First-Order Vacuum Polarizability from the Principle of Causality

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(Received April 2, 1956)

A driving current induces in the vacuum a response current, but the response must not precede its cause. This application of the principle of causality enables one to derive the Pauli-Rose result for first-order vacuum polarization from the most elementary calculation in quantum electrodynamics: the absorption of a *single* photon (with $-k_\alpha k^\alpha > 4k_m^2$) by a negative-energy electron. This calculation is also given.

INTRODUCTION AND SUMMARY

AN appreciable part of the Lamb-Retherford shift arises from the polarization of the vacuum (Fig. 1) by the nuclear charge. The first-order part of the polarization can be expressed in the following terms: a *driving* charge-current,

$$j_p(x)=\int j_p(k)e^{ikx}d^4k, \tag{1}$$

induces in the vacuum a driven charge-current,

$$j_{p,\,\text{induced}}(x)=\int j_{p,\,\text{induced}}(k)e^{ikx}d^4k. \tag{2}$$

Invariance arguments connect the two quantities by a *polarization coefficient* (Fig. 2),

$$\begin{gathered} j_{p,\,\text{induced}}(k)/j_{p,\,\text{driving}}(k)=-h(k^2)=-h_1-ih_2, \\ h(-k)=h^*(k), \end{gathered} \tag{3}$$

that can depend only on the quantity $k^2=k_\alpha k^\alpha=-\Omega^2$. Of interest for the Lamb shift is the inductive part, h_1, of this response coefficient, which can be read from the results of Pauli and Rose,[1] obtained by subtraction methods.

One can avoid direct use of subtraction theory to find h_1 if one will calculate the absorptive part, h_2, and apply the principle of causality,[2]

$$h_1(\Omega_r)=\frac{2}{\pi}(1+\Omega_r^2)\lim_{\Omega_i\to 0^+}\int_0^\infty \frac{\nu h_2(\nu)d\nu}{(\nu^2+1)[\nu^2-(\Omega_r+i\Omega_i)^2]}+h_0. \tag{4}$$

Here the constant h_0 is fixed by the renormalization requirement, $h_1(\Omega_r)=0$ for $\Omega_r=0$. The causality requirement states that the vacuum should not respond before the driving current starts.

The absorptive part of the polarization comes out of an extremely simple calculation, remarkable both because it encounters the radiative perturbation at the most elementary possible level, and because the direct calculation seems not to have been made before. The process amounts to absorption of a single photon by a negative energy electron—but a photon that does not satisfy the relation $k^2=0$. In physical terms, one envisages two uniform distributions of charge of the same magnitude and opposite sign, one at rest, the other oscillating back and forth in the z direction with the same phase everywhere in space, and with a frequency $c\Omega$. The charge is zero. The current flows in a single direction, is independent of position, but depends upon time. Under the influence of this oscillatory current, electrons make transitions from negative to positive energy states at an easily calculable rate. The absorption gives rise to an energy dissipation described by the coefficient

$$h_2=(e^2/2\hbar c)SC^{-3}(C^2-\tfrac{1}{3}S^2), \tag{5}$$

where $C=\cosh\theta=\Omega/2k_m$, $S=\sinh\theta$, and $k_m=mc/\hbar$. From this result and the dispersion formula there follows immediately the Pauli-Rose result for the

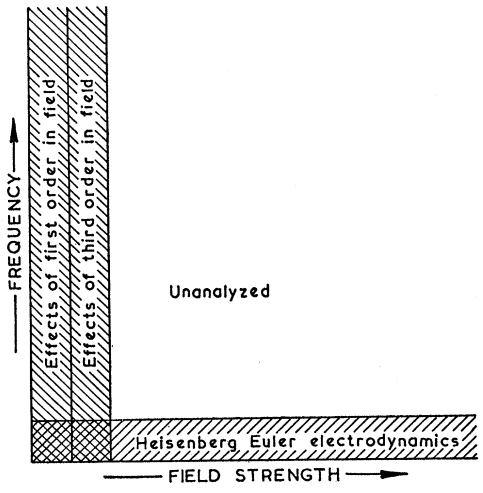

FIG. 1. Cases of vacuum polarization which have been analyzed so far with any completeness, except for vacuum polarization in a strong Coulomb field, for which see E. Wichmann and N. M. Kroll, Phys. Rev. **96**, 232 (1954). Third-order effects: R. Karplus and M. Neuman, Phys. Rev. **80**, 380 (1950). Low-frequency, strong field: W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936).

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[2] J. S. Toll, thesis, Princeton, 1952 (unpublished). We write $\Omega=\Omega_r+i\Omega_i$.

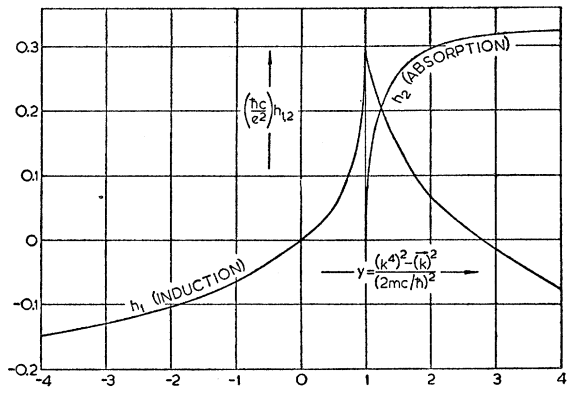

FIG. 2. Polarization of the vacuum as a function of frequency. Inductive part of the polarization coefficient, h_1; absorptive part, h_2. The dimensionless variable y has the value 1 at the threshold for the production of real pairs and more generally has the value $y=(k_4^2-k_1^2-k_2^2-k_3^2)/4k_m^2=-k_\alpha k^\alpha/4k_m^2=-k^2/4k_m^2=\Omega^2/4k_m^2$. Here the driving current is proportional to $\exp(ik_\alpha x^\alpha)$, and $k_m=mc/\hbar$.

inductive response,

$$h_1=\left(\frac{e^2}{\pi\hbar c}\right)\left\{\frac{5}{9}-\left(\frac{4}{3x^2}\right)-\frac{1}{3}\left(1+\frac{4}{x^2}\right)^{\frac{1}{2}}\left(1-\frac{2}{x^2}\right)\times\ln\left|\frac{(1+4/x^2)^{\frac{1}{2}}+1}{(1+4/x^2)^{\frac{1}{2}}-1}\right|\right\},\quad(6)$$

where $x^2=k^2/k_m^2$, with its well-known consequences for the Lamb shift.

DETAILS

A single photon cannot produce a pair, according to the laws of conservation of momentum and energy. Hence it is easy to see why this simplest of all radiative absorption processes receives no attention. However, the situation is different when a free electromagnetic wave is replaced by a forced electromagnetic disturbance of the form (1). Then the quantum energy and quantum momentum associated with the wave will suffice to raise an electron from a negative energy state to a positive energy state, provided that the condition is satisfied,

$$(k_4^2-k_1^2-k_2^2-k_3^2=-k_\alpha k^\alpha)=-k^2>4k_m^2\ (=4m^2c^2/\hbar^2).\quad(7)$$

Such a wave appears simplest in a frame of reference moving with the velocity $\mathbf{v}=c(k_1,k_2,k_3)/k^4$. There the strength of the wave depends not at all upon position, and oscillates in time with the frequency $c\Omega=c(-k_\alpha k^\alpha)^{\frac{1}{2}}$. There the electron is seen to jump from a negative energy state to a positive energy state without change of momentum; or the two components of the pair divide the available energy equally. Only the direction of the line of separation of negaton and positon and their polarizations remain free.

Switching to the preferred Lorentz frame, and recalling that the current

$$j_m(x)=J_me^{ikx}+\text{comp. conj.}\quad(8)$$

has zero divergence,

$$\partial j^\alpha/\partial x^\alpha=0,\quad\text{or}\quad J^\alpha k_\alpha=0,\quad(9)$$

we conclude that the driving disturbance is a pure current, $J^4=0$, which can be taken to oscillate in the z direction:

$$j_p=0,$$

except for

$$j_3=2J_3\cos k_4x^4.\quad(10)$$

The vector potential—in Lorentz gauge—satisfies the wave equation

$$\partial^2A_p/\partial x^\alpha\partial x_\alpha=-4\pi j_p/c,$$

with the solution

$$A_p=(4\pi J_p/ck^2)e^{ikx}+\text{c.c.}\quad(11)$$

for the driving potential. This potential produces the perturbation

$$-e\boldsymbol{A}=-(4\pi e\boldsymbol{J}/ck^2)e^{ikx}+\text{c.c.}\quad(12)$$

in the Lagrangian of the electron. Here $\boldsymbol{V}$ is an abbreviation for the "Feynman slash" spin matrix $V_\alpha\gamma^\alpha$ associated with any four-vector V_p.

The driven electron starts in the initial state

$$\langle x\,|\,i\rangle=(k_m/|\,b^4\,|)^{\frac{1}{2}}u_be^{ibx}.\quad(13)$$

Here b^4 is negative and close to $-\frac{1}{2}k^4$. The electron ends in the final state

$$\langle x\,|\,f\rangle=(k_m/a^4)^{\frac{1}{2}}u_ae^{iax},\quad(14)$$

where a^4 is positive and close to $\frac{1}{2}k^4$. Here both state functions are normalized to one electron per unit volume. The spinors u_a and u_b have the normalization

$$\begin{aligned}&u_a^*u_a=a^4/k_m;\quad u_a^\dagger u_a=1;\\&u_b^*u_b=|\,b^4\,|/k_m;\quad u_b^\dagger u_b=-1;\\&\textstyle\sum_{\text{polarization}}uu^\dagger=(k_m-i\boldsymbol{k})/2k_m.\end{aligned}\quad(15)$$

Here the Pauli conjugate of a spinor s has been designated as $s^\dagger=s^*i\gamma^4$.

In the absence of the perturbation, the state function at a new time would be given in terms of the initial state function at the original time by the integral $\int\langle x_2\,|\,t_2;t_1\,|\,x_1\rangle d^3x_1\langle x_1\,|\,i\rangle$, where $|\,t_2;t_1\,|$ is the normal free-electron propagator. The perturbation changes this propagator to first order by the amount

$$\delta\langle x_2\,|\,t_2;t_1\,|\,x_1\rangle=-(ie/\hbar c)\int\langle x_2\,|\,t_2;t_3\,|\,x_3\rangle\times\boldsymbol{A}(x_3)d^4x_3\langle x_3\,|\,t_3;t_1\,|\,x_1\rangle.\quad(16)$$

As a consequence there is a certain probability amplitude, φ, for the transition from the initial state to the final state:

$$\varphi_{fi}=\delta\langle f|t_2;t_1|i\rangle = \int\langle f|x_2\rangle d^3x_2\delta\langle x_2|t_2;t_1|x_1\rangle d^3x_1\langle x_1|i\rangle. \quad (17)$$

Direct integration gives the result

$$\varphi_{fi}=-(ie/\hbar c^2)k_m(a^4)^{-\frac{1}{2}}(|b^4|)^{-\frac{1}{2}}(4\pi/k^2)(2\pi)^4 \times(u_a^\dagger Ju_b)\delta_L(a^1-b^1)\cdots\delta_L(a^4+|b^4|-k^4). \quad (18)$$

The integration was allowed to run over the limited region $\Delta x^1=L^1, \cdots, \Delta x^4=L^4$ where the perturbation is considered to act. Consequently the functions δ_L are not the accurate Dirac δ functions. They satisfy the usual normalization requirement,

$$\int_{-\infty}^{+\infty}\delta_L(a^1-b^1)da^1=1, \quad (19)$$

but they have finite spreads, so that they satisfy also conditions of the form

$$\int\delta_L{}^2(a^1-b^1)da^1=L^1/2\pi, \quad (20)$$

conditions which may be written symbolically in the form

$$\delta_L{}^2(a^1-b^1)\text{ ``}=\text{''}\ (L^1/2\pi)\delta_L(a^1-b^1). \quad (21)$$

The probability for creation of a pair per unit of interaction volume and per unit of interaction time in the specified state is therefore

$$|\varphi_{fi}|^2/(L^1L^2L^3L^4/c) = (e^2/\hbar^2c^3)(k_m{}^2/a^4|b^4|)(4\pi/k^2)^2(2\pi)^4(u_a^\dagger Ju_bu_b^\dagger J^*u_a) \times\delta_L(a^1-b^1)\cdots\delta_L(a^4+|b^4|-k^4). \quad (22)$$

We sum over spin polarizations of the initial and final electron states, encountering the quantity

$$\mathrm{Tr}\Big(\sum_{\mathrm{pol},a}u_au_a^\dagger J\sum_{\mathrm{pol},b}u_bu_b^\dagger J^*\Big) = (1/4k_m{}^2)\ \mathrm{Tr}(k_m-i\mathbf{a}\cdot\boldsymbol{\gamma}+ia^4\gamma^4)(\mathbf{J}\cdot\boldsymbol{\gamma}) \times(k_m-i\mathbf{a}\cdot\boldsymbol{\gamma}-ia^4\gamma^4)(\mathbf{J}^*\cdot\boldsymbol{\gamma}) = (2/k_m{}^2)[(a^4)^2(\mathbf{J}\cdot\mathbf{J}^*)-(\mathbf{J}\cdot\mathbf{a})(\mathbf{J}^*\cdot\mathbf{a})]. \quad (23)$$

Let θ denote the angle between the direction of the driving current and the direction of emission of the negative electron. Then the emission probability plainly varies as $1-(v^2/c^2)\cos^2\theta$, being greatest at right angles to the current.

We multiply the probability (22) of transition between specified states by the number of positive energy states (normalized as in (15) to one electron per unit volume),

$$da^1da^2da^3/(2\pi)^3=|\mathbf{a}|a^4da^4d\Omega_a/(2\pi)^3, \quad (24)$$

and by a similar expression for the number of negative energy states and integrate. The integration over b^1 gives nothing except when b^1 is close to a^1; similarly for the other space components of the wave vector b. Thus the three space δ functions in (22) integrate at once. In addition, equality of the momenta of a and b implies the energy equality $|b^4|=a^4$. Hence the fourth δ function becomes

$$\delta(2a^4-k^4)=\tfrac{1}{2}\delta[a^4-\tfrac{1}{2}k^4].$$

We obtain for the total transition rate the result,

(number of pairs per cm^3 sec)

$$=(4e^2/\hbar^2c^3)(|\mathbf{a}|/a^4)(k^4)^{-4}\int[(a^4)^2(\mathbf{J}\cdot\mathbf{J}^*) -(\mathbf{J}\cdot\mathbf{a})(\mathbf{J}^*\cdot\mathbf{a})]d\Omega_a = (\pi/\hbar c)(e^2/\hbar c)(\mathbf{J}\cdot\mathbf{J}^*/ck_m{}^2)SC^{-5}(C^2-S^2/3).$$

Each act of pair creation extracts from the driving field the energy $k^4\hbar c=2a^4\hbar c=2\hbar ck_mC$. The rate of dissipation of energy to the vacuum is therefore

(ergs lost per cm^3 sec)

$$=2\pi(e^2/\hbar c)(\mathbf{J}\cdot\mathbf{J}^*/ck_m)SC^{-4}(C^2-\tfrac{1}{3}S^2), \quad (25)$$

a quantity that we have now to express in terms of the dissipative part of the vacuum polarization coefficient, h_2.

The driving field has the value

$$\mathbf{E}_{\mathrm{driving}}=-\partial\mathbf{A}/\partial x^4=(-4\pi i\mathbf{J}/ck^4)e^{ikx}+\mathrm{c.c.} \quad (26)$$

The time average rate of dissipation of energy is

(ergs lost per cm^3 sec)

$$=\langle\mathbf{E}_{\mathrm{driving}}\mathbf{j}_{\mathrm{induced}}\rangle =\langle[(-4\pi i\mathbf{J}/ck^4)e^{ikx}+\mathrm{c.c.}][-h\mathbf{J}e^{ikx}+\mathrm{c.c.}]\rangle \quad (27) =(8\pi\mathbf{J}\cdot\mathbf{J}^*/ck^4)h_2.$$

Comparing (27) and (25), we find for the dissipation coefficient the expression of Eq. (5). It is difficult to imagine a more elementary instance of the absorptive properties of the vacuum.

In passing from the dissipation coefficient, h_2, to the induction coefficient, h_1, the essential point is the demand that the complex function h_1+ih_2 have no poles when extended into the upper half of the complex k^4 or Ω plane.[2] The reason for the demand may be recalled. (a) A driving current (1) that is zero before $x^4=0$ is described by a Fourier transform $j_p(k^4)$ that

has no poles in the upper half of the complex plane: then the path of integration in (1) is deformable—for negative x^4—to a line arbitrarily far above the real axis, where the integrand is arbitrarily small. (b) The integral (2) for the induced current will likewise vanish if the complex coefficient $h(k^4)$ likewise has no poles in the upper half-plane. As a consequence of demanding this property for h, we know that the value of h at a point k^4 just above the real axis is given by the Cauchy integral,

$$h(k^4)=(1/2\pi i)\int h(x^4)(x^4-k^4)^{-1}dx^4, \qquad (28)$$

where the path of integration is a closed loop that goes around the point k^4 in the positive sense and remains above the real axis. Thinking of the upper part of the loop as a semicircle, we want to expand this semicircle to an infinite radius and want then to have a zero contribution from this region of the complex plane that has no physical interest. But h does not fall off fast enough to allow the contribution on that semicircle to be neglected. Consequently we replace h in the discussion by $h/[1+(k^4)^2]$. Of course the "1" in this formula could be replaced by any dimensionally correct number, but this detail is irrelevant, for the value of the constant will drop out in the end. Having made this change in (28), we take the real part on both sides of the equation, with the result (4). The arbitrary constant comes from the value of the complex function h at the pole $k^4=i$. It is obvious that such a constant can always be added to the polarization coefficient without disturbing the causal relation between driving and induced currents.

The *value* of the adjustable constant comes from another requirement: that the polarization vanish at low frequencies. A nonzero polarization at low frequencies would disturb all laboratory measurements of charge by a common factor and would falsely suggest the possibility of distinguishing observationally at low frequencies between the "real charge" and the "induced charge." To recognize the impossibility of making this distinction, we demand that the constant h_0 be so adjusted as to make h vanish at low frequencies. Thus we find

$$\begin{aligned}h_1(\Omega)&=(4/\pi)(e^2/\hbar c)k_m^2\int_0^\infty(1+\Omega^2)(4k_m^2C^2+1)^{-1}\\&\quad\times(4k_m^2C^2-\Omega^2)^{-1}S^2(1-S^2/3C^2)d\varphi\\&\quad-(\text{same integral calculated for }\Omega=0)\\&=(e^2/\pi\hbar c)\int_0^1 x^2(1-x^2/3)\\&\qquad\times(x^2-1+4k_m^2/\Omega^2)^{-1}dx, \qquad (29)\end{aligned}$$

an integral which leads directly to the Pauli-Rose formula (6).

At low frequencies, (29) and (6) reduce to the well-known Uehling[3] value,

$$h_1=(e^2/15\pi\hbar c)(\Omega^2/k_m^2)=-(e^2/15\pi\hbar c)(k^2/k_m^2). \qquad (30)$$

In consequence, as will be recalled, a point charge,

$$\begin{aligned}j_{\text{driving}}{}^4(x)/c&=Ze\delta(x^1)\delta(x^2)\delta(x^3)\\&=(Ze/8\pi^3)\int\exp(i\mathbf{kx})d^3k, \qquad (31)\end{aligned}$$

induces a charge

$$j_{\text{induced}}{}^4(x)/c=-(Ze/8\pi^3)\int h_1\exp(i\mathbf{kx})d^3k,$$

which in turn produces a potential

$$A_{\text{induced}}{}^4(x)=-(Ze/8\pi^3)\int(4\pi h_1/k^2)\exp(i\mathbf{kx})d^3k.$$

The atomic electron, interacting with this potential, experiences an energy shift

$$\begin{aligned}\Delta E&=\langle -eA_{\text{induced}}{}^4(x)\rangle\\&=Ze^2|\psi(0)|^2(4\pi h_1/k^2)_{k=0}\\&=-(4Ze^4/15\hbar ck_m^2)|\psi(0)|^2, \qquad (32)\end{aligned}$$

a well-known part of the Lamb-Retherford displacement.

The causality-principle analysis of vacuum polarization has three features: It shows in a very convincing way that all questions of principle about the inductive response of the vacuum to weak fields are concentrated in a single constant; it gives a physical reason for renormalizing this constant to zero; and it conceals the fact that straightforward calculation via relativistic electron theory yields for this constant an infinite value. *Note added in proof.*—We have just learned in the course of interesting discussions with Dr. S. Zienau that he has applied the dispersion formula to the problem of vacuum polarization in unpublished lectures at Liverpool in 1953. See also J. Toll and J. A. Wheeler, Phys. Rev. **81**, 654 and 655 (1951) for other applications of causality to the problem of vacuum polarization; and M. Gell-Mann, Proceedings of the Sixth Rochester Conference on High Energy Nuclear Physics, 1956, for the question of whether all of quantum electrodynamics can be transliterated into the language of dispersion theory.

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