Nucleon Transfer and Virtual Coulomb Excitation*

G. BREIT AND M. E. EBEL
Yale University, New Haven, Connecticut
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A semiquantitative explanation of the experiments of Reynolds and Zucker on transfer reactions arising in the collision of N^{14} with N^{14} is arrived at, making use of the effect of Coulomb excitation to virtual levels. These excitations are shown to be important even though a permanent separation of nuclei in Coulomb excited states may be energetically impossible. The process is of sufficient generality to have application to other similar reactions. An analysis of the experimental material shows that the process is probably important in the case of the $N^{14}+N^{14}$ reactions for all energies below that required for contact between nuclear surfaces and that it is probably the only important process at bombarding energies below 15 Mev.

I. INTRODUCTION AND NOTATION

AN attempt[1] to explain the experimental results of Reynolds and Zucker[2] on $N^{14}+N^{14}$ transfer reactions on the basis of tunneling of nuclear particles from one nucleus to the other gave a too rapid variation of the reaction cross section with energy, the disagreement with experiment being by a factor of about 35. It also gave angular distributions in qualitative disagreement with experiment. In both cases the results are independent of assumptions regarding the nuclear radius as long as the radius is not so large as to give contact of nuclear surfaces. Since it is very improbable that the latter may take place, one has to suppose that some other process is modifying the transfer process. The modification resulting from static distortion of the shape of the nucleus has been estimated and according to reference 1 can at most reduce the factor 36 to 18. It has been concluded[1] therefore that some other process is taking place. The tunneling from the ground state just referred to has made little use of specialized assumptions regarding the nature of nuclear forces. The essential assumption made is that they have a short range. For this reason one expects a strong drop of yield at low energies and an associated dominance of large deflections. The assumptions made regarding the character of the ground configuration enter the calculations referred to mainly in the determination of numerical energy-independent coefficients. These assumptions have, to be sure, been too specialized, the existing evidence being in better agreement[3] with an intermediate rather than extreme $j-j$ coupling model. Nevertheless, the only essential change which would affect the character of the energy or angular dependence of the cross section is a large change in the nuclear radius. The change has to be so large that at 10-Mev bombarding energy there is an appreciable force on a nucleon in one nucleus due to nucleons in the other. Such a change would make it incorrect to use the barrier penetration factor for leakage through regions of negative kinetic energy of the transferred particle. It appears unlikely that such phenomena can take place as a result of specifically nuclear forces for energies at which the N^{14} nuclei are never closer than 1.37×10^{-12} cm to each other.

The electric field of one nucleus at the other brings in, however, a long-range interaction without making radical changes in the assignment of nuclear radii. It offers a possibility for increasing the yield at low energies and for distant collisions provided it is capable of exciting the nucleus on which it acts to a state in which the wave function of the nucleon to be transferred decreases appreciably more slowly outside the nucleus than in the ground state. The present paper is concerned with this process.

In Sec. II the relative coordinates appropriate to the problem are considered. The Hamiltonian is written in forms showing the way in which the terms group themselves so as to correspond to the introduction of internal functions, the energy responsible for virtual Coulomb excitation, and the perturbing part responsible for the transfer. In Sec. III the separation of variables is discussed and approximations simplifying the treatment are introduced. In Eqs. (3.4) and (3.5) there occurs an initial state wave function Ψ^{CN}, the transfer from which takes place on account of the perturbing effect of V_1 according to Eq. (3.5) or, more explicitly, Eq. (4). The Ψ^{CN} contains parts corresponding to the ordinary Coulomb wave describing the relative motion of two charged particles and other parts corresponding to Coulomb excitation having taken place in one of them. The calculation of the latter is made in a special but reasonably typical case in the remainder of this section beginning with Eq. (5). The function Ψ^{CN} is replaced by a somewhat simpler quantity ψ which does not refer to the whole system but applies to one of the heavy aggregates having been replaced by a point charge. The internal coordinates of the particle doing the Coulomb excitation are clearly unessential to this part of the consideration. The result

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[1] G. Breit and M. E. Ebel, Phys. Rev. **103**, 679 (1956).
[2] H. L. Reynolds and A. Zucker, Phys. Rev. **101**, 166 (1956).
[3] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953); B. Jancovici and I. Talmi, Phys. Rev. **95**, 289 (1954).

of this work is Eq. (7.5), which gives the Coulomb-excited wave at arbitrary points in configuration space.

In Sec. IV the two special cases of excitations resulting in bare dissociation and in the formation of states lying high in the continuum are considered. The sum over partial waves occurring in Eq. (7.5) is carried out making some approximations and results in Eq. (9.2) for one case and Eq. (11.3) for the other. In Sec. V the wave mechanical calculations are compared with corresponding estimates by means of a semiclassical theory (SCT) in which the motion of the heavy aggregates is considered by means of classical mechanics while the excitation process is treated quantum-mechanically. An approximate agreement between the two ways is found. Numerical estimates are then made employing the f sum rule for dipole transition probabilities along the lines of Levinger and Bethe.[4] This procedure makes the conclusions essentially independent of the specialization to s-p transitions which was made in the preceding section. It is then found that with a conservative estimate of the chance of transfer due to virtual state formation one may expect a factor of about 5 in favor of transfer by this process in a comparison with tunneling from the ground state. This discussion is found in connection with Eq. (15). A set of limitations and omissions in the treatment is discussed in items (a) to (i) in the text following Eq. (15). In view of the difficulty of taking into account all of these omissions properly, the remainder of the paper is concerned with an attempt to infer from the experimental material the relative importance of the factors entering the interpretation which have not been explicitly calculated. It is found that reasonable agreement with experiment is obtainable by bringing in some of the effects of different degrees of adiabaticity of the collisions depending on the collision time. This view is not in disagreement with the approximate resonance caused by the nearly equal binding energy of the last neutron in N^{14} and N^{15}. The part of the continuum mainly responsible for the excitation matters in determining the energy dependence and the angular distribution.

The phenomenon of virtual state formation by Coulomb excitation is a special case of a more general situation which was discussed by one of the writers[5] in connection with Li^7 reactions. According to this view, proximity of the two colliding nuclei can cause transitions to other configurations within one of the nuclei which can influence the course of the reaction and in particular can give effects somewhat similar to an effective increase in nuclear radius, especially for improbable reactions. Qualitatively the effect of the virtual-state formation worked out in the present paper has resulted in the increase in the reaction cross section which is large for large impact parameters, as though the nuclear radius were increased.

From a formal viewpoint, the consideration of virtual-state formation differs from the more familiar type such as enters the theory of bremsstrahlung[6] in that in the present case the starting point is very far from being a plane wave. It is in fact a wave highly distorted by the action of the Coulomb field. On account of the fact that the masses of the colliding nuclear aggregates are large, one can picture the collision process approximately by means of classical mechanics. In doing so one arrives at an approximate and at first sight paradoxical situation in which the energy required for virtual-state formation can be borrowed from a classical kinetic energy in an amount larger than the kinetic energy. This paradox is apparent rather than real because the classical orbit picture is only a convenient substitute for the more complete consideration of the wave function.

The present paper is not intended as a quantitative treatment. As has been mentioned many effects have been left out of the considerations. In addition to those discussed below there are several others such as that of higher multipoles which is likely to become more important at short distances and neglect of effects of particle identity. A complete treatment is probably impossible without a thorough study of the photodisintegration cross section, and the analysis of the experiments would be more certain with the addition of measurements of angular distributions at lower energies. It appears probable nevertheless that virtual Coulomb excitation plays perhaps the main role in the transfer reactions for $N^{14}+N^{14}$ at energies below the Coulomb barrier and probably also in transfer reactions for other similar nuclei.

The following symbols are used in the paper:

$\mathbf{r}_p$ = coordinate of proton with respect to nucleus to which it is initially attached.
$\mathbf{r}$ = relative coordinate for motion of nuclei.
Z_1e, Z_2e = charges of colliding nuclei in initial state.
m = reduced mass for the colliding particles in initial state.
M = reduced mass for the relative motion of the proton with respect to the nucleus to which it is initially attached.
Ψ^{CN} = wave function for the system neglecting possibility of transfer.
$\Psi^{(1)}$ = first order correction to Ψ^{CN} resulting from transfer.
ψ = wave function describing proton and relative motion of nuclei.
ψ^c = Coulomb wave function for motion of nuclei.

[4] J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950).

[5] G. Breit, Rev. Sci. Instr. **9**, 63 (1938). On hearing an account of the present work and in reply to a question, Professor E. P. Wigner recalled an unpublished consideration of S. M. Dancoff [U. S. Atomic Energy Commission Report AECD-2853, May, 1950] regarding spontaneous alpha emission in which an excitation of the residual nucleus of a temporary type, and therefore somewhat similar to that considered here, has been studied.

[6] H. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).

v_i, w_μ = wave functions describing initial and excited states of proton.
E_w = energy of states w_μ measured with respect to v_i.
v, v_w = relative velocities of nuclei in initial and final states, respectively.
$\eta = Z_1Z_2e^2/\hbar v$.
$\hbar k$, $\hbar k_w$ = relative momenta of nuclei in initial and final states, respectively.
$e_L = [\Gamma(L+1+i\eta)/\Gamma(L+1-i\eta)]^{\frac{1}{2}}$.
$Y_{l\mu}$ = spherical harmonic of order l.
$b_L{}^\mu$ = coefficient used in the addition of angular momenta, defined by Eq. (5.9).
F_L, G_L = respectively, the regular and irregular solutions of the differential equation for $r\times$radial function in a Coulomb field.
$C_{ow} = \{2\pi\eta_w/[-1+\exp(2\pi\eta_w)]\}^{\frac{1}{2}}$.
$a_w = \hbar^2/(Z_1Z_2me^2)$ is the Bohr length for the collision.
$d = (ra_w/2)^{\frac{1}{2}}$.
$\beta^2 = (2m/\hbar^2)(E_w-E)$.
r_0 = separation of nuclei at the perihelion of the classical orbit.
$2a'$ = distance of closest approach of the nuclei.
ϵ, w = parameters of classical orbit.
$\xi = E_w\eta/2E$.
b = nuclear radius.
p = impact parameter of the collision.

II. RELATIVE COORDINATES

On account of the change in the grouping of the particles which results as a consequence of the transfer, it is necessary to present in detail the grouping of terms in the Hamiltonian made in treating the relative coordinates. A particle of mass m_3 is supposed to be attached initially to a mass m_2 and to be transferred, as a result of the collision, to mass m_1. The masses m_1 and m_2 are those of nuclear aggregates such as N^{13} and N^{14}. One introduces for the final state the coordinates

$$\mathbf{r}_{31}=\mathbf{r}_3-\mathbf{r}_1,\quad \mathbf{R}_{13}=\mathbf{R}_{31}=(m_1\mathbf{r}_1+m_3\mathbf{r}_3)/(m_1+m_3),$$
$$\mathbf{R}=(m_1\mathbf{r}_1+m_2\mathbf{r}_2+m_3\mathbf{r}_3)/(m_1+m_2+m_3), \quad (1)$$

as well as momenta

$$\mathbf{p}_{31}=-\mathbf{p}_{13}=\mu_{13}[(\mathbf{p}_3/m_3)-(\mathbf{p}_1/m_1)]$$
$$=(m_1\mathbf{p}_3-m_3\mathbf{p}_1)/(m_1+m_3),$$
$$\mathbf{p}_{2,13}=[m_2(m_1+m_3)/(m_1+m_2+m_3)]$$
$$\times[(\mathbf{p}_2/m_2)-(\mathbf{p}_1+\mathbf{p}_3)/(m_1+m_3)]$$
$$=[(m_1+m_3)\mathbf{p}_2-m_2(\mathbf{p}_1+\mathbf{p}_3)]/(m_1+m_2+m_3), \quad (1.1)$$

with

$$\mu_{13}=m_1m_3/(m_1+m_3) \quad (1.2)$$

standing for the reduced mass of 1 and 3. The alternative expressions for the momenta are given so as to show the entrance of velocities as well as the direct meaning in terms of momenta. The kinetic energy of the system, discarding the term depending on the total momentum, is

$$T=p_{31}{}^2/(2\mu_{13})+p_{2,13}{}^2/(2\mu_{2,13}),$$

where

$$\mu_{2,13}=m_2(m_1+m_3)/(m_1+m_2+m_3)$$

is the reduced mass for the relative motion of 2 with respect to the center of mass of m_1 and m_3. The Hamiltonian may be written as

$$H=p_{13}{}^2/(2\mu_{13})+p_{2,13}{}^2/(2\mu_{2,13})$$
$$+V_{23}(\mathbf{r}_{2,13}+m_1\mathbf{r}_{13}/(m_1+m_3))+V_{13}(\mathbf{r}_{13})$$
$$+V_{12}(m_3\mathbf{r}_{13}/(m_1+m_3)-\mathbf{r}_{2,13})+H_{11}+H_{22} \quad (1.3)$$

with $\mathbf{r}_{2,13}$ standing for the displacement vector from the center of mass of 1 and 3 to 2. The last two terms in (1.3) represent respectively the parts of the Hamiltonian pertaining respectively to the internal coordinates of particles 1 and 2.

Since in the applications the ratios m_3/m_1, m_3/m_2 are reasonably small, terms of order m_3/m_1, m_3/m_2 will be neglected and the Hamiltonian will be written in the approximate forms

$$H_f\cong p_{13}{}^2/(2\mu_{13})+p_{2,13}{}^2/(2\mu_{2,13})+V_{23}(|\mathbf{r}_{2,13}+\mathbf{r}_{13}|)$$
$$+V_{13}(r_{13})+V_{12}(|\mathbf{r}_{2,13}|)+H_{11}+H_{22} \quad (1.4)$$

for the final state and

$$H_i\cong p_{23}{}^2/(2\mu_{23})+p_{1,23}{}^2/(2\mu_{1,23})+V_{23}(r_{23})$$
$$+V_{13}(|\mathbf{r}_{1,23}+\mathbf{r}_{23}|)+V_{12}(|\mathbf{r}_{1,23}|)+H_{11}+H_{22} \quad (1.5)$$

for the initial. The unessential specialization consisting in the replacement of displacement vectors by their absolute value is now being made. For simplicity the dependence on the vector directions has been suppressed. It may be introduced without inherent complication but at the expense of a more complicated notation.

In each of Eqs. (1.4) and (1.5) there enters a different set of relative coordinates. In Eq. (1.4) particle 3 is considered as attached to m_1 and the mass m_2 is considered as moving with respect to the center of mass of m_1+m_2. In these respects the situation is as in the exact Eq. (1.3) but an approximation is made in (1.4) regarding the potential energies V_{23} and V_{12}. The effect of the approximation is that the problem contains only the coordinates $\mathbf{r}_{13}$ and $\mathbf{r}_{2,13}$ in addition to the internal coordinates of aggregates m_1 and m_2. The coordinates $\mathbf{r}_{13}$, $\mathbf{r}_{2,13}$ enter V_{23} in a combination such as would enter a problem in which a particle p is displaced from m_2 by a vector $\mathbf{r}_p$ and from m_1 by $\mathbf{r}_p-\mathbf{r}$ which corresponds to setting $\mathbf{r}_{13}=\mathbf{r}-\mathbf{r}_p$, $\mathbf{r}_{2,13}=-\mathbf{r}$. With these identifications the potential energy terms $V_{23}+V_{13}+V_{12}$ occur in Eq. (1.4) as though one were dealing with the particle p in the fields of m_1 and m_2 and as though the effective mass for $\mathbf{r}$ were $\mu_{2,13}$ and that for $\mathbf{r}-\mathbf{r}_p$ were μ_{13}. For the initial motion the approximation of Eq. (1.5) corresponds to $\mathbf{r}_{32}=\mathbf{r}_p$, $\mathbf{r}_{1,23}=\mathbf{r}$, $\mathbf{r}_{1,23}+\mathbf{r}_{23}=\mathbf{r}-\mathbf{r}_p$; the effective mass for $\mathbf{r}_{23}$ is μ_{23}, that for $\mathbf{r}$ is

$\mu_{1,23}$. Equations (1.4), (1.5) could have been set down on an intuitive basis but the approximations made would then not be as readily accessible to inspection. Since at the larger distances between 3 and the other two particles the surviving interaction is the Coulombic one, the important terms in the present problem are V_{23}, V_{13}. For these the important interaction energy depends on distance only and the simplified notation is therefore justifiable.

The charges of the colliding nuclei are taken to be Z_1e, Z_2e, respectively. For the sake of concreteness particle 3 will be taken to be a proton and will be referred to by the subscript p. The proton is attached initially to m_2. In the $\mathbf{r}$, $\mathbf{r}_p$ notation,

$$\begin{aligned} V_{23}(r_{23}) &= (Z_2-1)e^2/r_p + V_2(r_p), \\ V_{13}(r_{13}) &= Z_1e^2/|\mathbf{r}_p-\mathbf{r}| + V_1(|\mathbf{r}-\mathbf{r}_p|), \\ V_{12}(r_{12}) &= Z_1(Z_2-1)e^2/r, \end{aligned} \tag{2}$$

where

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2.$$

Here V_1, V_2 are potentials of non-Coulombic origin. For the initial state, in terms of the identifications discussed right after Eq. (1.5), one may arrange the contributions to V as follows

$$\begin{aligned} V=\frac{Z_1Z_2e^2}{r}+Z_1e^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}_p|}-\frac{1}{r}\right) \\ +\left(V_2(r_p)+\frac{(Z_2-1)e^2}{r_p}\right) \\ +V_1(|\mathbf{r}-\mathbf{r}_p|)+V_{11}+V_{22}. \end{aligned} \tag{2.1}$$

The first term represents the Coulomb energy of two point charges Z_1e, Z_2e at a distance r from each other. It may be used in the definition of the Coulomb functions for the initial stage and may be removed from the wave equation by employing wave functions containing the Coulomb functions as factors. The second term, containing Z_1e^2, brings in a potential energy such as occurs in the theory of Coulomb excitation and will be referred to as the Coulomb excitation potential. It gives rise to Coulomb excitation of proton p by Z_1e while p is attached to m_2. The third term contributes to the potential energy of p in the field of m_2 and has to be taken into account in the definition of the eigenfunctions describing the condition of p before transfer. The term $V_1(|\mathbf{r}-\mathbf{r}_p|)$ brings about the possibility of transfer.

For the final grouping of particles the following arrangement of terms in the potential energy is natural.

$$\begin{aligned} V=\frac{(Z_1+1)(Z_2-1)e^2}{r}+(Z_2-1)e^2\left(\frac{1}{r_p}-\frac{1}{r}\right) \\ +\left(V_1(|\mathbf{r}-\mathbf{r}_p|)+\frac{Z_1e^2}{|\mathbf{r}-\mathbf{r}_p|}\right) \\ +V_2(\mathbf{r}_p)+V_{11}+V_{22}. \end{aligned} \tag{2.2}$$

The Coulomb functions describing the relative motion of the heavy particles contain now the product $(Z_1+1)(Z_2-1)$ instead of Z_1Z_2 for the initial state. In the final state there exists also the possibility of Coulomb excitation of the final nucleus under the influence of the charge $(Z_2-1)e$ which is left on that nucleus which has lost the proton, and the remaining terms have a meaning similar to that described for corresponding terms in relation to (2.1). It is seen that in the arrangement of contributions to V there occur the combinations

$$V_1^{\text{eff}}=V_1(|\mathbf{r}-\mathbf{r}_p|)+\frac{Z_1e^2}{|\mathbf{r}-\mathbf{r}_p|}, \tag{2.3}$$

and

$$V_2^{\text{eff}}=V_2(r_p)+(Z_2-1)e^2/r_p. \tag{2.4}$$

The first of these represents the effective potential for the proton after transfer when it is attached to m_1; the second gives similarly the effective potential of the proton before transfer while it is still attached to m_2. The Coulomb potential energies which enter these expressions are to be modified for small $|\mathbf{r}-\mathbf{r}_p|$ and r_p on account of the space distribution of nuclear charge. Since no explicit use of these parts of the potential energies will be made in the present paper, this modification is not consistently indicated but it is nevertheless useful to remember that these contributions to V_1^{eff} and V_2^{eff} are not literally correct. The main object in exhibiting the forms of V_1^{eff} and V_2^{eff} is, however, to show how these quantities are related to the parts of the Hamiltonian which are not absorbed in the definitions of the Coulomb functions for heavy particles, the Coulomb excitation potential, and the potentials used in the definitions of internal functions. In the initial grouping shown in (2.1) the $\mathbf{r}_p$ is contained only in $V_1(|\mathbf{r}-\mathbf{r}_p|)$ after the other potentials are absorbed in the introduction of the three kinds of functions just mentioned. This perturbation potential is, however, not V_1^{eff} but

$$V_1^{\text{eff}}-\frac{Z_1e^2}{|\mathbf{r}-\mathbf{r}_p|}, \tag{2.3'}$$

and is therefore considerably more attractive than the effective potential needed for the description of the proton after capture.

III. SEPARATION OF VARIABLES

In accordance with (2.1), it is convenient to introduce Ψ^{CN} by means of

$$(H^{CN}-E)\Psi^{CN}=0, \tag{3}$$

with H^{CN} containing all but the V_1 part of the potential energy so that

$$H^{CN}=H-V_1(|\mathbf{r}-\mathbf{r}_p|). \tag{3.1}$$

By means of the identification $\mathbf{r}_p=\mathbf{r}_{32}$, $\mathbf{r}=\mathbf{r}_{1,23}$ which has been discussed in connection with (1.4), the kinetic

energy part of (1.5) becomes

$$(-\hbar^2/2\mu_{23})\Delta(\mathbf{r}_p)-(\hbar^2/2\mu_{1,23})\Delta(\mathbf{r}),$$

where the quantity in parentheses after the Δ indicates the variables with respect to which differentiations are made in the Δ. The expression for the kinetic energy will be written by introducing symbols

$$m=\mu_{1,23}, \quad M=\mu_{23}, \tag{3.2}$$

so that m is the reduced mass for the colliding particles in the initial state while M is the reduced mass for the relative motion of the proton with respect to the nucleus to which it is initially attached. This notation does not give a convenient distinction between initial and final reduced masses. Since in the applications contemplated here the mass of the transferred particle is relatively small, one has, however,

$$\mu_{13}\cong M, \quad \mu_{2,13}\cong m, \tag{3.3}$$

and a distinction between these reduced masses and those for the initial state would not be altogether consistent with the approximations made in obtaining Eqs. (1.4), (1.5) from the more accurate initial equations such as (1.3).

The function Ψ^{CN} is the solution of the problem if one neglects the possibility of transfer. The superscripts on this symbol are intended to indicate the Coulomb wave solution by means of C and the entrance of heavy nuclei in the solution by N. The Coulomb repulsion of the heavy colliding particles and the possibility of Coulomb excitation of the proton from the ground state of the composite system 2, 3 are taken into account in Ψ^{CN}. The plan of the calculation is to employ Ψ^{CN} in a perturbation procedure in which $V_1(|\mathbf{r}-\mathbf{r}_p|)$ is considered as the perturbing part of the Hamiltonian. Representing the first order correction to Ψ^{CN} by $\Psi^{(1)}$ one has

$$\Psi=\Psi^{CN}+\Psi^{(1)}, \tag{3.4}$$

and since Ψ^{CN} satisfies (3) one has

$$(H-E)\Psi^{(1)}=-V_1\Psi^{CN}. \tag{3.5}$$

In solving for $\Psi^{(1)}$ one can use on the left side of this equation either the arrangement of V such as in (2.1) or else the one corresponding to the final state and shown in (2.2). The second of these has the advantage of showing more directly the relation to the final state. Employing it, one has to solve

$$\left\{-\frac{\hbar^2}{2M}\Delta(\mathbf{r}_p)+V_1^{\text{eff}}-\frac{\hbar^2}{2m'}\Delta(\mathbf{r}')+(Z_2-1)e^2\left(\frac{1}{r_p}-\frac{1}{r'}\right)+\frac{(Z_1+1)(Z_2-1)e^2}{r'}+V_2(r_p)+H_{11}+H_{22}-E\right\}\Psi^{(1)}=-V_1(|\mathbf{r}-\mathbf{r}_p|)\Psi^{CN}. \tag{4}$$

Here the difference in the meaning of the coordinates and masses for the final state from that for the initial is indicated by primes. This distinction will be dropped, however, in view of the smallness of this difference which results from the fact that $m_3\ll m_1$, $m_3\ll m_2$. In treating Eq. (4), the following approximations will be made. In the first place, $V_2(r_p)$ will be dropped. This approximation is partially justifiable because the main effect of $\Psi^{(1)}$ is to cause the appearance of the proton at m_1 and in this region $V_2(r_p)$ is small. The approximation is not completely justifiable, however, because $\Psi^{(1)}$ originates at m_2 as a result of Coulomb excitation by Z_1e and is thus not necessarily negligible in the region of appreciable V_2. A second approximation will be made in neglecting the term $(Z_2-1)e^2(1/r_p-1/r)$, corresponding to disregarding the effect of Coulomb excitations after transfer. This procedure is also not completely justifiable because $\Psi^{(1)}$ has appreciable values in the region of small r_p. In fact, looking at the reaction in the reverse direction of time, Coulomb excitation of the final states may be expected to be important if it matters for the initial states. The approximations made amount, therefore, to an unsymmetric treatment of initial and final states resulting in the omission of terms arising in the time reverse reaction in a manner similar to those kept.

With these approximations and conventions, one may expand

$$\Psi^{(1)}=\sum_j u_j(\mathbf{r}_p-\mathbf{r})\varphi_j(\mathbf{r}), \tag{4.1}$$

where

$$\{-(\hbar^2/2M)\Delta(\mathbf{r}_p)+V_1^{\text{eff}}(\mathbf{r}_p-\mathbf{r})+H_{11}+H_{22}-E_j\}\times u_j(\mathbf{r}_p-\mathbf{r})=0. \tag{4.2}$$

Here the u_j are eigenfunctions for the final nucleus obtained by transferring the proton to m_1. The entrance of internal coordinates of m_1 and m_2 is understood but not shown in u_j. The index j distinguishes not only the states of the proton but also those of the aggregate systems m_1, m_2. The right side of Eq. (4) may be expanded as

$$-V_1(|\mathbf{r}-\mathbf{r}_p|)\Psi^{CN}=\sum_j u_j(\mathbf{r}_p-\mathbf{r})\chi_j(\mathbf{r}), \tag{4.3}$$

where the χ_j are functions of $\mathbf{r}$ only which are otherwise defined by the above expansion. Combining Eqs. (4), (4.1), (4.2), and (4.3), there results the approximate equation

$$\left\{-\frac{\hbar^2}{2m}\Delta(\mathbf{r})+\frac{(Z_1+1)(Z_2-1)}{r}e^2-(E-E_j)\right\}\varphi_j(\mathbf{r})=\chi_j(\mathbf{r}). \tag{4.4}$$

The requirement will be made that the $\varphi_j(\mathbf{r})$ be asymptotic to constant multiples of $\exp(i\kappa_j r)$ for $E-E_j>0$ and to constant multiples of $\exp(-\alpha_j r)$ for $E-E_j<0$ with $\kappa_j>0$, $\alpha_j>0$. These requirements together with the equations just mentioned determine $\varphi_j(\mathbf{r})$.

The amplitudes of the $\varphi_j(\mathbf{r})$ at $r=\infty$ determine the probabilities of the transfer reaction taking place. According to (4.4) these depend on the magnitudes of the $\chi_j(\mathbf{r})$, and according to (4.3) the $\chi_j(\mathbf{r})$ owe their existence to Ψ^{CN}. The latter consists partly of a term representing the incident wave with the initial grouping of particles in their ground states. For this term, the factor $V_1(|\mathbf{r}-\mathbf{r}_p|)$ on the left side of (4.3) exercises a strong selection because of the limited range of V_1. The left side of (4.3) vanishes unless Ψ^{CN} has appreciable values for sufficiently small $|\mathbf{r}-\mathbf{r}_p|$, and Ψ^{CN} cannot have such values unless r is sufficiently small. Otherwise the leakage of p out of the aggregate m_2+m_3 is not sufficient to make it lie close enough to m_1. The situation is radically different, however, for the parts of Ψ^{CN} for which the aggregate m_2+m_3 has been raised to an excited state, for now the leakage of m_3 is more pronounced and p can find itself in the region of small $|\mathbf{r}-\mathbf{r}_p|$ even if r is large. It becomes necessary to compare therefore the relative sizes of contributions to Ψ^{CN} arising from the ground state and from excited states of the colliding nuclei. In particular, if Ψ^{CN} contains a part for which p is in an excited state relative to m_2 while the remainder of m_2 is in its ground state, then the u_j can also contain the ground state of m_2 and the φ_j is energetically possible at large r whenever this is the case for the part of $\Psi^{(1)}$ in which all of m_2+m_3 is in its ground state. The main requirement on the calculation to be performed therefore is to compare the values of the contributions to Ψ^{CN} arising from p being in its ground state or in an excited state with respect to m_2, while the internal condition of m_2 is unexcited.

The problem is thus essentially reduced to the consideration of the relative magnitudes of parts of Ψ^{CN} just described. For large r, these values are much smaller for the highly excited states of p. Thus, for example, for $r=\infty$ these parts of Ψ^{CN} vanish whenever the excitation energy is higher than the initially available kinetic energy of the colliding nuclei. For states in which the process is energetically possible but the excitation energy is high, the probability of Coulomb excitation is low because of the nearly adiabatic character of the collision. Neither of these considerations excludes, however, the existence of an appreciable probability of an excited state of p at smaller r.

The calculation of Ψ^{CN} will be made employing the form (2.1) for the potential energy. The perturbing part of the Hamiltonian responsible for Coulomb excitation is

$$H'=Z_1e^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}_p|}-\frac{1}{r}\right). \tag{5}$$

In order to deal with a simple case, the dipole type of excitation only will be considered so that the relevant part of H' will be contained in the approximation

$$H'=(4\pi/3)Z_1e^2\sum_{\mu,p}(r_p/r^2)Y^*{}_{1\mu}(\theta_p,\varphi_p)Y_{1\mu}(\theta,\varphi), \tag{5.1}$$

where (r_p,θ_p,φ_p) and (r,θ,φ) stand respectively for the polar coordinates of $\mathbf{r}_p$ and $\mathbf{r}$, with θ denoting the colatitude and φ the azimuth. The Coulomb wave without Coulomb excitation will be denoted by ψ^c and the nucleus with mass m_1 will be considered for the present as a single particle with charge Z_1e. The latter simplification is unessential since in the present part of the problem it amounts to the omission of the factor in the wave function which contains the internal coordinates of m_1. The subscript μ covers the values -1, 0, 1 and the $Y_{l\mu}$ are spherical harmonics of order l used with the normalization

$$\int|Y_{l\mu}|^2d\Omega=1,$$

the integration being over all solid angles. The phases are in the standard convention[7] giving simple forms for angular momenta. A complete treatment of the problem should involve the consideration of the complete wave function of the aggregate m_2+m_3. For simplicity, however, the effect of changing the central-field states of particle p alone will be discussed for the present. The spin of p will be left out of the consideration and p will be taken to be in an s state. The initial state of p will be denoted by $v_i(\mathbf{r}_p)$, the three final substantes to which it may be excited for a given energy level by $w_\mu(\mathbf{r}_p)$. Considering these three substates only, the wave function is

$$\psi=\psi^cv_i+\sum_\mu w_\mu\psi^{(1)}{}_\mu. \tag{5.2}$$

Substitution of ψ into the wave equation gives then the approximate equation

$$(H^{(0)}+E_w-E)\psi^{(1)}{}_\mu=-H'{}_{\mu i}\psi^c, \tag{5.3}$$

where the Hamiltonian for the system consisting of m_1 and the aggregate m_2+m_3 is $H^{(0)}+H'$ and where the energy of the degenerate level with eigenfunctions w_μ is E_w when measured with respect to the ground state of m_2+m_3 while

$$H'{}_{\mu i}=\int w_\mu^*(\mathbf{r}_p)H'v_i(\mathbf{r}_p)d\mathbf{r}_p. \tag{5.4}$$

On account of the presence of (r,θ,φ) in H' the $H'{}_{\mu i}$ is a function of these quantities. On expanding ψ^c in partial waves, one encounters on the right side of (5.3) the combinations $\psi^cY_{1,\mu}$, the $Y_{1,\mu}$ coming from the last factor in (5.1). Employing for ψ^c a Coulomb modified plane wave of unit density

$$\psi^c=\sum_L i^L(2L+1)P_L(\cos\theta)e_LF_L(kr)/(kr), \tag{5.5}$$

with symbols as in the list of notation, one finds

$$\psi^cY_{1,\mu}=(4\pi)^{\frac{1}{2}}\sum_L\frac{i^{L-1}}{kr}[(2L-1)^{\frac{1}{2}}e_{L-1}b_{L-1}{}^\mu F_{L-1}$$
$$-(2L+3)^{\frac{1}{2}}e_{L+1}b_{L+1}{}^\mu F_{L+1}]Y_{L,\mu}, \tag{5.6}$$

[7] B. L. van der Waerden, *Gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, 1932).

where

$$e_L=[\Gamma(L+1+i\eta)/\Gamma(L+1-i\eta)]^{\frac{1}{2}},$$

$$b_{L-1}{}^{\mu}=\frac{(3/4\pi)^{\frac{1}{2}}}{[(2L-1)(2L+1)]^{\frac{1}{2}}}\times\left(\frac{[L(L+1)]^{\frac{1}{2}}}{2^{\frac{1}{2}}},\ L,\ \frac{[L(L+1)]^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right), \quad (5.7)$$

$$b_{L+1}{}^{\mu}=\frac{(3/4\pi)^{\frac{1}{2}}}{[(2L+1)(2L+3)]^{\frac{1}{2}}}\times\left(-\frac{[L(L+1)]^{\frac{1}{2}}}{2^{\frac{1}{2}}},\ L+1,\ -\frac{[L(L+1)]^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right), \quad (5.8)$$

for $\mu=(1, 0, -1)$, respectively. The coefficients b enter through

$$Y_{1,\mu}Y_{L,0}=\beta_1(L,\mu)Y_{L+1,\mu}+\beta_{-1}(L,\mu)Y_{L-1,\mu},$$
$$\beta_1(L-1,\mu)=b_{L-1}{}^{\mu}, \quad \beta_{-1}(L+1,\mu)=b_{L+1}{}^{\mu}. \quad (5.9)$$

The notation in terms of the $b_{L-1}{}^{\mu}$, $b_{L+1}{}^{\mu}$ would be confusing, being incomplete, if it were not for the fact that in the present paper $b_{L-1}{}^{\mu}$ always refers to a β_1 and $b_{L+1}{}^{\mu}$ to a β_{-1}.

From (5.1) and (5.4), it follows that

$$H'_{\mu i}=KY_{1\mu}{}^{*}(\theta,\varphi)/r^2, \quad (6)$$

with

$$K=[(4\pi)^{\frac{1}{2}}/3]Z_1e^2\langle r_p\rangle, \quad (6.1)$$

where

$$\langle r_p\rangle=\int_0^{\infty}R_w(r_p)R_i(r_p)r_p{}^3dr_p, \quad (6.2)$$

and the radial functions R_i, R_w for the initial and final states of p are normalized according to

$$\int_0^{\infty}R_w{}^2(r_p)r_p{}^2dr_p=\int_0^{\infty}R_i{}^2(r_p)r_p{}^2dr_p=1. \quad (6.3)$$

For states in the continuum, the normalization integral is understood to extend through the radius of a quantizing sphere. Introducing (6) in (5.3) and remembering that

$$Y_{1\mu}{}^{*}=(-)^{\mu}Y_{1\mu},$$

one has

$$(H^{(0)}+E_w-E)\psi^{(1)}{}_{-\mu}=(-)^{\mu+1}(K/r^2)Y_{1\mu}\psi^{c}. \quad (6.4)$$

The right side of this equation is a sum of products of radial functions multiplied by the $Y_{L\mu}$ as is seen from (5.6). Equation (6.4) may be solved therefore by identifying parts containing the same $Y_{L\mu}$ on both sides and solving the resultant radial equations subject to the appropriate boundary conditions. The latter have to represent the requirement that the $\psi^{(1)}{}_{\mu}$ contain outgoing waves only, since the incident state is being dealt with here. The equations are of the type

$$\left(-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}-k_w{}^2+\mathfrak{U}(r)+\frac{L(L+1)}{r^2}\right)X=F_l(kr)/(kr^3) \quad (7)$$

with

$$k_w{}^2=(2m/\hbar^2)(E-E_w)$$

and $\mathfrak{U}(r)$ standing for the term introduced by the Coulomb energy. The solution of (7) is

$$X=[1/(kk_wr)]\int_0^{\infty}\mathcal{G}_L(r,r')F_l(kr')(1/r'^2)dr', \quad (7.1)$$

with

$$\mathcal{G}_L(r,r')=H_L(k_wr_>)F_L(k_wr_<), \quad H_L=G_L+iF_L, \quad (7.2)$$

the functions F_L, G_L being here Coulomb functions for the state having energy $E-E_w$ and

$$\begin{aligned}(r_>,r_<)&=(r,r'), \quad (r>r')\\ (r_>,r_<)&=(r',r), \quad (r'>r).\end{aligned} \quad (7.3)$$

The notation is here abbreviated by omitting explicit indication of the argument of the Coulomb functions which is independent of r. No confusion will arise, however, if it is remembered that whenever k_wr rather than kr enters the Coulomb function the second argument corresponds to an energy $E-E_w$ rather than E. If $E-E_w<0$, the definition of H_L given in Eq. (7.2) has to be changed so as to make H_L exponentially decaying at large r but preserving the validity of

$$H_LdF_L/dr-F_LdH_L/dr=k_w, \quad (7.4)$$

which is needed for the correctness of (7.1). By means of (7.1), (5.6), and (6.4), one obtains

$$\psi^{(1)}{}_{-\mu}=(-)^{\mu+1}(4\pi)^{\frac{1}{2}}\frac{2m}{\hbar^2}K\sum_L\frac{i^{L-1}Y_{L\mu}}{kk_wr}\times\left((2L-1)^{\frac{1}{2}}e_{L-1}b_{L-1}{}^{\mu}\int_0^{\infty}\mathcal{G}_L(r,r')\times F_{L-1}(kr')\frac{dr'}{r'^2}-(2L+3)^{\frac{1}{2}}e_{L+1}b_{L+1}{}^{\mu}\times\int_0^{\infty}\mathcal{G}_L(r,r')F_{L+1}(kr')\frac{dr'}{r'^2}\right). \quad (7.5)$$

According to (7.2), the $\mathcal{G}_L$ contains in one of its terms the product $G_L(k_wr_>)F_L(k_wr_<)$. When $r\cong r'$, the smallness of F_L within the Coulomb and centrifugal barriers is largely compensated for by the largeness of G_L. The large barrier-penetration effects do not enter therefore for values of r for which the factor multiplying $\mathcal{G}_L(r,r')$ in the integrand has an appreciable value. It is thus immediately obvious that the value of $E-E_w$ does not enter nearly as critically as it does for larger values of r.

For these the $G_L(k_w r)$ enters only with the larger argument, and since for large r G_L is asymptotic to a sine function of unit amplitude the compensation of the smallness of F_L cannot take place.

IV. SPECIAL CASES

The case $E \to E_w$ will be considered first. The parameter η_w which enters the Coulomb functions is

$$\eta_w = Z_1 Z_2 e^2/\hbar v_w,$$

where

$$v_w = [2(E-E_w)/m]^{\frac{1}{2}}$$

is the classical velocity of relative motion corresponding to the residual energy $E-E_w$ when p has been excited to the energy E_w. In the limit $E \to E_w$, the classical velocity vanishes and $\eta_w \to \infty$. One has accordingly[8]

$$F_L(k_w r) \cong 2^{-\frac{1}{2}} \eta_w^{-1} \{[1+L^2/\eta_w^2] \times [1+(L-1)^2/\eta_w^2] \cdots [1+1/\eta_w^2]\}^{\frac{1}{2}} \times C_{0w} (r/a_w)^{\frac{1}{2}} I_{2L+1}(x), \quad (8)$$

$$G_L(k_w r) \cong -(2^{\frac{3}{2}}/C_{0w})(r/a_w)^{\frac{1}{2}} \times \{[1+L^2/\eta_w^2][1+(L-1)^2/\eta_w^2] \cdots [1+1/\eta_w^2]\}^{-\frac{1}{2}} K_{2L+1}(x),$$

where $I_\nu(x)$, $K_\nu(x)$ are Bessel functions of imaginary argument in the notation of Whittaker and Watson,[9]

$$x = (8r/a_w)^{\frac{1}{2}} = (8k_w r \eta_w)^{\frac{1}{2}},$$
$$C_{0w} = [2\pi\eta_w/(-1+\exp(2\pi\eta_w)]^{\frac{1}{2}}, \quad (8.1)$$

while

$$a_w = \hbar^2/(Z_1 Z_2 m e^2) \quad (8.2)$$

is the Bohr length for the collision. The values (8) give

$$F_L(k_w r) G_L(k_w r') \cong -2k_w (rr')^{\frac{1}{2}} I_{2L+1}(x) K_{2L+1}(x'), \quad (8.3)$$

where x' is the value of x corresponding to r'. For the distance of closest approach in the collision of N^{14} and N^{14} at 10-Mev bombarding energy the argument of I and K is about 36, so that asymptotic expressions for large values of the argument apply. Employing, therefore,

$$I_{2L+1}(x) \sim e^x/(2\pi x)^{\frac{1}{2}}, \quad K_{2L+1}(x) \sim -(\pi/2x)^{\frac{1}{2}} e^{-x},$$

one has

$$F_L(k_w r) G_L(k_w r') \cong 2^{-\frac{3}{2}} k_w a_w^{\frac{1}{2}} (rr')^{\frac{1}{4}} \exp(x-x'). \quad (8.4)$$

The coefficient of the exponential may also be written as

$$k_w (rr')^{\frac{1}{2}}/(xx')^{\frac{1}{2}}.$$

For $r=r'$, one has

$$F_L(k_w r) G_L(k_w r) \cong 2^{-\frac{3}{2}} k_w (a_w r)^{\frac{1}{2}}. \quad (8.5)$$

[8] Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1936); G. Breit and M. H. Hull, Jr., Phys. Rev. **80**, 392, 561 (1950).

[9] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, London, 1920), third edition, Chap. XVII.

The large effects of barrier penetration are contained in the factor $C_{0w} e^x$ in F_L and the reciprocal of this factor in G_L. These factors cancel out completely in (8.5), and in (8.4) for moderate $x-x'$ there is only a weak effect left. Since (8.4) enters for $r'>r$ only, the effect of the factor $\exp(x-x')$ is to decrease $\mathcal{G}_L(r,r')$ and to give it a cusp-like behavior as r' is varied while r is fixed. The integrals in (7.5) can be evaluated approximately by making use of this circumstance. One has

$$|x-x'| \cong |r'-r|/d, \quad d=(ra/2)^{\frac{1}{2}}, \quad (8.6)$$

as follows by differentiation. For 10-Mev bombarding energy in the case of $N^{14}+N^{14}$ at the distance of closest approach, $r=14.0(7)\times10^{-13}$ cm, while $a=8.4(7)\times10^{-15}$ cm for these two nuclei and $d=0.77\times10^{-13}$ cm. On the other hand, for the incident state and the above r, one has $kr=18.2$. A quarter-wave change corresponds roughly to a change of $\pi/2$ in kr which corresponds to a change of 1.2×10^{-13} cm in r. Thus d is only slightly smaller than a quarter wavelength and the cusp may not be considered as infinitely sharp. On the other hand, d is relatively small compared with r so that one may treat all factors multiplying $\exp(-|x-x'|)$ in the integrands of (7.5) except F_L as constants. Proceeding in this manner one finds

$$\int_0^\infty \mathcal{G}_L(r,r')[F_L(kr')/r'^2]dr' \cong (1/(2r\eta_w))[F_L(kr) + k^2 d^2 F_L''(kr)]. \quad (8.7)$$

In obtaining this formula, the integrations over $r-r'$ have been taken from $-\infty$ to $+\infty$ rather than over the actual range. The error caused by this approximation is doubtless not as important as some of the other inaccuracies on account of the smallness of d/r. The second term in brackets on the right side of (8.7) gives a rough correction to the first term. Since (8.7) contains the infinite quantity η_w in the denominator, the right-hand side vanishes. On the other hand, (7.5) contains in the denominator the vanishing k_w and the right side of (7.5) is finite and does not vanish. Disregarding the second term in brackets in (7.5) and another circumstance to be discussed presently, it is possible to perform the summation over L in (7.5) as follows:

$$(2L-1)^{\frac{1}{2}} e_{L-1} b_{L-1}^{\mu} \int_0^\infty \mathcal{G}_L(r,r') \times [F_{L-1}(kr')/r'^2]dr' - (2L+3)^{\frac{1}{2}} e_{L+1} b_{L+1}^{\mu} \times \int_0^\infty \mathcal{G}_L(r,r')[F_{L+1}(kr')/r'^2]dr' \cong [(2L-1)^{\frac{1}{2}} e_{L-1} b_{L-1}^{\mu} F_{L-1}(kr) - (2L+3)^{\frac{1}{2}} e_{L+1} b_{L+1}^{\mu} F_{L+1}(kr)]/(2r\eta_w), \quad (9)$$

and substituting into (7.5) one needs the sum

$$\sum_L \frac{i^{L-1}Y_{L\mu}}{kr}[(2L-1)^{\frac{1}{2}}e_{L-1}b_{L-1}{}^{\mu}F_{L-1}(kr) - (2L+3)^{\frac{1}{2}}e_{L+1}b_{L+1}{}^{\mu}F_{L+1}(kr)]$$
$$=\sum_L \frac{i^L}{kr}(2L+1)^{\frac{1}{2}}[\beta_1(L,\mu)Y_{L+1,\mu} + \beta_{-1}(L,\mu)Y_{L-1,\mu}]e_LF_L(kr)$$
$$=[\textstyle\sum_L i^L(2L+1)^{\frac{1}{2}}e_LY_{L,0}F_L(kr)/kr]Y_{1,\mu}$$
$$=(4\pi)^{-\frac{1}{2}}Y_{1,\mu}\psi^c. \quad (9.1)$$

Employing this value in (7.5), one obtains, on making use of $\eta_w k_w = 1/a$,

$$\psi^{(1)}{}_{-\mu} \cong (-)^{\mu+1}\frac{mK}{\hbar^2 r/a}Y_{1,\mu}\psi^c$$
$$=(-)^{\mu+1}\frac{(4\pi)^{\frac{1}{2}}}{3Z_2 r}\langle r_p\rangle Y_{1,\mu}\psi^c. \quad (9.2)$$

This value of $\psi^{(1)}{}_{-\mu}$ is approximate only, several approximations having been made as is clear from (8.7) and the related discussion. According to (9.2) the Coulomb excited wave is decidedly smaller than ψ^c, but the factor involved is essentially the moderately small $\langle r_p\rangle/(Z_2 r)$ rather than the very strong barrier-penetration factor.

The last of the two expressions for $\psi^{(1)}$ given in Eq. (9.2) does not contain Z_1 except for its appearance in ψ^c. On the other hand, the first expression shows that if r/a is kept constant, $\psi^{(1)}{}_{\mu}/\psi^c$ is proportional to K and hence to Z_1. The occurrence of K for fixed r/a is not surprising since the radial functions in the partial wave analysis of $\psi^{(1)}{}_{\mu}$ have a shape dependent on r/a only. The disappearance of Z_1 from $\psi^{(1)}{}_{\mu}/\psi^c$ in the second of the two forms in (9.2) requires explanation, however, because the effect of Z_1 is present in a concealed form. It enters in the applicability of the asymptotic expansions for $I_{2L+1}(x)$, $K_{2L+1}(x)$ used in obtaining (8.4) which requires that x be large enough. Since, for fixed r, this x is proportional to $Z_1{}^{\frac{1}{2}}$, the last form in Eq. (9.2) is only apparently independent of Z_1. It gives the quantity in the limit of sufficiently large Z_1. It should be mentioned that in the derivation of (9.2) the asymptotic forms used for I_{2L+1}, K_{2L+1} become inapplicable for sufficiently large L. For this reason, as well as the approximations related to neglecting the second term in square brackets in (8.7), one has to regard Eq. (9.2) as a crude approximation. On the other hand, the large L are not important for close collisions which are of main interest for the nucleon transfer cross section.

The effect will next be considered for very high excitations. The radial equation for F_L becomes

$$\left(\frac{d^2}{dr^2}-\beta^2-\frac{2}{ar}-\frac{L(L+1)}{r^2}\right)F_L=0, \quad (10)$$

with

$$\beta^2>0. \quad (10.1)$$

One can introduce for this equation two solutions somewhat analogous to the case of positive kinetic energy and referred to by the same letters:

$$F_L{}^{(1)}=\frac{\Gamma(L+1+(1/a\beta))}{2\Gamma(2L+2)}M_{\kappa,m}(2\beta r),$$
$$H_L{}^{(1)}=W_{\kappa,m}(2\beta r), \quad \kappa=-1/(a\beta), \quad (10.2)$$

where $M_{\kappa,m}(z)$, $W_{\kappa,m}(z)$ are solutions of the confluent hypergeometric equation in Whittaker's notation. For large r,

$$F_L{}^{(1)}\sim\tfrac{1}{2}(2\beta r)^{1/a\beta}e^{\beta r}, \quad H_L{}^{(1)}\sim(2\beta r)^{-1/a\beta}e^{-\beta r}, \quad (10.3)$$

so that

$$H_L{}^{(1)}dF_L{}^{(1)}/dr-F_L{}^{(1)}dH_L{}^{(1)}/dr=\beta. \quad (10.4)$$

Instead of Eq. (7), one deals in this case with

$$\left(-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\beta^2+\mathfrak{U}(r)+\frac{L(L+1)}{r^2}\right)X=\frac{F_L(kr)}{kr^3}, \quad (10.5)$$

which has the solution

$$X=\frac{1}{k\beta r}\int_0^\infty \mathcal{G}_L{}^{(1)}(r,r')[F_l(kr')/r'^2]dr', \quad (10.6)$$

with

$$\mathcal{G}_L{}^{(1)}(r,r')=H_L{}^{(1)}(r_>)F_L{}^{(1)}(r_<). \quad (10.7)$$

Employing the approximations (10.3), which are justified by the assumption of largeness of β, one has

$$\mathcal{G}_L{}^{(1)}(r,r')\cong\tfrac{1}{2}\exp(-\beta|r'-r|). \quad (11)$$

Here it was assumed that the lack of symmetry in this cusp-like function is negligible, again in conformity with the assumption of high excitation. This value of $\mathcal{G}_L{}^{(1)}$ takes the place of (8.4) in the case of bare ionization, i.e., zero relative kinetic energy of heavy particles. One should compare therefore

$$\int_{-\infty}^{+\infty}\tfrac{1}{2}\exp[-\beta(r'-r)]dr=\frac{1}{\beta}$$

with

$$2^{-\frac{3}{2}}k_w(ar)^{\frac{1}{2}}\int_{-\infty}^{+\infty}\exp(-|r'-r|/d)dr=\tfrac{1}{2}k_w ra.$$

On account of the difference in these quantities, the right side of (9.2) needs multiplication by

$$2/(\beta k_w ra). \quad (11.1)$$

In addition it is necessary to consider the reason for the occurrence of the k_w in the denominator of (7.5), which is readily seen to be the presence of k_w on the right side of (7.4). Since in the analogous Eq. (10.4) of the new case, k_w is replaced by β, there occurs a $1/\beta$ in (10.6) instead of the $1/k_w$ in the corresponding (7.1). The application of (10.6) to obtain an equation taking place of (7.5) for the new case gives therefore a similar right hand side with $1/k_w$ replaced by $1/\beta$ and $\mathcal{G}_L$ replaced by $\mathcal{G}_L^{(1)}$. In addition to the factor (11.1), there is therefore the factor k_w/β to be introduced. The net factor is

$$2/(\beta^2 ra), \tag{11.2}$$

and the approximation corresponding to (9.2) is

$$\psi^{(1)}{}_{-\mu} \cong (-)^{\mu+1} \frac{2mK}{\hbar^2\beta^2 r^2} Y_{1,\mu}\psi^c = (-)^{\mu+1} \frac{2(4\pi)^{\frac{1}{2}}\langle r_p\rangle}{3Z_2\beta^2 ar^2} Y_{1,\mu}\psi^c. \tag{11.3}$$

As in the previous case the barrier penetration effects have disappeared. These are still present in (10.3) but the relative smallness of the $F_L^{(1)}$ at small r is not the determining factor since it enters with $H_L^{(1)}$ to give the $\mathcal{G}_L^{(1)}$ in which there is exact compensation of the penetrability effects when $r=r'$. The effect of the excitation energy is contained in (10.3) in the factor $1/\beta^2$ only. This type of dependence can be expected from the general form of the first-order perturbation in a wave function which is obtainable by the Rayleigh-Schrödinger perturbation method and contains as a factor the reciprocal of the excitation energy. The calculation could have been performed in fact by employing a quantizing sphere and reducing the problem to the calculation of the wave function of a stationary state. Such a calculation would involve, however, a sum over all possible β rather than just those corresponding to conservation of energy and would be more complicated than the one presented here. The first of the two forms for $\psi^{(1)}{}_{-\mu}$ shows proportionality to Z_1 which is contained in K.

V. COMPARISON WITH THE SCT AND NUMERICAL ESTIMATES

It is of interest to compare the results obtained so far with the approximation of considering the relative motion of the two heavy aggregates by means of classical mechanics. High excitation energies give then small probabilities of ordinary Coulomb excitation. As has been mentioned, however, an important influence in this smallness of the effect is the increasingly adiabatic character of the collision as the excitation energy is increased. For this reason smallness of Coulomb excitation after the particles have separated does not imply its smallness for small distances of approach. The general situation is in fact that appreciable probabilities of Coulomb excitation at small distances are followed by a de-excitation in which the energy of excitation is gradually returned to the kinetic energy of the colliding particles.

In approximating a quantum problem by a classical one, the question of correlation of initial and final states frequently arises and in the sense of Bohr's correspondence principle one usually takes some kind of mean of initial and final states in the classical calculations in order to obtain agreement with quantum quantities. In the present case the initial state in the above sense is the whole ψ of (5.2) by means of which one can obtain $\Psi^{(1)}$ of (3.4), which gives rise to the final state in which the neutron has been transferred from one nucleus to the other. The results under immediate discussion are concerned with $\psi^{(1)}{}_{\mu}$, which is part of ψ, and the usual averaging procedure between initial and final states does not apply therefore to the consideration of the initial and final velocity of the heavy aggregates in the determination of $\psi^{(1)}{}_{\mu}$. For this reason it is simplest to calculate the probability of virtual state formation employing the initial relative velocity. On the other hand, this simplified procedure is an approximate one only because it would not give correctly the probability of real Coulomb excitation for which the $\psi^{(1)}{}_{\mu}$ at large r have to be considered as the final states.

On the SCT there enters a matrix element of the time-dependent perturbing Hamiltonian

$$H_{\mu i}'(t) = KY_{1,\mu}{}^*(\theta_t,\varphi_t)/r_t^2, \tag{12}$$

where r_t, θ_t, φ_t are the values of r, θ, φ at time t. The amplitude of the excited state w_μ is

$$A_\mu = \frac{1}{\hbar}\int_{-\infty}^{t} H'_{\mu i}(t)\exp(iE_{\mu i}t/\hbar)dt, \tag{12.1}$$

where $E_{\mu i}$ is the energy of the state w_μ minus the energy of v_i. On account of the degeneracy with respect to μ, one may take

$$E_{\mu i} = E_w = \hbar\omega. \tag{12.2}$$

It is convenient to parametrize the hyperbolic motion by[10]

$$\begin{aligned} x &= a'(\epsilon+\cosh w), \quad & y &= a'(\epsilon^2-1)^{\frac{1}{2}}\sinh w, \\ r &= a'(1+\epsilon\cosh w), \quad & t &= (a'/v)(w+\epsilon\sinh w), \end{aligned} \tag{12.3}$$

where x is the coordinate measure along the major axis of the hyperbola, y in a direction perpendicular to x and ϵ is the eccentricity. For the closest collisions, $\epsilon=1$ and if one may set $\omega=0$, one has for the perihelion, on making use of

$$\int_0^\infty \frac{dw}{1+\cosh w} = 1, \tag{12.4}$$

$$A_0 = \frac{k\langle r_p\rangle}{\sqrt{3}Z_2}, \quad A_1 = A_{-1} = 0, \quad (\epsilon=1,\ \omega=0,\ w=0). \tag{12.5}$$

[10] K. A. Ter-Martirosyan, J. Exptl. Theoret. Phys. (U.S.S.R.) **22**, 284 (1952).

The effect of the excitation energy is to change the integral in (12.4) to

$$\int_0^\infty \frac{e^{i\omega t}}{1+\cosh w}dw=\int_0^\infty \frac{\exp[i\xi(w+\sinh w)]}{1+\cosh w}dw, \quad (12.6)$$

with

$$\xi=\omega a'/v=E_w\eta/(2E), \quad (12.7)$$

where the value of η is that for the initial velocity, with some uncertainty inherent in the application of classical mechanics to the present problem. The exact treatment of (12.6) is tedious. A crude but apparently fair approximation for high excitations can be obtained as follows. Introducing

$$u=w+\sinh w, \quad (13)$$

one has

$$\int_0^\infty \frac{\exp[i\xi(w+\sinh w)]}{1+\cosh w}dw=\int_0^\infty e^{i\xi u}\left(\frac{dw}{du}\right)^2 du. \quad (13.1)$$

By numerical trial it is seen that

$$(dw/du)^2\cong 0.25/[1+(u/2.75)^2] \quad (13.2)$$

the approximation being reasonably good between $u=0$ and $u=6.5$. In this region $(dw/du)^2$ decreases from 0.25 to 0.032, and the principal part of the effective range of values of this factor is covered therefore by (13.2). Employing this approximation, one obtains from (12.6) and (13.1)

$$\begin{aligned}\int_0^\infty \frac{e^{i\omega t}}{1+\cosh w}dw &\cong \frac{121}{64}\left\{\frac{\pi}{5.5}e^{-2.75\xi}\right.\\ &\quad -\frac{i}{5.5}e^{2.75\xi}\,\mathrm{Ei}(-2.75\xi)\\ &\quad \left.+\frac{i}{5.5}e^{-2.75\xi}\,\bar{\mathrm{E}}\mathrm{i}(2.75\xi)\right\}\\ &\sim 1.89\left\{\frac{\pi}{5.5}e^{-2.75\xi}\right.\\ &\quad \left.+\frac{i}{2.75}\left[\frac{1}{2.75\xi}+\frac{2!}{(2.75\xi)^2}+\cdots\right]\right\}, \quad (13.3)\end{aligned}$$

where

$$-\mathrm{Ei}(-x)=\int_x^\infty (e^{-t}/t)dt, \quad \bar{\mathrm{E}}\mathrm{i}(x)=P\int_{-\infty}^x (e^t/t)dt,$$

and P stands for "principal value of." For sufficiently large ξ, Eq. (13.3) reduces to

$$\int_0^\infty \frac{e^{i\omega t}}{1+\cosh w}dw\cong\frac{0.25i}{\xi}. \quad (13.4)$$

Combining this result with (12.5),

$$A_0\cong 0.145k\langle r_p\rangle/(Z_2\xi), \quad (13.5)$$

while the other two classical amplitudes are zero as in (12.5). The calculations so far have been made for the special case $\epsilon=1$, i.e., for orbits corresponding to head on collisions and the limits of integration for w correspond to the perihelion. Without these specializations one can observe that for close approach and high excitations

$$\left|\int_{-\infty}^{+\infty}\frac{e^{i\omega t}dw}{1+\epsilon\cosh w}\right|$$

is much smaller than

$$\left|\int_{-\infty}^{w}\frac{e^{i\omega t}dw}{1+\epsilon\cosh w}\right|,$$

and that therefore

$$\int_{-\infty}^{w}\frac{e^{i\omega t}dw}{1+\epsilon\cosh w}\cong-\int_w^\infty\frac{e^{i\omega t}dw}{1+\epsilon\cosh w}.$$

On the other hand, if one sets $u=w+\epsilon\sinh w$, the integral in the last equation is

$$\begin{aligned}\int_w^\infty e^{i\xi u}\left(\frac{dw}{du}\right)^2 du&=\int_w^\infty\left(\frac{dw}{du}\right)^2 d\left(\frac{e^{i\xi u}}{i\xi}\right)\\ &\cong-\frac{e^{i\xi u}}{i\xi}\left(\frac{dw}{du}\right)^2=\frac{ie^{i\xi u}}{\xi(1+\epsilon\cosh w)^2}. \quad (13.6)\end{aligned}$$

According to (12.3), the excitation amplitude is expected therefore to be proportional to $1/r^2$. This dependence on r is similar to that of $\psi^{(1)}{}_{-\mu}$ in Eq. (11.3). The factor ψ^c in that equation also contains r, but since the classical estimate is made in terms of the ratio of the probability of excitation to the probability of finding the particles in a certain relative position, the comparable quantum estimate must be made by taking the factor multiplying ψ^c in (11.3). The entrance of r in ψ^c should not be counted therefore in the comparison with (13.6), and the comparison indicates approximately the same variation of the probability of excitation with distance in the two cases.

The relative probabilities of excitation will now be compared at the perihelion, i.e., $r=2a'$. According to (11.3), the factor multiplying ψ^c is in this case, disregarding signs,

$$\frac{K|Y_{1,\mu}|}{|\Delta E|(2a')^2}, \quad (13.7)$$

which is to be compared with A_0 of (13.5). Employing the value of K from (6.1), the ratio of the quantum-mechanical amplitude to A_0 reduces to unity if one sets $|Y_{1,\mu}|$ in (13.7) equal to $(3/4\pi)^{\frac{1}{2}}$. This number is just the value of $|Y_{1,\mu}|$ which was used in obtaining

(12.5) on the SCT, the reason for using it having been that in the plane of the orbit only $Y_{1,0}$ is distinct from zero. The correspondence between the SCT calculation and the one by the quantum treatment of the relative motion is seen to be rather close for the case of high excitations.

The quantum calculation may be expected to be relatively better for high excitations than for those barely resulting in ionization, because for high excitations the treatment of the Green's function as having a very sharp cusp is more nearly justified. The approximation of Eq. (13.4) for the SCT also involves the assumption of large ξ and is therefore more questionable when applied to excitation resulting in bare ionization than to high excitations. The comparison of the quantum and SCT calculations by means of the approximations worked out in the present paper is not as significant therefore as the comparison for high excitations. Making the comparison for $r=2a'$ by employing Eq. (9.2) for the quantum case and Eq. (13.5) for the SCT and setting $Y_{1,0}=(3/4\pi)^{\frac{1}{2}}$ in the quantum calculation, the ratio

$$A_{\text{quant}}/A_0 \cong \Delta E/E. \tag{13.8}$$

In the application to $N^{14}+N^{14}$, one is interested especially in a bombarding energy of 10 Mev, corresponding to $E=5$ Mev, and the excitation energy is the tightness of binding of a nucleon, which is about 10 Mev. The right-hand side of (13.8) is thus of the general order of unity in this case. The quantum estimate gives a slower decrease of the excitation amplitude with distance than the classical one. There is also some penetration into the region of negative kinetic energy for the heavy aggregates implied in ψ^c which is absent in the SCT. These effects are in the direction of making the quantum effect relatively the more important one. It should be observed however that the term $k^2d^2F_L''(kr)$ in (8.7) has been omitted in the estimates and that through the oscillatory region F_L'' has a sign opposite to that of F_L. This correction is in the direction of decreasing the quantum quantity but the effect of this correction is least important in the vicinity of the classical turning point. It is seen therefore that the SCT gives a fair approximation to the quantum result.

Returning to the consideration of Eq. (4), one sees that on its right side there is present in Ψ^{CN} a contribution caused by the presence of Coulomb excited waves having their origin in $\sum_\mu w_\mu\psi^{(1)}{}_\mu$ of Eq. (5.2). This part of Ψ^{CN} gives rise to contributions to $\Psi^{(1)}$ of Eqs. (3.4) and (4). Since the w_μ are more highly excited than the ground state function v_i, the tunnelling factor in the nucleon transfer calculations[1,11] varies for them less critically and transfer from these states may be expected to take place more readily once they are present. No attempt will be made here to take their effect into account exactly, the main object being to consider the qualitative implications of the virtual state Coulomb excitation effects. The ratio of the high excitation form of $\psi^{(1)}{}_\mu$ to the form derived for bare removal of a nucleon gives, by comparison of (11.3) with (9.2),

$$\frac{Z_1Z_2e^2/r}{E_w-E} \cong \frac{\text{amplitude for high excitation}}{\text{amplitude for excitation to bottom of continuum}}. \tag{13.9}$$

For closest approach, the numerator in the above formula is the kinetic energy in the center-of-mass system. For $N^{14}+N^{14}$ at a bombarding energy of 10 Mev and for $E_w-E=20$ Mev, the above ratio is 1/4. The comparison indicates therefore that the excitations to the lower part of the continuum are the more important ones.

In order to make numerical estimates, it is necessary to remove the specialization in the calculations which was made in considering s-p excitations only. These will not actually be important for the valence shell of N^{14}, since this is a p-shell. The $\langle r_p\rangle$ of Eq. (6.1) thus has to be interpreted in terms of the dipole transition matrix element by setting

$$(e\langle r_p\rangle)^2=\sum_f[|\mathfrak{M}_{fi}{}^{(x)}|^2+|\mathfrak{M}_{fi}{}^{(y)}|^2+|\mathfrak{M}_{fi}{}^{(z)}|^2], \tag{14}$$

where $\mathfrak{M}$ is the electric dipole operator and the sum is extended over the sublevels of the upper state. The dipole operator is that of the whole nucleus and includes the effect of the electric field on all nuclear protons. The correctness of the above identification can be verified by observing that for the s-p transition

$$Z_{\mu=0,i}=3^{-\frac{1}{2}}\langle r_p\rangle,$$

and employing spectroscopic stability. The replacement of $\langle r_p\rangle$ by its generalization by means of (14) takes care of a decrease in the effect caused by the near equality of charge-to-mass ratio of the proton and the remainder of the nucleus as well as the increase caused by the presence of seven protons instead of one in the N^{14} nucleus. An estimate of the right side of Eq. (14) can be made by means of the Thomas, Reiche, Kuhn sum rule which is usually stated in terms of f values. According to Levinger and Bethe,[4] this sum rule assumes the form

$$\int_0^\infty \sigma(E_\gamma)dE_\gamma=\frac{2\pi^2e^2\hbar}{Mc}\sum_f f_{if} = 0.060\frac{NZ}{A}\ \text{Mev-barn}, \tag{14.1}$$

where in the subscript i and f stand for the initial and final states, respectively, and the sum is taken over final states including sublevels. The last quantity listed in

[11] Breit, Hull, and Gluckstern, Phys. Rev. **87**, 74 (1952); G. Breit, Phys. Rev. **102**, 549 (1956); G. Breit and M. E. Ebel, reference 1; M. E. Ebel, Phys. Rev. **103**, 958 (1956).

the above equation does not include exchange-effect modifications[4,12] of the sum rule. The first quantity is the energy integral of the photodisintegration cross section, approximate values of which are available from experiment. The quantity of the middle entry in (14.1) contains numbers of equivalent harmonic oscillators of dispersion theory, f_{if}, for the nucleus as a whole, with e, M standing respectively for the proton charge and mass. The harmonic oscillators are also supposed to have charge and mass values e and M. In the treatment of the present paper, the effect of the whole nucleus was represented for simplicity by that of a single proton. One obtains the correspondence between the $|\mathfrak{M}_{fi}|^2$ and the f_{if} therefore by employing the relation between these quantities for charge e and mass M. This relation is[13]

$$e^2 f_{if}=\frac{2ME_{ji}}{3\hbar^2}(|\mathfrak{M}_{fi}^{(x)}|^2+|\mathfrak{M}_{fi}^{(y)}|^2+|\mathfrak{M}_{fi}^{(z)}|^2). \quad (14.2)$$

Insertion in (14.1) gives

$$\frac{4\pi^2}{3\hbar c}\sum_f E_{fi}\sum_{xyz}|\mathfrak{M}_{fi}^{(x)}|^2=0.060\frac{NZ}{A}\text{ Mev-barn}. \quad (14.3)$$

Lumping the whole effect in the vicinity of one energy E_w and letting $\Delta E=E_{fi}$ and making use of (14) one has approximately from the above equation

$$\frac{4\pi^2}{3}\frac{e^2}{\hbar c}\Delta E\langle r_p\rangle^2\cong 0.060\frac{NZ}{A}\text{ Mev-barn}, \quad (14.4)$$

which gives

$$\langle r_p\rangle^2=0.63\frac{NZ}{A\Delta E}\text{ barn}, \quad (14.5)$$

provided ΔE is measured in Mev. For $\Delta E=10$ Mev, one obtains

$$\langle r_p\rangle=4.7\times10^{-13}\text{ cm }(\Delta E=10\text{ Mev}) \quad (14.6)$$

as the equivalent value of $\langle r_p\rangle$.

A crude comparison with the tunnel penetration effect can now be made by writing

$$\sigma_{vs}/\sigma_{tun}\cong a_1 a_2, \quad (15)$$

where a_1 is the probability of virtual state (vs) formation at the perihelion, and a_2 the ratio of tunnel penetration probability from the virtual state to that from the ground state.

For excitation to an excitation energy of 10.55 Mev as measured from ground, the neutron is just ready to be detached. The value of ξ is 9.6 and the SCT factor correcting the amplitude for finite excitation is $1.89/[9.6\times(2.75)^2]=0.026$ otherwise obtainable from (13.4) as 0.25/9.6. The value of a_1 on the SCT without the correction factor for excitation is, according to (12.5), $[k\langle r_p\rangle/(3^{\frac{1}{2}}Z_2)]^2$. For a bombarding energy of 10 Mev, one has $k=10^{14}/(7.72\text{ cm})$ and the above-mentioned parameter is 0.251. The resultant value of a_1 is therefore $(0.026)^2\times0.251=1.7\times10^{-4}$. As a first approximation, one may use for a_2 the barrier penetration, tunneling from the ground state, factor which is given[14] by Eq. (28.1) of reference 1. For 10-Mev bombarding energy $\alpha=7.13\times10^{12}$ cm^{-1}, $2a'-2b=7.31\times10^{-13}$ cm, and $2\alpha(2a'-2b)=10.42$. Assuming for the present that the virtual state is equally effective in its transfer except for absence of the barrier penetration, it is $e^{10.42}=3.36\times10^4$ times more effective. The gain over tunneling from the ground state is therefore $3.36\times10^4\times1.7\times10^{-4}=5.7$. This estimate indicates that the virtual state formation process is the more important of the two considered here. The estimate as made is defective regarding the following matters.

(a) As has been considered in connection with Eq. (3.4) the virtual state formation process should enter in the consideration of the final Ψ^{CN} as well as in that of the initial. The initial wave without Coulomb excitation is coupled by this process to the Coulomb excited virtual states of $N^{13}+N^{15}$. This coupling gives rise to an additional term for the final amplitude which is of the same order as that estimated. The phase relation of this contribution to the one estimated has not been ascertained. Assuming the phase relation to be random as in quadrature, one expects a factor of about 2 in addition to the factor 5.7.

At first sight it would appear that a similar circumstance neglected in the estimates is that both nuclei, rather than one, are excited in the initial state. Since each excitation leads to a transfer in a definite direction this omission, apart from interference effects caused by particle identity, cannot be distinguished from the consideration of counting recoils. Since the tunneling from the ground state estimate as used here does not include these either, this effect can be omitted.

(b) The application of the correction factor for excitation by means of the SCT estimate and the employment of the SCT estimate for the condition close to bare removal of a nucleon are not quantitatively reliable. The agreement of the quantum result (9.2) with SCT was obtained for example by replacing $Y_{1,\mu}$ in (9.2) by $(3/4\pi)^{\frac{1}{2}}$ and counting only one μ instead of taking angular integrals. Secondly, there is a factor $1/r$ in (9.2) which decreases the probability of virtual state formation at large r. For the latter reason one might expect the virtual state process to be overestimated, if it were not for the fact that the collision for the virtual state part of Ψ is less adiabatic since the probability of the presence of the virtual state varies

[12] E. Feenberg, Phys. Rev. 49, 328 (1936); A. J. F. Siegert, Phys. Rev. 52, 787 (1937).

[13] S. A. Korff and G. Breit, Revs. Modern Phys. 4, 471 (1932).

[14] A factor 2 should be inserted in the exponential on the right side of Eq. (28.1) of reference 1. Furthermore, the nuclear radius, denoted in reference 1 by a, is indicated in the present work by b, a being reserved for the Bohr length of the collision.

with r. For the latter reason the transfer via the virtual state should be more effective.

(c) The comparison of virtual state transfer with ground state transfer is crude because the $\psi^{(1)}{}_\mu w_\mu$ penetrate to the other nucleus very effectively, and the application of perturbation theory such as has been used for the ground tunneling can give only a rough approximation for the $\psi^{(1)}{}_\mu w_\mu$ part of ψ. The w_μ factor can be expected to be strongly perturbed by the second nucleus, making the estimates as carried out here have a qualitative significance only. This is especially true on account of the approximate equality of binding energies of the last neutron in N^{14} and N^{15}.

(d) The effect of virtual-state formation may have been overestimated because the virtual state function consists partly of components in which the wrong kind of nucleon is excited. Also the virtual state function may have parts in which the residual nucleus is excited, making it necessary for a rearrangement of particles to take place in it before the nucleon transfer is completed.

(e) On the other hand, the excitation of the residual nucleus can be expected to lead to an enlargement in its radius and a consequent possibility of participation of a second nucleon in the transfer.

(f) The possibility of interference from different parts of the continuum or from the discrete part of the excitation spectrum has been neglected in the lumping procedure which gave rise to (14.4). For the effect under study it is the sum of the $\mathfrak{M}_{fi}$ that matters while the f sum rule is concerned mainly with the sum of the $|\mathfrak{M}_{fi}|^2$. If one distributes the contribution to the f sum into two parts, each one half of the original, there are two contributions to the $\sum\mathfrak{M}_{fi}$, each $1/\sqrt{2}$ of the original. If these are in phase, as they are expected to be according to (11.3) and (9.2), for the same μ, the value of $\sum\mathfrak{M}_{fi}$ is $2/\sqrt{2}=\sqrt{2}$ times the original and the probability of transfer is increased by a factor 2. There exists therefore a possibility of an increase in the effect for this reason.

(g) The effect of correction for the exchange character of nuclear forces has not been included. This effect has been included by Levinger and Bethe[4] in their estimates of the theoretically expected photodisintegration and it increases the effect. In the comparison with experiment carried out by Levinger and Bethe there is evidence for the existence of the effect, at least to the extent of providing an empirical correction to their estimates of the integrated photodisintegration cross section made by means of the f sum rule.

(h) The comparison has been made for the perihelion of the collision leading to largest deflection. This is the most favorable condition for ground state tunneling. The virtual state process is for this reason underestimated, since for it proximity of the heavy aggregates does not enter as critically.

(i) Dipole effects only have been considered so far.

The presence of these effects makes it clear that the estimates presented have no claim to quantitative validity. On the other hand, they show that virtual state formation by Coulomb excitation is more important at low bombarding energies than leakage through the region of negative kinetic energy for nucleons in their ground state.

An exact calculation of the energy dependence and of the angular distribution will not be attempted in the present paper. A few qualitative considerations will, nevertheless, be carried through. Their object is to see whether virtual state formation can conceivably fit the observations of Reynolds and Zucker.[2] The problem of capture of the nucleon from a virtual state by the nucleus carrying charge Z_1 has some analogies to that of the capture of an electron by an ion traveling through a gas. The principal viewpoints regarding this problem have been treated by Thomas,[15] Bohr,[16] and Brinkman and Kramers.[17] The important part of the process can be pictured by regarding the electron as released by the parent atom as soon as the forces exerted on it by the atom and ion are equal and counting it as attached to the ion whenever its energy in the system of the ion is negative at the instant of release. Applying this picture to the transfer of the nucleon from the virtual state to (Z_1,m_1), conditions for the application of the criterion are simple for virtual states in the continuum. The condition for release is satisfied whenever the nucleon is inside the capturing nucleus or even considerably outside it. A wide range of energies in the continuum satisfies the condition for the energy being negative at the capturing nucleus. As the two nuclei approach, the virtual state wave function increases in amplitude and some virtual state nucleons thus move through the capturing nucleus. As an approximation, it will be supposed that the fraction $(\pi b^2)/(4\pi r_0^2)$ of all virtual state nucleons is intercepted and captured. Here r_0 is the distance of closest approach for a given orbit. This criterion leaves out of account many relevant features of the process, in particular the influence of the adiabatic character of the collision on the lack of effectiveness of the transfer. The anisotropy of the virtual state also was neglected. Such factors can be included in a later improvement, however. The probability of transfer for an orbit will thus be taken provisionally as

$$a_1(b^2/4r_0^2). \tag{15.1}$$

In terms of the impact parameter p, the cross section corresponding to the annular region between p and $p+dp$ is $2\pi pdp$. Close to nuclear dissociation the probability of the virtual state is obtainable from (11.3). Taking $\sum_\mu|\psi^{(1)}{}_\mu|^2$, one obtains $|\langle r_p\rangle/(Z_2r)|^2|\psi^c|^2$, which indicates that $|\langle r_p\rangle/(Z_2r_0)|^2$ is the probability

[15] L. H. Thomas, Proc. Roy. Soc. (London) **A114**, 661 (1927).
[16] N. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **18**, No. 8 (1948).
[17] H. C. Brinkman and H. A. Kramers, Proc. Acad. Sci. Amsterdam **33**, 973 (1930).

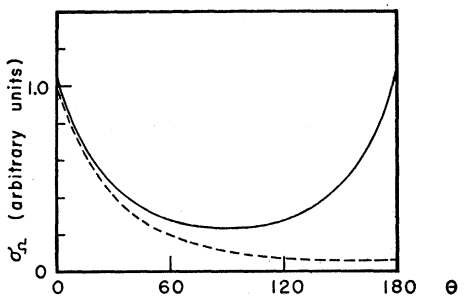

FIG. 1. Relative values of differential transfer cross section $\sigma\Omega$ for low virtual state excitation plotted against angle θ in degrees. Dashed curve neglects recoils, solid curve includes recoils but neglects interference.

of there being a virtual state at the perihelion. At $E_{\text{bomb}}=10$ Mev, employing Eq. (14.6), the chance of a virtual state is then, on substituting numbers,

$$(1/440)(2a'/r_0)^2. \tag{15.2}$$

By means of (12.3), one finds

$$2\pi p dp=\frac{a'^2 d(2\pi\cos\theta)}{4\sin^4(\theta/2)}=\frac{a'^2 d\Omega}{4\sin^4(\theta/2)}, \tag{15.3}$$

where $d\Omega$ is the solid angle for the deflected particles corresponding to the annular region $(p, p+dp)$. Combining the value of the area with (15.1) and (15.2), the chance of transfer per unit solid angle of scattered direction is

$$\sigma_\Omega=\frac{b^2}{17600[1+\sin(\theta/2)]^4}. \tag{15.4}$$

One has

$$\int[1+\sin(\theta/2)]^{-4}d\Omega=(2\pi/3), \tag{15.5}$$

and employing $b=1.5\times10^{-13}A^{\frac{1}{3}}$ cm, $b^2/17600=7.4\times10^{-29}$ cm^2 so that $\int\sigma_\Omega d\Omega=15.6\times10^{-29}$ cm^2. This value is larger than the experimental σ for either neutrons or protons at this bombarding energy, both of which are between 10^{-29} and 10^{-28} cm^2. Since there are several factors which can make the cross section larger, there appears to be no objection to supposing that the partly adiabatic nature of the collision has some effect and also allowing some of the virtual states to be in the region of high excitation energies, for which, it will be remembered, the virtual state formation is less probable according to (13.9).

The estimate of the total cross section is not in the way of assuming that at 10-Mev bombarding energy the transfer is mainly caused by virtual state formation. The energy dependence expected for the mechanism used in obtaining (15.4) will now be estimated. There is a factor $1/E^2$ present in a'^2 of (15.3). Similarly there is a factor $1/E$ in every r_0 which entered in the factors (15.1) and (15.2). In (15.1) there is thus a factor E^2. In (15.2) there was at an earlier stage a factor $1/r^2$ giving rise to E^2. There is a net factor E^2 which gives a factor 2.2 between 10 and 15 Mev. This factor is smaller than the experimental $9.0\times10^{-28}/5.0\times10^{-29}=18$. One can try to explain the discrepancy by assuming that at 15 Mev the virtual state process is subordinate to nucleon tunneling which could be supposed to have increased by a larger factor than the experimental σ in this energy range. It has been found[1] in fact that the nucleon tunneling should increase much more rapidly with energy than the observed σ.

Such an explanation cannot be reconciled with the angular dependence of σ at 16.3 Mev. The latter shows a depression at small angles. On the other hand, the above model gives a pronounced peak at small angles which is the residue of the large small-angle Rutherford scattering. In Fig. 1 are shown two plots. One of them is the factor $1/[1+\sin(\theta/2)]^4$ of (15.5), which corresponds to the angular variation of $\sigma_{\Omega,\text{trans}}$ forgetting about recoil particles. The second represents $1/[1+\sin(\theta/2)]^4+1/[1+\cos(\theta/2)]^4$ and includes the effect of recoils. The experience in reference 1 shows that the effect of interference between the two effects is not major and it will be neglected therefore. Even a factor 4 at $\theta=90°$, which is a maximum possible estimate for the interference effect and is larger than the factor 2/3 calculated in reference 1 can only succeed in flattening the curve to an approximately horizontal line but does not reproduce the observed rise at intermediate angles. A dominance of the nuclear tunneling effect at 19.2 and 21.1 Mev would also give peaks of σ both at $\theta=0°$ and $\theta=180°$ which is in disagreement with experiment. This part of the argument is not as strong as the first because there could conceivably be interference between direct tunneling and virtual state transfer. Also the higher energies are not clearly cases of transfer at a distance. The lower energy observations appear to be sufficient, however, to exclude the model.

The situation changes completely if one uses the dependence on r to be expected from the high-excitation conditions. The extra power of $1/r$ in the amplitude for virtual state formation present in (11.3) gives an extra factor $1/r_0^2$ in the differential cross section which becomes proportional to

$$\sin^2(\theta/2)/[1+\sin(\theta/2)]^6, \tag{15.6}$$

and taking into account recoils but neglecting interference between the two amplitudes one expects σ to be proportional to

$$\frac{\sin^2(\theta/2)}{[1+\sin(\theta/2)]^6}+\frac{\cos^2(\theta/2)}{[1+\cos(\theta/2)]^6}. \tag{15.7}$$

Plots of these quantities are shown in Fig. 2. They are seen to be reasonably similar to the experimental σ. The data plotted are for the case of neutron transfer; however, it may be expected that neutron transfer and proton transfer should be approximately equal in this

model. Since nearly the same type of distribution is observed at 16.3, 19.2, and 21.1 Mev, it appears probable that the virtual-state formation process is dominant through most of the energy range. At the higher energies the direct tunneling process does not differ from the experimental curve in as pronounced a way as at low energies. One cannot exclude some direct tunneling at 19.2 Mev, but at 16.3 Mev the proportion of cases taking place by direct tunneling must be small. In comparing total yields at 10 and 15 Mev, it is impossible to assume therefore that at 15 Mev direct tunneling is dominant.

The dependence on energy corresponding to (15.6) and (15.7) differs from that for the first model through the inclusion of an extra factor $1/r$ in (11.3) as compared with (9.2). There is therefore an extra $(1/r_0)^2$ in the expression for the transfer probability. Since r_0 contains a' and a' is inversely proportional to E, there appears an extra factor E^2 corresponding to an E^4 rather than E^2 dependence of σ on E. The expected factor between 10 and 15 Mev is $(1.5)^4=5.1$ as compared with the observed ratio of 18. There remains a discrepancy of a factor 3.5. This may be partly accountable for by the participation of direct tunneling at the higher energy. Such participation cannot be excluded because the direct and virtual state processes give rise to the same final states, so that interference terms between the two types of angular distributions cannot be excluded. It is improbable that one could account for the whole factor 3.5 on this basis, since this would require dominance of direct tunneling at 15 Mev which is hard to reconcile with the angular distribution. The factor to be accounted for is appreciably smaller, however, than for the first model. It is seen therefore that high rather than low excitation is favored by experiment.

The outstanding discrepancy is the factor 3.5 in the total yield for the high excitation model. This factor cannot be considered as an objection to the explanation in terms of virtual state formation on account of the following omissions in the considerations. In the first place, the virtual states must be appreciably distorted by the presence of the capturing nucleus. The distortion is more pronounced at the higher energies. At 10 Mev the distance between closest points of nuclear surface $2a'-2b=7.3\times10^{-13}$ cm while $2a'=14.1\times10^{-13}$ cm. At 15 Mev, $2a'=9.4\times10^{-13}$ cm and the distance between closest points is only 2.6×10^{-13} cm. It would not be surprising if both the direct tunneling and virtual state formation estimates were seriously affected. In particular the virtual state could conceivably form a more intimate combination with the capturing nucleus than has been considered, leading to a relatively higher yield. Secondly the influence of the adiabatic nature of the collision on the probability of transfer from the virtual state has not been taken into account. In this connection there enters the collision time of the process. Approximating the radial motion at the perihelion by a uniformly accelerated one and defining the collision time as the time necessary to move from r_0 to $2r_0$, the collision time is

$$\Delta t=\{2r_0/[v^2-(Z_1Z_2e^2/mr_0)]\}^{\frac{1}{2}}.$$

For close collisions, $\Delta t=2r_0/v$. For distant collisions, $\Delta t\cong 2^{\frac{1}{2}}(r_0/v)[1+a'/(2p)]$. Approximately, therefore, Δt is proportional to r_0. For close collisions at 10 Mev, one has $\Delta t=181\ e^2/mc^3=169\times10^{-23}$ sec. The time needed for a nucleon with 20-Mev kinetic energy to traverse the nuclear radius $b=3.62\times10^{-13}$ cm is 5.75×10^{-23} sec. The collision time is long compared to this time. It is also long compared with the time $\hbar/\Delta E$ where ΔE is the excitation energy. If we take for the latter ΔE a nominal 5 Mev, the time is $\hbar/(10mc^2)=13.7e^2/mc^3=12.6\times10^{-23}$ sec, which is again short compared with Δt. The latter fact means that Coulomb excitation is small compared with virtual state formation at the perihelion. The former indicates that a potential well will behave adiabatically during the collision time. It would appear at first sight that no capture is to be expected. In the consideration of capture, however, there enters the time of passage between the nuclei and return which is $2\times14.1\times10^{-13}$ cm/3.62×10^{-13} cm$=7.8$ times longer than the time to traverse b. The value of Δt is furthermore too large because the amplitude of the virtual state decreases by a factor 1/4 during that time and in addition there is a decrease at the capturing nucleus by a factor 1/2 caused by the variation of the amplitude of the function in space. The relevant times are therefore comparable, leading to the expectation of an appreciable transfer probability but with a dependence on the collision time. This dependence produces effects similar to those caused by going from the low-excitation to the high-excitation model. It subdues the effectiveness of small-angle collisions and produces therefore the same kind of difference as that between Fig. 2 and Fig. 1. Besides it subdues the effectiveness of low energies on account of the more nearly adiabatic character of the collisions in this energy range. This is the desired type of effect for the high-excitation model which left a factor of about 3.5 in the energy dependence to be explained.

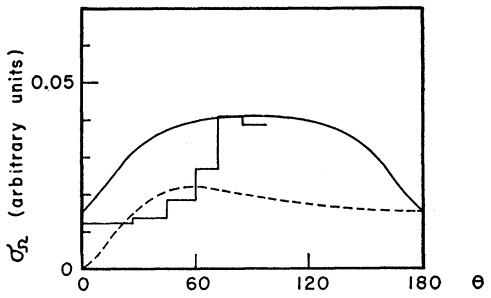

FIG. 2. Relative values of differential transfer cross section σ_Ω for high virtual state excitation plotted against angle θ in degrees. Dashed curve neglects recoils, solid curve includes recoils but neglects interference. The histogram shows experimental distribution[2] at 16.3 Mev.

In the above considerations, free use was made of classical mechanics in making estimates regarding the nature of the collisions. Among them there was the estimate of the period of the whole system with fixed positions of the heavy aggregates which enters a comparison with the collision time. In a quantum calculation there enters in place of this the time corresponding to the transition frequency between the initial and final states. In the case of $N^{14}+N^{14}\rightarrow N^{13}+N^{15}$, this time is especially long because of the approximate resonance in neutron energies, i.e., the near equality of binding of the last neutron in N^{14} and N^{15}. In mass units $N^{14}+N^{14}-(N^{13}+N^{15})=0.00028=0.261$ Mev. The value of $\hbar/0.261$ Mev is 240×10^{-23} sec which is longer than the collision time of 169×10^{-23} sec but of the same order. The qualitative situation is not changed by the quantum estimate. It suggests, however, that characteristic differences may be found in reactions with different energy evolutions.

It appears from the above discussion that the measurements of the differential cross sections suggest the way in which virtual state formation enters the collision process. The suggested picture is either excitation to the lower part of the continuum strongly influenced by the degree to which a given orbit performs an adiabatic rather than shock-type collision; or else excitation to higher energies with a smaller influence of the degree of adiabaticity appears at present to be equally acceptable.

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Energy Distribution of Mass-97 Fission Fragments from Thermal-Neutron Fission of Uranium-235

BERNARD L. COHEN, ANNA FONER COHEN, AND CHARLES D. COLEY
Oak Ridge National Laboratory, Oak Ridge, Tennessee

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The energy distribution of mass-97 fragments from thermal neutron induced fission of U^{235} was measured with a high-resolution magnetic spectrograph. The fragments originate in a thin plating of U^{235} near the center of the Oak Ridge National Laboratory graphite reactor and travel 16 feet to a wedge magnet which analyzes and focuses them at the focal plane 6 feet beyond. There they are caught in an aluminum foil which, after the irradiation, is cut into strips each of which is radiochemically analyzed for Zr^{97}. The $H\rho$ distribution is complicated by the large energy width which leads to overlapping of momentum distributions from successive charges. An analysis of the shapes of the $H\rho$ distributions obtained with different relative charge populations determines that the width of the energy distributions is $(11.4\pm0.8)\%$ corrected for broadening due to prompt neutron emission. This result is in agreement with measurements of the distributions of the number of neutrons per fission but is in sharp disagreement with the predictions of Fong's theory of the fission process. The most probable energy is 174.7 ± 2 Mev for the mass-97 fission mode, and about 164.5 ± 3 Mev for the mass-91 mode.

INTRODUCTION

THE energy spectrum of fission fragments of a given mass has been investigated by several groups[1] using ionization chamber techniques; they have obtained roughly Gaussian distributions with the most probable total kinetic energy release, $\bar{E}$, about 155 Mev, and full width at half-maximum, $\Delta E/\bar{E}$, about 15%. Recent investigations of ionization defects for fission fragments[2] have indicated that the former result is much too small, and experiments using both time-of-flight[3] and calorimeter[4] techniques have measured $\bar{E}$ to be about 167 Mev. The situation regarding the width of the distributions is considerably less certain. By comparing the results of time of flight and ionization chamber experiments, Leachman[3] concluded that the experimental dispersion in the latter experiments was about 9%, which would reduce $\Delta E/\bar{E}$ to about 12%. However, Fong,[5] analyzing the same data, found $\Delta E/\bar{E}$ to be only 8%. From measurements of range distributions in various gases, Good and Wollan[6] found $\Delta E/\bar{E}=6\%$; however, their corrections for foil thickness were quite large and appear to be inconsistent with other data. The general consensus has been that, due to the large dispersion and many other inaccuracies inherent in ionization chamber and absorption techniques, these experiments do not establish a lower limit to the widths of the energy distributions. It was decided, therefore, to measure these widths by magnetic analysis. In the process, new measurements of $\bar{E}$ were obtained.

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