF A MINIMUM SATELLITE

The motion of a free particle, such as a satellite, will in accordance with the theory of relativity be given by the geodesic

$$\frac{d^2\xi^\sigma}{ds^2}+\left\{\begin{matrix}\sigma\\ \mu\nu\end{matrix}\right\}\frac{d\xi^\mu}{ds}\frac{d\xi^\nu}{ds}=0. \tag{1}$$

Considering the earth to be a mass point, we can evaluate the Christoffel three-index $\left\{\begin{matrix}\sigma\\ \mu\nu\end{matrix}\right\}$ using the expression for the line element given by Schwarzschild:

$$ds^2=-dr^2(1-2m/r)^{-1}-r^2d\theta^2 -r^2\sin^2\theta d\phi^2+(1-2m/r)dt^2. \tag{2}$$

In considering effects over a very small region, the term involving the cosmological constant $(-\frac{1}{3}\Lambda r^2dt^2)$ can be omitted. Note also that in free space, i.e., for $m/r=0$, Eq. (2) reduces to the usual expression from special relativity. Evaluating Eq. (1) and choosing $\theta=\pi/2$, leads to

$$\frac{d^2(1/r)}{d\phi^2}+\frac{1}{r}=\frac{m}{h^2}+\frac{3m}{r^2}, \tag{3}$$

where $m=GM_E$, $h=r^2d\phi/ds=$ const (integral of angular momentum), and $ds=$ element of proper time as measured by satellite clock.

Note that $3m/r^2 \ll m/h^2$ and is absent in the Newtonian orbit equation. Its presence causes the orbit to be nonreentrant. Omitting $3m/r^2$, we obtain the well-known orbit expression

$$1/r=mh^{-2}[1+\epsilon\cos(\phi-\omega)], \tag{4}$$

where ϵ is the eccentricity of the ellipse, while ω determines the perigee position. Substituting back into (3), we obtain

$$\frac{1}{r}=mh^{-2}\left[1+\epsilon\cos\left(\phi-\omega-\frac{3m^2}{h^2}\phi\right)\right]. \tag{5}$$

The rotation of the line of apsides per revolution, i.e., $\phi=2\pi$, is then given by the difference $\delta=2\pi(3m^2/h^2)$. For an orbit at 330 km, the numerical answer is

$$\delta=6\pi\left(\frac{5.98\times10^{27}\text{ g}}{1.35\times10^{28}}\right)^2 \times\frac{(5400\text{ sec})(3.3\times10^{-11})^{-2}}{(6.7\times10^8\text{ cm})^4(2\pi)^2}=1.24\times10^{-8}$$

radians per revolution (about 90 minutes). This works out to 7.25×10^{-5} rad (or 15 sec of arc) per year, about 35 times as large as the advance of perihelion of the planet Mercury. It is of course much smaller than the effects caused by the oblateness of the earth. In any case, we observe only the product $\epsilon\delta$. For $\epsilon\sim0.1$, we would observe an advance of 1.5 sec per year.