Application of Adiabatic Functions to $N^{14}+N^{14}$ Reactions*

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Nucleon tunneling in reactions of the type $N^{14}(N^{14},N^{13})N^{15}$ and $N^{14}(N^{14},C^{13})O^{15}$ is calculated by using adiabatic wave functions constructed by the well-depth eigenfunction method. The procedure is applicable only if resonance effects in the transfer do not enter; in this case the results are shown to be equivalent with those obtained in the more detailed considerations of the energy matrix method.

I. INTRODUCTION

CONSIDERABLE interest has recently been attached to nuclear reactions between two nuclei such as N^{14}, in which a nucleon is transferred from one nucleus to the other. In particular, the role of simple tunnelling of the nucleon in $N^{14}(N^{14},N^{13})N^{15}$ and $N^{14}(N^{14},C^{13})O^{15}$ has been analyzed in some detail.[1] A special circumstance enters in the case of these reactions; namely, the binding energies of the last neutron in N^{14} and N^{15} are approximately equal, as are those of the last proton in N^{14} and O^{15}. The difference is about 0.3 Mev in each case. On the other hand, if such a resonance condition does not exist, it may be convenient to treat the tunneling process in terms of adiabatic functions, employing the well-depth eigenfunction method.[2] It is the purpose of this note to show that such a treatment of the $N^{14}+N^{14}$ reactions leads to the same approximate result as the energy matrix method of BE if the bombarding energy is so low that the change in energy of the adiabatic states from their values in the isolated nuclei is small compared to the difference of the energies of the states in the two nuclei, i.e., the resonance condition does not obtain.

In the present treatment, the nucleon is considered to move in a potential which is the sum of two potentials, each describing its interaction with one of the nuclei. In this respect these considerations are less general than those in BE, in which the nucleon behavior was described in terms of conditions at the nuclear surface. The reaction is considered in the semiclassical approximation, i.e., the two potential wells are assumed to move along the hyperbolic orbits of the classical problem. The validity of this approximation has been discussed in BE and in the references quoted therein.

The calculation proceeds in two steps. First, adiabatic wave functions which are solutions of the Schrödinger equation for one nucleon in two potential wells are obtained, with the fixed separation of the two nuclei treated as a parameter. As the separation becomes large, these wave functions approach functions describing conditions in which the nucleon is bound to one or the other of the nuclei. The one exception to this is the case of exact degeneracy arising when the two potential wells are identical. In this latter circumstance the asymptotic wave functions would be symmetric and antisymmetric combinations of the single-well eigenfunctions. The following treatment does not consider such a case.

The second step invokes the adiabatic approximation, in which the previously constructed functions are used as approximate time-dependent wave functions, the time entering through the dependence of the wave function on the separation of the two nuclei. The variation of the adiabatic functions with time is known explicitly if the motion of the nuclei is treated classically. The probability of a system which is in a state corresponding to the localization of the nucleon in one well at time $t=-\infty$ making a transition to a state corresponding to the nucleon being in the other well, is then calculated by the use of time-dependent perturbation theory. The result is found to agree approximately with that obtained from the energy matrix treatment of BE, within the limit of validity of the present treatment, which is just the range of bombarding energies in which the perturbation treatment as applied converges rapidly. The considerations regarding the effects of spin and of the identity of particles are just the same as those made in BE for the energy matrix approach. The notation used will be substantially that of B and BE. The fact that the adiabatic wave function and the energy matrix treatments agree regarding the entrance of penetrability factors may be regarded as additional evidence for conclusions reached by BE. In particular, the expected dependence of cross section on energy and on angle is very similar in the two approximations and the evidence regarding the existence of a process operating in addition to ordinary tunneling is thus supported by the present considerations.

II. ADIABATIC WAVE FUNCTIONS

It is desired to construct wave functions describing a single particle of positive separation energy interacting with two nuclei. These interactions are represented by potential energies $(\hbar^2/2M)V_1(\mathbf{r})$ and $(\hbar^2/2M)V_2(\mathbf{r})$,

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[1] G. Breit and M. E. Ebel, Phys. Rev. **103**, 679 (1956). This work will henceforth be referred to as BE.

[2] G. Breit, Phys. Rev. **102**, 549 (1956). This work will henceforth be referred to as B.

M being the nucleon mass; that is, solutions of the Schrödinger equation

$$[\Delta-\kappa^2-V_1(\mathbf{r})-V_2(\mathbf{r})]\psi(\mathbf{r})=0 \tag{1}$$

corresponding to states of energy $-\hbar^2\kappa^2/2M$ are sought. For simplicity, nucleon and nuclear spins are ignored, the results being easily generalized to include their effects. Equation (1) is solved by using the method of well-depth eigenfunctions outlined in B. The results of B which will be used below may be summarized as follows.

The potentials $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ are assumed not to overlap. Then it is useful to consider solutions to the related integral equations

$$\begin{aligned} u_{1n}(\mathbf{r})&=\lambda_{1n}\int K(\mathbf{r},\mathbf{r}';\kappa^2)V_1(\mathbf{r}')u_{1n}(\mathbf{r}')d\mathbf{r}', \\ u_{2m}(\mathbf{r})&=\lambda_{2m}\int K(\mathbf{r},\mathbf{r}';\kappa^2)V_2(\mathbf{r}')u_{2m}(\mathbf{r}')d\mathbf{r}', \end{aligned} \tag{1.1}$$

and

$$u(\mathbf{r})=\lambda\int K(\mathbf{r},\mathbf{r}';\kappa^2)[V_1(\mathbf{r}')+V_2(\mathbf{r}')]u(\mathbf{r}')d\mathbf{r}', \tag{1.2}$$

where the kernel satisfies

$$(\Delta-\kappa^2)K(\mathbf{r},\mathbf{r}';\kappa^2)=\delta(\mathbf{r}-\mathbf{r}'), \tag{1.3}$$

and is regular for $|\mathbf{r}|$, $|\mathbf{r}'|$ approaching infinity. If the eigenvalues λ, λ_{1n}, λ_{2m} were all unity, then solutions to Eq. (1.2) would be the desired eigenfunctions, and the solutions to Eq. (1.1) would be the ordinary Schrödinger eigenfunctions of the single-potential problem. As it is, solutions to Eq. (1.2) will be obtained in terms of the functions u_{1n} and u_{2m}, and the corrections for $\lambda\neq 1$ discussed in B will be shown to be negligible in the present application. The quantities $V_1(\mathbf{r})u(\mathbf{r})$ and $V_2(\mathbf{r})u(\mathbf{r})$ are expanded as

$$\begin{aligned} V_1(\mathbf{r})u(\mathbf{r})&=\textstyle\sum_n b_{1n}V_1(\mathbf{r})u_{1n}(\mathbf{r}), \\ V_2(\mathbf{r})u(\mathbf{r})&=\textstyle\sum_m b_{2m}V_2(\mathbf{r})u_{2m}(\mathbf{r}). \end{aligned} \tag{1.4}$$

The coefficients b_{1n}, b_{2m} satisfy the coupled equations

$$\begin{aligned} b_{1n}/\lambda&=(b_{1n}/\lambda_{1n})-\textstyle\sum_m V_{1nm}b_{2m}/\lambda_{2m}, \\ b_{2m}/\lambda&=(b_{2m}/\lambda_{2m})-\textstyle\sum_n V_{2mn}b_{1n}/\lambda_{1n}, \end{aligned} \tag{1.5}$$

with the matrix elements

$$\begin{aligned} V_{1nm}&\equiv(u_{1n},V_1u_{2m}), \\ V_{2mn}&\equiv(u_{2m},V_2u_{1n})=\lambda_{1n}(V_{1nm}/\lambda_{2m})^*, \end{aligned} \tag{1.6}$$

in the usual notation. The normalization of the u's has been taken to be

$$(u_{1n},V_1u_{1\nu})=-\delta_{n\nu};\quad (u_{2m},V_2u_{2\mu})=-\delta_{m\mu}. \tag{1.7}$$

These matrix elements are a measure of the ability of the nucleon to penetrate the barrier between V_1 and V_2, and for cases of practical importance will be small compared to one. These results from B will now be used to obtain the wave function for the particular system under consideration.

Since the barrier penetration is assumed to be small, a first-order calculation of the tunneling suffices. In this case only the states between which the transfer is supposed to take place enter into consideration, and the solutions to Eq. (1.2) are found to be

$$\begin{aligned} u_+&=N_+[u_{1n}-Au_{2m}], \\ u_-&=N_-[Au_{1n}+u_{2m}], \end{aligned} \tag{2}$$

with the definition

$$A\equiv|V_{1nm}|\lambda_{1n}/(\lambda_{2m}-\lambda_{1n}). \tag{2.1}$$

The eigenvalue $1/\lambda_+$ associated with u_+ differs from $1/\lambda_{1n}$ by terms of second order in the barrier penetration, while $1/\lambda_-$ differs from $1/\lambda_{2m}$ also by terms of this order. Accordingly, if the energy of u_+ is chosen such that $\lambda_{1n}=1$, and of u_- such that $\lambda_{2m}=1$, the well-depth eigenfunctions u_+ and u_- differ from energy eigenfunctions by terms involving the square of V_{1nm}. The quantity $\lambda_{2m}-\lambda_{1n}$ may be related to the energy difference of the levels of the isolated nuclei using Eq. (12) of B. There results

$$\lambda_{2m}-\lambda_{1n}=(2M/\hbar^2)[(u_{1n},u_{1n})(u_{2m},u_{2m})]^{\frac{1}{2}}(E_1-E_2), \tag{2.2}$$

and thus

$$A=(\hbar^2/2M)|V_{1nm}|\lambda_{1n}/\{(E_1-E_2)[(u_{1n},u_{1n})(u_{2m},u_{2m})]^{\frac{1}{2}}\}. \tag{2.3}$$

With this choice of the energies of the two states, u_{2m} in u_+ and u_{1n} in u_- are not identical with the energy eigenfunctions. For the particular transfer considered here, these differences are however small and, since they affect only terms which already have small coefficients, they may be neglected. The adiabatic functions may be rewritten in terms of the single-well energy eigenfunctions φ_{1n}, φ_{2m} as

$$\begin{aligned} \varphi_a&=N_a[\varphi_{1n}(\mathbf{r})-A(\mathbf{R})\varphi_{2m}(\mathbf{r})], \\ \varphi_b&=N_b[A(\mathbf{R})\varphi_{1n}(\mathbf{r})+\varphi_{2m}(\mathbf{r})]. \end{aligned} \tag{2.4}$$

The functions of Eq. (2.4) are normalized such that

$$(\varphi_a,\varphi_b)=0;\quad (\varphi_a,\varphi_a)=(\varphi_b,\varphi_b)=1. \tag{2.5}$$

These relations, together with the similar normalization of φ_1 and φ_2, imply that N_a and N_b are unity in this approximation. The wave functions given in Eq. (2.4) are nearly the same as those which would be obtained by taking θ as small in Eqs. (6.5) and (6.5′) of BE, the difference consisting in the presence of additional corrections to A resulting from the fact that $E_1\neq E_2$.

III. PROBABILITY OF TRANSFER

The adiabatic wave functions of Eq. (2.4) may now be used to obtain the probability of transfer of a nucleon from nucleus 1 to nucleus 2. Setting $A=0$ in Eq. (2.4), it is seen that φ_a describes a state which for large separation of the nuclei has the nucleon in nucleus 1, while φ_b corresponds to the nucleon being asymptotically in 2. Thus it is sufficient to calculate the transition probability from state φ_a to φ_b.

The wave function $\Psi(t)$ for the system is written as

$$\Psi(t)\simeq a(t)\varphi_a(\mathbf{r},\mathbf{R})\exp(-iE_at/\hbar) + b(t)\varphi_b(\mathbf{r},\mathbf{R})\exp(-iE_bt/\hbar), \quad (3)$$

indicating explicitly the dependence of φ_a and φ_b upon the nuclear separation $\mathbf{R}$, which is in turn a function of time. Substitution of this wave function in the time-dependent Schrödinger equation leads in the usual way to the coupled equations

$$\begin{aligned} da/dt+b(\varphi_a,d\varphi_b/dt)\exp(i\omega t)=0,\\ db/dt-a(\varphi_a,d\varphi_b/dt)\exp(-i\omega t)=0, \end{aligned} \quad (3.1)$$

with

$$\omega=(E_a-E_b)/\hbar=(E_1-E_2)/\hbar. \quad (3.2)$$

The quantity $(\varphi_a,d\varphi_b/dt)$ entering here may be calculated from Eq. (2.4), since

$$(\varphi_a,d\varphi_b/dt)=(d\mathbf{R}/dt)\cdot(\varphi_a,\nabla_R\varphi_b). \quad (3.3)$$

The variation of φ_b with $\mathbf{R}$ will be due to two effects, first, the dependence of A upon the barrier penetration, and second, the variations of φ_{1n} and φ_{2m} themselves. These latter variations are small, and will be neglected as is done in BE. The consideration of the former gives

$$(\varphi_a,\nabla_R\varphi_b)=(\varphi_1-A\varphi_2,\ \nabla_RA\varphi_1)=\nabla_RA \quad (3.4)$$

or

$$(\varphi_a,d\varphi_b/dt)=dA/dt, \quad (3.5)$$

with A given by Eq. (2.3). The initial conditions for a and b which apply in the solution of Eq. (3.1) are

$$a(-\infty)=1;\quad b(-\infty)=0. \quad (3.6)$$

Since it is assumed that A remains small over all the orbit, Eq. (3.1) may be solved subject to the assumption that $|a(t)|=1$. There results

$$\begin{aligned} |b(t)|&\simeq\left|\int_{-\infty}^{t}\frac{dA}{dt'}\exp(-i\omega t')dt'\right|\\ &=\left|\int_{-\infty}^{t}A(t')\omega\exp(-i\omega t')dt'\right|. \end{aligned} \quad (3.7)$$

Finally, the use of Eq. (2.3) for A gives for the probability of transfer

$$|b(+\infty)|^2=(\hbar/2M)^2 \times\left|\int_{-\infty}^{\infty}\{\lambda_{1n}|V_{1nm}|/[(u_{1n},u_{1n})(u_{2m},u_{2m})]^{\frac{1}{2}}\}dt\right|^2. \quad (3.8)$$

The exponential factor in the integrand has been dropped, since according to the estimates in BE, this changes the value of the cross section by at most 2%.

All that remains in establishing the connection between this treatment and that of the energy matrix method is to relate the matrix element V_{1nm} to the value of β occurring if the nucleon-nucleus interaction is described by a potential. The quantity entering is

$$\lambda_{1n}V_{1nm}=\lambda_{1n}\int u_{1n}^*(\mathbf{r})V_1(\mathbf{r})u_{2m}(\mathbf{r})d\mathbf{r}. \quad (4)$$

Using the differential equation satisfied by $u_{1n}^*(\mathbf{r})$, Eq. (4) becomes

$$\lambda_{1n}V_{1nm}=\int_{R1}[(\Delta-\kappa^2)u_{1n}^*(\mathbf{r})]u_{2m}(\mathbf{r})d\mathbf{r}, \quad (4.1)$$

the integral being extended over the region R_1 in which $V_1\neq0$. If Green's theorem is applied to this integral, there results

$$\lambda_{1n}V_{1nm}=\int_{S1}[u_{2m}\nabla_nu_{1n}^*-u_{1n}^*\nabla_nu_{2m}]dS, \quad (4.2)$$

since u_{2m} satisfies the free-field equation with energy κ^2 in the region R_1.

If the potential V_1 is spherically symmetric, as will be assumed, the surface integral serves to select out of all parts of $u_{2m}(r)$ that part which has the same angular dependence as $u_{1n}(r)$. The quantity which enters in the transition probability is homogeneous in the wave functions u_{1n}, u_{2m}; normalization factors will therefore be dropped. For the case in which a p-nucleon is transferred, the wave function $u_{1n}(r)$ and the p part of an analysis of u_{2m} in spherical harmonics about nucleus 1 are given in Eq. (1) and Eq. (1.2) of BE. The effect of transfer from various p-sublevels is included by using Eq. (24) of BE as the radial part of u_{2m}. Substitution of these wave functions in Eq. (4.2), and performing the indicated differentiations, gives

$$\lambda_{1n}V_{1nm}=[aR_{1n}(a)]^2 \times[\kappa a/(1+\kappa a)]^2(1/R)\exp[-\kappa(R-2a)], \quad (4.3)$$

where $R_{1n}(a)$ is the radial part of u_{1n} evaluated at the nuclear radius a. By virtue of the Hermitian property of V_{1nm} expressed in Eq. (1.6), the factor occurring in Eq. (3.8) becomes

$$\begin{aligned} |\lambda_{1n}V_{1nm}|/[(u_{1n},u_{1n})(u_{2m},u_{2m})]^{\frac{1}{2}}&\\ =(1/\lambda_1\lambda_2)^{\frac{1}{2}}[\kappa a/(1+\kappa a)]^2(1/R)&\\ \times\exp[-\kappa(R-2a)],& \end{aligned} \quad (4.4)$$

where Eq. (27.1) of BE, *viz.*,

$$1/\lambda_1 = |aR_{1n}(a)|^2/(u_{1n},u_{1n}), \qquad (4.5)$$

has been applied. Equation (4.4) is in fact just the expression for β given in Eq. (24.1) of BE. The same result for the transition probability is thus obtained if the condition of resonance between the states of the two nuclei may be ignored. The results for cross sections and angular distributions quoted in the previous work may therefore be carried over directly to this treatment in the absence of resonance. If Eq. (2) were written in terms of a transformation angle $\frac{1}{2}\tan^{-1}(2A)$, the integral for the transition probability would involve $\int\frac{1}{2}(\tan^{-1}2A)dt$, instead of $\int A dt$. The difference between the two results would only be less than 2% at 10 Mev and less than 30% at 15 Mev.

It may be remarked that, by employing exact adiabatic functions but omitting the process of linearly combining them to improve convergence as done by BE, a slightly incorrect answer is obtained. On the other hand, if the inexact form listed in Eq. (2) is employed and the linear combinations are not taken, the energy matrix answer of BE is reproduced, the errors of the two approximations being of a compensating nature.

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Nuclear Levels in S^{32}, S^{34}, Cl^{35}, and Cl^{37}†

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Barium-chloride targets have been bombarded with protons accelerated by an electrostatic generator to an energy of 7.04 Mev. Charged reaction products (alpha particles and elastically and inelastically scattered protons) were observed at 90 and 130 degrees to the proton beam with a high-resolution magnetic analyzer.

The following ground-state Q values have been measured: $Cl^{35}(p,\alpha)S^{32}$, $Q=1.863\pm0.008$ Mev; $Cl^{37}(p,\alpha)S^{34}$, $Q=3.026\pm0.008$ Mev. Levels were observed in S^{32} at 2.237, 3.780, 4.287, 4.465, and 4.698 Mev; in S^{34} at 2.127, 3.302, 3.915, 4.073, 4.114, 4.621, 4.685, and 4.876 Mev; in Cl^{35} at 1.221, 1.763, 2.645, 2.695, 3.006, (3.165), 4.058, 4.113, and 4.174 Mev; in Cl^{37} at 0.838, 1.728, (3.087), and (3.105) Mev.

I. INTRODUCTION

In a previous paper,[1] investigations were described of the level schemes of the nuclei S^{33}, S^{35}, Cl^{36}, and Cl^{38}. Barium-chloride targets were bombarded with deuterons, and the alpha particles and protons from the (d,α) and (d,p) reactions on chlorine were magnetically analyzed.

The same technique has been applied to the charged reaction products from the proton bombardment of these targets. This provides the level schemes of the nuclei S^{32}, S^{34}, Cl^{35}, and Cl^{37} through the (p,α) and (p,p') reactions on the chlorine isotopes Cl^{35} and Cl^{37}.

Very little was known of the Cl^{35} and the Cl^{37} level schemes; only three levels were known in S^{34}, while seven levels[2] had been found in S^{32}.

II. EXPERIMENTAL PROCEDURE

Protons were accelerated to an energy of 7.037 Mev with the MIT–ONR electrostatic generator.[3] Energies of charged reaction products emitted from the target at angles of 90 or 130 degrees to the proton beam were determined with a broad-range magnetic spectrograph.[4] The preparation of $BaCl_2$ targets has been described in the Cl+d paper.[1]

Four different bombardments were performed; one on a thick target at $\theta=130°$; one on a thin target at $\theta=130°$; and two on a thin target at $\theta=90°$ with different spectrograph field settings so as to focus either the high-energy (3.8 to 9.0 Mev) or the low-energy part (2.5 to 5.8 Mev) of the secondary particles on the nuclear emulsion serving for particle detection.

The assignment of particle groups to the responsible isotope was made by observing the energy difference

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