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Vanishing of Renormalized Charges in Field Theories with Point Interaction

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A short survey of the results obtained by applying the theory of Landau, Abrikosov, and Khalatnikov to pseudoscalar meson theory is presented. An independent deduction of the explicit expressions for the Green's functions and vertex part is obtained on the basis of simple renormalizability considerations. The relation thus obtained between g_0^2, g_c^2 and the momentum cutoff Λ is such that the theory inevitably leads to the result that for point interaction (i.e., in the limit $\Lambda\to\infty$) the renormalized charge g_c must equal zero.

It is shown that if two cutoffs Λ_p and Λ_k (corresponding to the nuclear and meson momenta) are introduced, this result can rigorously be proved for any value of g_0^2, provided that the limits are moved apart sufficiently rapidly when $\Lambda_k\to\infty$. In the course of the proof an estimation is made of the terms neglected in the zero approximation in the vertex part equation, these terms corresponding to diagrams with intersecting meson lines and nucleon loops.

It is shown that for two different ways of carrying out the limiting process, namely,

(a) for $\Lambda_k\to\infty$, and $[\ln(\Lambda_k^2/m^2)][\ln(\Lambda_p^2/\Lambda_k^2)]^{-1}\ll 1$,
(b) for $\Lambda_k\to\infty$, and only $[\ln(\Lambda_p^2/\Lambda_k^2)]^{-1}\ll 1$,

the contribution of these diagrams is vanishingly small for any g_0^2.

In the second case, the contributions from an infinite set of meson-meson scattering diagrams are summed and it is found that the total contribution is of the same order as that from the simplest diagrams of this process.

The theory with pseudovector coupling, which is not renormalizable when expanded into the usual perturbation theory series, is also considered. It is shown that renormalization can be carried out without expanding into a series; the renormalized charge in this case also vanishes. This result has been rigorously obtained only for a special type of limiting process $\Lambda_k\to\infty$, namely if the inequality

(c) $\Lambda_k^2/m^2[\ln(\Lambda_p^2/\Lambda_k^2)]^{-1}\ll 1$

is obeyed.

In conclusion, a short discussion on the possibility of an experimental proof of the inconsistency of field theory is presented.

1. INTRODUCTION

A NEW approach to a solution of the quantum field theory equations has been suggested by Landau, Abrikosov, and Khalatnikov.[1] Point interaction was treated by these authors as the limit for $\Lambda\to\infty$, of a nonlocal interaction "smeared out" over a radius $1/\Lambda$, the bare coupling constant g_0^2 generally being considered to depend on Λ.

If one also assumes[2] that $g_0^2\ll 1$, it becomes possible to expand any quantity (Green's function, vertex part, etc.) for large momenta, when $-p^2\gg m^2$, m being the nuclear mass (or electron mass in electrodynamics) in a series of the form:

$$f_0[g_0^2\ln(\Lambda^2/-p^2)]+g_0^2f_1[g_0^2\ln(\Lambda^2/-p^2)]+(g_0^2)^2f_2[g_0^2\ln(\Lambda^2/-p^2)]+\cdots. \quad (1)$$

An essential difference between this expansion and the usual perturbation theory series is that all terms proportional to various powers of the quantity $\kappa=g_0^2\ln(\Lambda^2/-p^2)$ are gathered together in the closed expressions $f_n(\kappa)$. The quantity κ cannot be considered small even for $g_0^2<1$, since $\ln(\Lambda^2/-p^2)$ may be arbitrarily large.[3] In principle the functions $f_n(\kappa)$ can be found with the aid of the Dyson-Schwinger integral equations. As a matter of fact, Landau, Abrikosov, and Khalatnikov determined in this way the zero-order term $f_0(\kappa)$ in the series expansion of the same type as (1) for the Green's function and the vertex part in electrodynamics.

Analogous calculations for pseudoscalar meson theory were carried out by Abrikosov, Galanin, and Khalatnikov.[4] The following expressions were obtained for the case $-p^2\gg m^2$:

$$\Gamma_5\simeq\gamma_5\alpha(g_0^2,\Lambda^2/-p^2),$$

$$G(p)\simeq\frac{1}{\gamma p}\beta(g_0^2,\Lambda^2/-p^2), \quad (2)$$

$$D(p)=\frac{1}{p^2}d(g_0^2,\Lambda^2/-p^2),$$

where,[5] in the case of symmetric pseudoscalar theory:

$$\alpha=Q^{1/5},\quad \beta=Q^{-3/10},\quad d=Q^{-4/5} \quad (3)$$

and for neutral theory:

$$\alpha=Q^{-1/5},\quad \beta=Q^{-1/10},\quad d=Q^{-2/5}, \quad (4)$$

where

$$Q=1+(5g_0^2/4\pi)\ln(\Lambda^2/-p^2). \quad (5)$$

The dependence of g_0^2 on Λ and g_c^2 was found[1] to be

$$g_0^2=\frac{g_c^2}{1-(5g_c^2/4\pi)L}, \quad (6)$$

[1] Landau, Abrikosov, and Khalatnikov, Doklady Acad. Nauk U.S.S.R. **95**, 497, 773, 1177 (1954).

[2] It will be seen from the following that this condition is not essential for the further exposition.

[3] The usual perturbation theory series,

$$\sum_{n=0}^{\infty}\sum_{\nu=0}^{n}C_{n\nu}(g_0^2)^n[\ln(\Lambda^2/-p^2)]^\nu$$

[see, for example, M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954)], follows from (1) if the $f_n(\kappa)$ in (1) are written in the form of a series:

$$f_n(\kappa)=\sum_{\nu=0}^{\infty}C_{n+\nu,\nu}\kappa^\nu.$$

[4] Abrikosov, Galanin, and Khalatnikov, Doklady Acad. Nauk U.S.S.R. **97**, 793 (1954).

[5] At large momenta, the vertex part $\Gamma_5(p,p-k)$ depends on the largest of the momenta p, k, and $p-k$. Henceforth it is assumed (if not otherwise stated) that p is the largest quantity.

where $L=\ln(\Lambda^2/m^2)$, or[6]

$$g_c^2=\frac{g_0^2}{1+(5g_0^2/4\pi)L}. \tag{7}$$

The structure of expression (7) is such that, irrespective of the mode of variation of g_0^2 with Λ (with the only restriction that $g_0^2\geqslant 0$, for the theory to be Hermitian), we obtain

$$g_c^2\to 0,$$

when $\Lambda\to\infty$ or $L\to\infty$. (If $g_0^2L\to\infty$ as $L\to\infty$, then $g_c^2\leqslant 4\pi/5L\to 0$; if, however, $g_0^2L\to N$, where N is any constant, then $g_c^2\leqslant N/L\to 0$).

Thus present field theory leads to the result that the renormalized nucleon charge is equal to zero.

At first glance, this conclusion may seem to be incorrect since, according to (6), g_0^2 increases with Λ. Beginning from some Λ, the conditions $g_0^2<1$ will be violated, and the series expansion in g_0^2 upon which (3), (4), and (7) are based will be invalid.

It is not difficult to see, however, that all the relations remain valid for any g_0^2 which is not small, provided that the "smearing out" of the interaction is carried out from the very beginning in the most general form,[7] the point interaction

$$g_0\bar{\psi}(x)\gamma_5\tau_\alpha\psi(x)\varphi_\alpha(x)$$

being considered as the limit for $\Lambda_k\to\infty$, $\Lambda_p\to\infty$ of the interaction

$$g_0\int F_{\Lambda_p\Lambda_k}(x-y,\,x-z)\bar{\psi}(x)\gamma_5\tau_\alpha\psi(y)\varphi_\alpha(z)dydz,$$

where $F_{\Lambda_p\Lambda_k}(x-y,\,x-z)$ is nonzero only if y and z are near to x, within regions with radii $1/\Lambda_p$ and $1/\Lambda_k$, respectively. For $\Lambda_k\to\infty$, $\Lambda_p\to\infty$,

$$F_{\Lambda_p\Lambda_k}(x-y,\,x-z)\to\delta(x-y)\delta(x-z)$$

and Λ_p is always greater than Λ_k (otherwise the results do not differ[7] from the single cutoff case). After renormalization the result should not depend on the method of approaching the limit, as the renormalized quantities do not contain any "smearing" parameters.

It can be shown (see reference 7 and below) that for the two-cutoff technique all relations remain exactly the same (in the momentum region $-p^2<\Lambda_k^2$) as in the one-cutoff case, the only difference being that instead of Λ the quantity Λ_k enters all the formulas and g_0^2 is replaced by

$$\bar{g}_0^2=\frac{g_0^2}{1+(g_0^2/\pi)(L_p-L_k)} \tag{8}$$

in symmetric theory, and by

$$\bar{g}_0^2=\frac{g_0^2}{1+(g_0^2/2\pi)(L_p-L_k)} \tag{9}$$

in neutral theory: here $L_p=\ln(\Lambda_p^2/m^2)$, $L_k=\ln(\Lambda_k^2/m^2)$.

The quantity $\bar{g}_0$ is arbitrarily small for *any value* of g_0 if

$$L_p-L_k=\ln(\Lambda_p^2/\Lambda_k^2)$$

is sufficiently large. Therefore the series expansion (1) (in which g_0^2 is replaced by $\bar{g}_0^2$) is always valid. Moreover, the first term [(2), (3), (4), or (7)] of this type of series (zero order in $\bar{g}_0^2$) will, for sufficiently small $\bar{g}_0^2$,[8] be equal to the total sum of the series (1) with any degree of accuracy.

Thus, relation (7), which may now be written as

$$g_c^2=\frac{\bar{g}_0^2}{1+(5\bar{g}_0^2/4\pi)L_k} \tag{10}$$

is correct, to any degree of accuracy, for *any* g_0, provided that $\ln(\Lambda_p^2/\Lambda_k^2)$ is sufficiently large.

For $\Lambda_k\to\infty$, or $L_k\to\infty$, it yields

$$g_c^2\to 0,$$

irrespectively of the mode of variation of g_0 with Λ.

Thus, modern meson theory for point interaction is inconsistent, as it leads to the absurd conclusion that no physical interaction exists (this statement also applies to electrodynamics,[9] to scalar coupling meson theory, and, as will be shown below, to pseudovector coupling theory).

In this paper, we shall give a simple deduction of the explicit expressions (2)–(5) for the Green's functions, and of formula (6) for the "bare" charge, on the basis of renormalizability considerations (Sec. 2). In Sec. 3, the two-cutoff case of Abrikosov and Khalatnikov will be considered. An estimation of the validity of formulas (2)–(9) of the zero-order approximation will be made in Secs. 4–7 by applying the two-cutoff technique.

For this purpose an estimation is made of the terms neglected in the zero-order (in $\bar{g}_0^2$) approximation, corresponding, in the vertex part equation, to diagrams with intersecting meson lines and nucleon loops.

It is shown that for two different ways of carrying out the limiting process, namely, (a) if in the limit $L_k\to\infty$, $L_p-L_k=\ln(\Lambda_p^2/\Lambda_k^2)$ is so large that

$$\text{(a)}\quad L_k/(L_p-L_k)\simeq L_k/L_p\ll 1$$

(the so-called "super-two-cutoff technique"), and (b)

[6] Relations between g_0^2 and g_c^2 similar to (6) and (7) were obtained by T. D. Lee [Phys. Rev. **95**, 1329 (1954)] for a special model of interaction fields. See also G. Källen and W. Pauli [Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **30**, No. 7 (1955)].

[7] A. A. Abrikosov and I. M. Khalatnikov, Doklady Acad. Nauk U.S.S.R. **103**, 993 (1955). Λ_p is the nucleon momentum cutoff and Λ_k is the meson momentum cutoff.

[8] For such a transition to the limit of point interaction in which $\ln(\Lambda_p^2/\Lambda_k^2)$ remains large, see I. Ya. Pomeranchuk, Doklady Acad. Nauk U.S.S.R. **104**, 51 (1955); **105**, 461 (1955).

[9] L. Landau and I. Pomeranchuk, Doklady Akad. Nauk U.S.S.R. **102**, 489 (1955); I. Pomeranchuk, Doklady Akad. Nauk U.S.S.R. **103**, 1001 (1955). Landau pointed out in the spring of 1954 that this difficulty might appear in present day theory; Fradkin in the fall of 1954 independently suggested that the renormalized charge must be equal to zero.

if only

$$\text{(b)}\quad (L_p-L_k)^{-1}\ll 1,$$

the contribution of these diagrams is vanishingly small for *any* g_0^2.

In case (b) the contributions from an infinite set of meson-meson scattering diagrams, which are essential for the problem, are summed, and it is found that the total contribution is of the same order as that from the simplest diagrams of this process.

Thus, result (7) [or (10)] and the conclusion that $g_c\to 0$ for $L_k\to\infty$ can be rigorously proved not only for limiting transition (a) but also for the case (b).

We shall (in Sec. 9) consider separately the theory with pseudovector coupling, in which the usual series expansion in g_0^2 cannot be renormalized.

If an even more special case than (b) is considered, namely the "super-two-cutoff" case for which

$$\text{(c)}\quad \frac{\Lambda_k^2}{m^2(L_p-L_k)}=\frac{\Lambda_k^2}{m^2\ln(\Lambda_p^2/\Lambda_k^2)}\ll 1,$$

i.e.,

$$\frac{\Lambda_p^2}{\Lambda_k^2}\gg\exp\left(\frac{\Lambda_k^2}{m^2}\right),$$

when $L_k\to\infty$, then it will not be difficult to prove that renormalization can also be carried out for pseudovector coupling and that the renormalized charge will be zero.

2. DEDUCTION OF THE GREEN'S FUNCTIONS AND VERTEX PART FORMULAS, (2)–(6), FROM THE RENORMALIZATION CONDITIONS

The expressions (2)–(5) possess, as one would expect, renormalizability properties:

$$\alpha(g_0^2,\Lambda^2/-p^2)=\alpha_c(g_c^2,-p^2/m^2)/\alpha_c(g_c^2,\Lambda^2/m^2),$$
$$\beta(g_0^2,\Lambda^2/-p^2)=\beta_c(g_c^2,-p^2/m^2)/\beta_c(g_c^2,\Lambda^2/m^2),$$
$$d(g_0^2,\Lambda^2/-p^2)=d_c(g_c^2,-p^2/m^2)/d_c(g_c^2,\Lambda^2/m^2),\quad (11)$$
$$g_0^2=g_c^2\alpha_c^2(g_c^2,\Lambda^2/m_c)\beta_c^2(g_c^2,\Lambda^2/m^2)d_c(g_c^2,\Lambda^2/m^2).$$

Indeed, qualities (11) can be obtained from (3) or (4) if one notices that $Q=Q_0\cdot Q_c$, where

$$Q_c=1-(5g_c^2/4\pi)\ln(-p^2/m^2),$$
$$Q_0=1+(5g_0^2/4\pi)\ln(\Lambda^2/m^2),$$

and if one determines α_c, β_c, and d_c similarly to α, β, and d, with Q_c substituted for Q. [The last of Eqs. (11) will then determine the dependence (6) of g_0^2 on Λ.]

We shall now show that the explicit forms (3) and (4) of functions α, β, and d can be obtained directly without solving the integral equations, exclusively on the basis of the renormalizability properties (6) and by using considerations similar to those presented in Gell-Mann and Low's paper[3] (see also papers referred to in reference 24).

We introduce the quantity

$$g^2(\xi)=g_0^2\alpha^2(g_0^2,L-\xi)\beta^2(g_0^2,L-\xi)d(g_0^2,L-\xi)$$
$$=g_c^2\alpha_c^2(g_c^2,\xi)\beta_c^2(g_c^2,\xi)d_c(g_c^2,\xi),\quad (12)$$

which may be called the effective charge for a given value of $-p^2$ and which is in fact independent of Λ. For convenience of notation, instead of $\Lambda^2/-p^2$ and $-p^2/m^2$ the logarithms of these quantities, $L-\xi=\ln(\Lambda^2/-p^2)$ and $\xi=\ln(-p^2/m^2)$, are written as the arguments of the functions α, α_c, etc. If now we denote $\alpha'=d\alpha/d\xi$, etc., it will not be difficult to see that the logarithmic derivatives of the functions α, β, d with respect to ξ may be expressed as functions exclusively of g^2:

$$\alpha'/\alpha=\alpha_c'/\alpha_c=F_1(g^2),$$
$$\beta'/\beta=\beta_c'/\beta_c=F_2(g^2),\quad (13)$$
$$d'/d=d_c'/d_c=F_3(g^2).$$

Consider, for example, the first of these equations. According to (12), $\xi=\xi(g_c^2,g^2)$, and therefore the quantity α_c'/α_c, which depends on g_c^2 and ξ, can be expressed in the form of a function of g_c^2 and g^2. Therefore the ratio α'/α, which equals α_c'/α_c, can be expressed as follows:

$$\alpha'(g_0^2,L-\xi)/\alpha(g_0^2,L-\xi)=F_1[g_c^2,g^2(g_0^2,L-\xi)].\quad (14)$$

It is emphasized here that, according to (12), $g^2=g^2(g_0^2,L-\xi)$.

If g_0^2 is fixed and L and ξ are varied in such a way that $L-\xi$ remains constant, then g_c^2, which according to (11) depends on g_0^2 and Λ, will vary, whereas the other quantities in (14) will not. Relation (14) will therefore be fulfilled only if F_1 does not depend on g_c^2 explicitly. In this case we arrive at the first of the Eqs. (13): the other equations can be proved in a similar manner.

One can determine the explicit form of functions F_1, F_2, and F_3 in (13) by assuming $\xi\to L$. In this region $\ln(\Lambda^2/-p^2)=L-\xi$ is not large, and for $g_0^2<1$, $g_0^2(L-\xi)$ is also a small quantity, i.e., the familiar perturbation theory can be applied. According to (12), for $\xi\to L$, $g^2\to g_0^2$, as then $\alpha=\beta=d=1$.

Simple calculations, carried out to logarithmic accuracy (i.e., by taking into account only the largest logarithmically diverging part of the integrals), yield in the first order of perturbation theory for symmetric pseudoscalar theory[10]:

$$\alpha=1+(g_0^2/4\pi)(L-\xi),$$
$$\beta=1-(3g_0^2/4\pi)(L-\xi),\quad g_0^2(L-\xi)<1,$$
$$d=1-(g_0^2/\pi)(L-\xi).$$

[10] After elimination of isotopic spin variables and matrices, we obtain for example for the vertext part

$$\Gamma_5(p,p-k)=\gamma_5-\frac{g_0^2}{\pi i}\int^L\gamma_5(\boldsymbol{p}-\boldsymbol{l}-m)^{-1}\times\gamma_5(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{l}-m)^{-1}\gamma_5(l^2-\mu^2)^{-1}d^4l,$$

where Feynman's notation has been used, with $\boldsymbol{p}\equiv\gamma p\equiv\gamma_0p_0-\boldsymbol{\gamma p}$, $k^2=k_0^2-\boldsymbol{k}^2$, $d^4k=(2\pi)^{-2}dk_0dk_1dk_2dk_3$; $iG_0(p)=i(\boldsymbol{p}-m)^{-1}$

Hence for $\xi \to L$ we get:

$$\alpha'/\alpha = -g_0^2/4\pi, \quad \beta'/\beta = 3g_0^2/4\pi, \quad d'/d = g_0^2/\pi.$$

Comparing with (13) we obtain to first order in g^2:

$$F_1(g^2) = -g^2/4\pi, \quad F_2(g^2) = 3g^2/4\pi, \quad F_3(g^2) = g^2/\pi.$$

Since, according to (8) and (9), $(g^2)'/g^2 = 2F_1 + 2F_2 + F_3$, we then obtain

$$[g^2(\xi)]'/g^2(\xi) = (5/4\pi)g^2(\xi).$$

Integrating this equation over ξ from ξ to L [and taking into account that, according to (12), $g^2(L) = g_0^2$], we get

$$g^2(\xi) = \frac{g_0^2}{1 + (5g_0^2/4\pi)(L-\xi)} = \frac{g_0^2}{Q(\xi)}. \qquad (15)$$

Inserting this expression for $g^2(\xi)$ and the expressions for the functions F_1, F_2, F_3 into (13), and integrating over ξ, we directly obtain (3).

Analogously, in neutral pseudoscalar theory the perturbation theory formulas yield:

$$\alpha = 1 - (g_0^2/4\pi)(L-\xi),$$
$$\beta = 1 - (g_0^2/8\pi)(L-\xi), \quad g_0^2(L-\xi) < 1,$$
$$d = 1 - (g_0^2/\pi)(L-\xi),$$

and we obtain $F_1(g^2) = g^2/4\pi$, $F_2(g^2) = g^2/8\pi$, $F_3(g^2) = g^2/2\pi$. The form of $g^2(\xi)$ in this case turns out to be the same as that in (15), and after integration of (13) over ξ the formulas (4) for α, β, and d immediately follow.

3. THE CASE OF TWO CUTOFFS[7]

Let us consider in greater detail the two-cutoff theory. We shall show that all the formulas are indeed the same as those for a single cutoff, provided that $\bar{g}_0^2$ is substituted in all formulas for g_0^2, and Λ_k for Λ.

After elimination of the isotopic spin variables and matrices (a trivial operation), the Dyson-Schwinger integral equations for symmetric[11] pseudoscalar theory take the form:

$$\Gamma_5(p, p-k) = \theta\gamma_5 - \frac{g_0^2}{\pi i}\int \Gamma_5 G(p-l) \times \Gamma_5 G(p-k-l)\Gamma_5 D(l) d^4l, \quad (16)$$

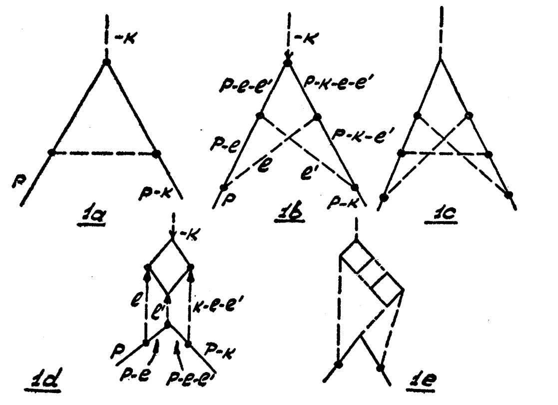

FIG. 1. Vertex part diagrams with intersecting meson lines (b,c) and nucleon loops (d,e).

$$\left[p - m - \frac{3g_0^2}{\pi i}\int \Gamma_5 G(p-k)\theta\gamma_5 D(k) d^4k\right] G(p) = 1, \qquad (17)$$

$$\left\{k^2 - \mu^2 + \frac{2g_0^2}{\pi i}\int \mathrm{Sp}[G(p)\Gamma_5 G(p-k)\theta\gamma_5 - (G(p)\Gamma_5 G(p-k)\theta\gamma_5)_{k^2=\mu^2}] d^4p\right\} D(k) = 1. \quad (18)$$

(Concerning the notations, see reference 10.) Here $\theta = \theta_{\Lambda_k\Lambda_p}(p, p-k)$ is the Fourier component of the "smear-out" function $F_{\Lambda_p\Lambda_k}(x-y, x-z)$ [for $\Lambda_k \to \infty$, $\Lambda_p \to \infty$, $\theta \to 1$]. By definition, θ vanishes if $-p^2 > \Lambda_p^2$, or $-k^2 > \Lambda_k^2$. At high momenta all quantities vary slowly (logarithmically) and therefore the detailed form of this function is unimportant. It may be considered that $\theta = 1$ if the momenta do not exceed the cut-off values, and otherwise $\theta = 0$.

Equations (17) and (18) are exact, while Eq. (16) approximate. The latter equation takes into account only the simplest diagram in Fig. 1(a), and does not[1] take account of more complex irreducible diagrams (in the usual Dyson sense) with intersecting meson lines, of the type shown in Fig. 1(b), 1(c), etc., or with nucleon loops, as in Fig. 1 (d), 1 (e), etc.[12] The possibility of disregarding these diagrams in the expansion of type (1) in the zero approximation with respect to g_0^2 (or $\bar{g}_0^2$, for two cutoffs) will be examined in detail below.

If $\Lambda_p = \Lambda_k = \Lambda$, the functions (2)–(3) will be solutions of Eqs. (16)–(18) for large momenta. In this case, all integrals break off at the momentum value $-p^2 = \Lambda^2$, and only the region in which p and k do not exceed this cutoff value is essential.

For two-cutoff values $\Lambda_p \gg \Lambda_k$, Γ_5 also vanishes if either of the momenta p or k exceeds the cut-off value (as then $\theta = 0$ and, according to (16), $\Gamma_5 = 0$). Thus, integrals (16) and (17) break off at the momentum Λ_k

corresponds to the nucleon line, and $4\pi i D_0(l) = 4\pi i(l^2-\mu^2)^{-1}$ to the meson; the S-matrix being $T\{\exp[g_0\int \bar{\psi}\gamma_5\tau_\alpha\psi\varphi_\alpha dx]\}$.

For integration with logarithmic accuracy, only the region in which l is much greater than any of the momenta p or k is important [i.e., on a logarithmic scale, the region $\xi \leqslant z \leqslant L$ where $z = \ln(-l^2/m^2)$]. Disregarding in this region the momenta p and k and the masses m and μ compared with l, we obtain for the integral the value

$$(-\gamma_5)\int_\xi^L (d^4k/k^4) = (-i/4)\gamma_5 \int_\xi^L dz = (-i/4)\gamma_5(L-\xi).$$

The value of α given in the text then follows.

[11] In neutral theory the integral term in the equation for Γ_5 has a positive sign, whereas in the equations for G and D the integrals enter without corresponding multipliers 3 and 2.

[12] All lines and points on Fig. 1 are thick and therefore refer to the exact functions G, D, and Γ.

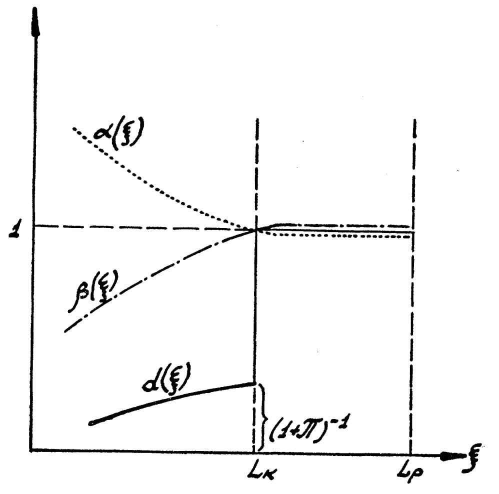

FIG. 2. Approximate behavior of the functions $\alpha(\xi)$, $\beta(\xi)$, and $d(\xi)$ in symmetric theory with two cutoffs.

and (18) breaks off at Λ_p. Moreover it follows from (18) that $d(\eta)=1$ if $-k^2>\Lambda_k^2$ or $\eta=\ln(-k^2/m^2)\geqslant L_k$, as in this case, the functions Γ_5 and θ in the integral term in (18) vanish.

For $-k^2<\Lambda_k^2$, i.e., $\eta\leqslant L_k$, the integral term in (18) does not vanish. Let $(k^2-\mu^2)\Pi$ denote that part of it which corresponds to integration over the region $\Lambda_k^2<(-p^2)<\Lambda_p^2$:

$$(k^2-\mu^2)\Pi=\frac{2g_0^2}{\pi i}\int_{\Lambda_k}^{\Lambda_p}\mathrm{Sp}[G(p)\Gamma_5G(p-k)\gamma_5-(G(p)\Gamma_5G(p-k)\gamma_5)_{k^2=\mu^2}]d^4p. \quad (19)$$

It will be shown below that in the most important region $\mu^2\ll(-k^2)\ll\Lambda_k^2$ the dimensionless (and positive) quantity Π is practically independent of k^2. Taking into account (19), Eq. (18) may be written in the form:

$$\left\{(k^2-\mu^2)(1+\Pi)+\frac{2g_0^2}{\pi i}\int^{\Lambda_k}\mathrm{Sp}[G(p)\Gamma_5G(p-k)\gamma_5-(G(p)\Gamma_5G(p-k)\gamma_5)_{k^2=\mu^2}]d^4p\right\}D(k)=1.$$

Substituting in this equation and also in (16) and (17)

$$D(k)=(1+\Pi)^{-1}\bar{D}(k),\quad [\text{or } d(\eta)=(1+\Pi)^{-1}\bar{d}(\eta)], \quad (20)$$

we obtain for Γ_5, G, and $\bar{D}$ the set of Eqs. (16), (17), and (18), that is, a set which is exactly similar to that in the case with a single cutoff momentum $\Lambda=\Lambda_k$, provided however that g_0^2 is replaced by

$$\bar{g}_0^2=(1+\Pi)^{-1}g_0^2. \quad (21)$$

It is evident that in the range $\xi\leqslant L_k$ the solutions for $\alpha(\xi)$, $\beta(\xi)$, and $d(\xi)$ will be the functions (3), (5) [or (4), (5) for neutral theory] obtained above, if one replaces Λ by Λ_k and g_0^2 by $\bar{g}_0^2$. For $\xi=L_k$ these functions equal unity; for $\xi>L_k$, in the range $L_k<\xi\leqslant L_p$, the functions $\alpha(\xi)$ and $\beta(\xi)$ remain equal to unity [since in this case there is no logarithmic integration region in integrals (16) and (17) and in the approximation considered here they should be neglected]. In distinction to α and β, the function $d(\eta)$ for $\eta=L_k$ undergoes a jump[7] from the value $(1+\Pi)^{-1}$ [in agreement with (20)] for $\eta=L_k-\epsilon$, $\epsilon\to 0$, to the value $d=1$ for[13] $\eta>L_k$. An approximate plot of $\alpha(\xi)$, $\beta(\xi)$, and $d(\xi)$ is shown in Fig. 2.

Substituting in (19) the values (2) of the functions Γ_5, D, G and, in accord with the foregoing, putting $\alpha=\beta=d=1$ in the range of integration, it is easy to compute the integral:

$$\frac{2g_0^2}{\pi i}\int_{\Lambda_k}^{\Lambda_p}\mathrm{Sp}\left[\frac{1}{p}\gamma_5\frac{1}{p-k}\gamma_5-\frac{1}{p}\gamma_5\frac{1}{p}\gamma_5\right]d^4p \approx\frac{k^2g_0^2}{\pi}\int_{L_k}^{L_p}d\xi=\frac{k^2g_0^2}{\pi}(L_p-L_k).$$

Hence

$$\Pi=(g_0^2/\pi)(L_p-L_k) \quad (22)$$

and (21) is the same as relation (8). In the neutral theory there is no multiplier 2 before the integral in (18) and Π in this case equals:

$$\Pi=(g_0^2/2\pi)(L_p-L_k).$$

Thus we obtain (9).

Relations (7), (8), and (9), which lead to the result that for point interactions the renormalized nucleon charge is zero, depend significantly on the explicit form (3), or (4), and (5), of the functions α, β, d. The form of these functions is in turn determined by Eqs. (16), (17), and (18). [We emphasize once again that the deduction of formulas (3) and (4) from the renormalization condition is equivalent to an asymptotic solution of the set of Eqs. (16)–(18)].

4. ESTIMATION OF THE TERMS NEGLECTED IN ZERO-APPROXIMATION THEORY

Consider now the theory with two-cutoff values Λ_k and Λ_p [in which g_0^2 in Eqs. (16)–(18) is replaced by $\bar{g}_0^2$]. We shall show that in this case the neglect in (16) of diagrams of the type shown in Fig. 1(b), (c), (d), (e), etc., with intersecting meson lines and nucleon loops, can be rigorously justified. In order to do this the terms emitted in (16) should be estimated. For this purpose the functions (2), (3), and (5) of the zero approximation [with respect to $\bar{g}_0^2$] should be used.

It can be shown in a general way that to a diagram with n intersecting meson lines [of the type shown in Fig. 1(b), 1(c), etc.] there corresponds in (16) a quantity of the order $(\bar{g}_0^2)^{n-1}$ in the sense of the expansion of type (1) [to be more exact, the series expansion (1) in which g_0^2 is replaced by $\bar{g}_0^2$ and Λ by Λ_k].

---

[13] This is due to the circumstance that the integral term in (18) suddenly vanishes when η exceeds L_k, which is a result of the assumption made above concerning the form of the function θ.

This is simply a result of the fact that on integration over n virtual meson momenta only one of n integrals is divergent (logarithmically) and the result of integration is thus proportional to $\ln(\Lambda_k^2/-p^2)$. Hence the contribution from such a diagram will be of the order (if free-field functions D, G, and Γ_5 are used in the estimation),

$$\gamma_5(\bar{g}_0^2)^n \ln(\Lambda^2/-p^2)=\gamma_5(\bar{g}_0^2)^{n-1}[\bar{g}_0^2 \ln(\Lambda^2/-p^2)], \quad (23)$$

i.e., of the order

$$(\bar{g}_0^2)^{n-1},$$

as $\bar{g}_0^2 \ln(\Lambda^2/-p^2)$ is, with respect to expansion into a series of type (1), a zero-order quantity. A more accurate estimation, obtained by substituting in the integrals expressions (3) or (4) for α, β, and $\bar{d}$, yields instead of $\bar{g}_0^2 \ln(\Lambda^2/-p^2)$ a numerical function of this quantity $F_n(\bar{g}_0^2 \ln(\Lambda^2/-p^2))$ of order unity. We shall demonstrate this for the simplest case, when $n=2$; the corresponding diagram is shown in Fig. 1(b).

Using expressions (2) for Γ_5, G, and $\bar{D}$ (with $D=\bar{D}/(1+\Pi)$), we find that the contribution of this diagram,

$$5\left(\frac{\bar{g}_0^2}{\pi i}\right)^2\int \Gamma_5 G(p-l)\Gamma_5 G(p-l-l')\Gamma_5 G(p-l-l'-k) \times\Gamma_5 G(p-k-l')\Gamma_5\bar{D}(l)\bar{D}(l')d^4ld^4l'$$

(in symmetric theory a factor 5 appears because of the isotopic spin variables: $\tau_\mu\tau_\nu\tau_a\tau_\mu\tau_\nu=5\tau_a$: it is absent in the neutral theory), may be represented in the form:

$$-\gamma_5\left(\frac{\bar{g}_0^2}{4\pi}\right)^2 5\int_\xi^{L_k} \alpha^5(z)\beta^4(z)\bar{d}^2(z)dz.$$

It is taken into account here that only the logarithmic region $\xi\leqslant z\leqslant L_k$ [$z=\ln-l^2/m^2$], in which l and l' are much greater than p and k, is important; and that, for a fixed value of l, the integral over l',

$$4i\int\frac{ll'}{(l+l')^2l'^4}d^4l'$$

converges, is equal to unity, and values of l' of the order of l are essential in it [correspondingly, we put

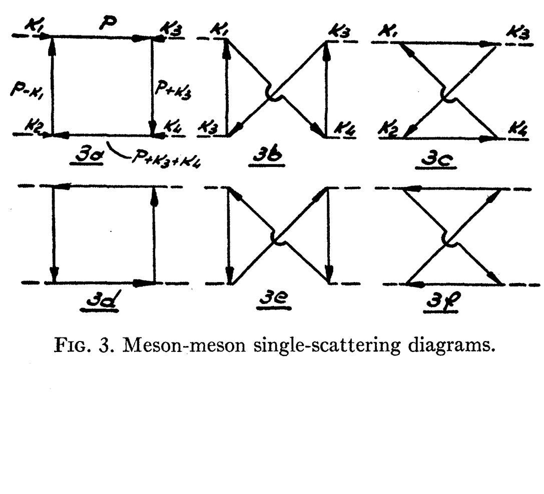

FIG. 3. Meson-meson single-scattering diagrams.

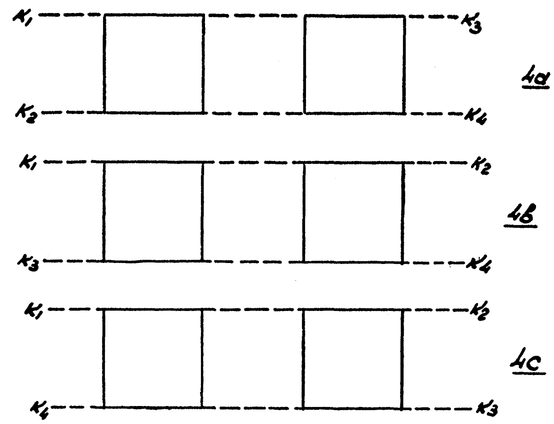

FIG. 4. Double-scattering diagrams. When each of the squares in Fig. 4(a) is replaced by one of the figures in Fig. 3, two series of 18 identical diagrams are obtained. Thus altogether there are $18\times 3=54$ different diagrams.

everywhere: $z'=\ln(-l'^2/m^2)\simeq z$ and $G(p-k-l-l')\simeq G(p-l-l')\simeq G(-l-l')\simeq-(l+l')^{-1}\beta(z)$, etc.]. Introducing the variable $\bar{q}=1+(5\bar{g}_0^2/4\pi)(L_k-z)$ and carrying out the integration, we obtain the result:

$$\gamma_5\bar{g}_0^2\frac{(-1)}{4\pi}\int_1^{\bar{Q}}\alpha^5(\bar{q})\beta^4(\bar{q})\bar{d}^2(\bar{q})d\bar{q}, \quad (24)$$

which confirms the estimate (23) for $n=2$, if $\bar{g}_0^2 \ln(\Lambda_k^2/-p^2)$ is replaced in (23) by a function $F_2(\bar{Q})$:

$$F_2(\bar{Q})=-\frac{1}{4\pi}\int_1^{\bar{Q}}\alpha^5\beta^4\bar{d}^2d\bar{q}=-\frac{5}{16\pi}(1-\bar{Q}^{-4/5}),$$

which is, for any $\bar{Q}=1+(5\bar{g}_0^2/4\pi)(L_k-\xi)$, a quantity of the order of unity or less.

Thus, in the zero approximation in $\bar{g}_0^2$, all diagrams with intersecting meson lines are indeed unimportant. It is more difficult to appraise the contribution in (16) of all possible diagrams of the type shown in Fig. 1(d), 1(e), etc., which possess an arbitrary number of nucleon loops.

Let us consider a diagram of the type shown in Fig. 1(d), which contains in the meson-meson scattering part any arbitrary diagram of this process represented in Figs. 3–6, etc. (say, diagram n, if the diagrams are numbered). The contribution from it [see Fig. 1(d)] may be written in the form;

$$-A\left(\frac{\bar{g}_0^2}{\pi i}\right)^2\int\Gamma_5 G(p-l)\Gamma_5 G(p-l-l')\Gamma_5\bar{D}(l) \times\bar{D}(l')\bar{D}(k-l-l')R_n(l,l',-k,k-l-l')d^4ld^4l',$$

if

$$(\bar{g}_0^2/4\pi i)R_n(k_1,k_2,k_3,k_4) \quad (25)$$

(with $k_1+k_2+k_3+k_4=0$) denotes the contribution corresponding to the nth meson-meson scattering diagram

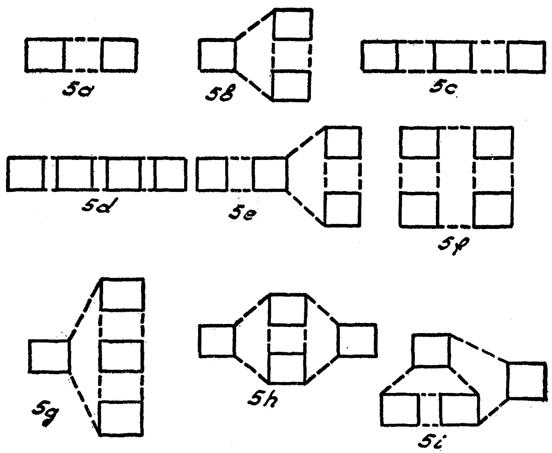

FIG. 5. Examples of "contractible" diagrams.

under consideration. A denotes a factor which depends on the isotopic spin variables[14]; this factor is absent in the neutral theory. Taking into account in the integral only the logarithmic region $\xi \leqslant z \leqslant L_k$, we obtain similarly to (24)

$$\gamma_5\left(\frac{\bar{g}_0^2}{4\pi}\right)^2 \int_\xi^{L_k} \alpha^3(z)\beta^2(z)\bar{d}^3(z)AR_n(z)dz$$
$$=\gamma_5\frac{\bar{g}_0^2}{20\pi}\int_1^{\bar{Q}} \alpha^3(\bar{q})\beta^2(\bar{q})\bar{d}^3(\bar{q})AR_n(\bar{q})d\bar{q}, \quad (26)$$

where $R_n(z)$ or $R_n(\bar{q})$ are the expressions for

$$R_n(l,\, l',\, -k,\, k-l-l'),$$

where l, l', and $k-l-l'$ are very large and of the same order of magnitude .[In this case the sum of any two momenta on which R_n depends will be of the same order of magnitude as l, l', or $k-l-l'$; it is shown below that under these conditions R_n depends only on a single variable, $R_n = R_n(z)$.]

We shall first consider the simplest diagrams for single and double scattering (Figs. 3 and 4). In accord with (25), let R_0 correspond to the contribution from the six diagrams in Fig. 3 for single scattering and

$$R_1 = R_{1a} + R_{1b} + R_{1c}$$

to the contribution from the diagrams of Fig. 4 for double scattering [a doubled number of these diagrams (108) can be obtained from Fig. 4 if each square in Fig. 4 is successively replaced by one of the diagrams in Fig. 3. Figure 4(a) then yields two series, each containing 18 identical diagrams]. For example, after separating, in accord with (25), the factor $\bar{g}_0^2/4\pi i$, the following quantity is found to correspond to Fig. 3(a):

$$R_{0a} = \frac{\bar{g}_0^2}{\pi i}\int^{\Lambda_p} \mathrm{Sp}[\Gamma_5 G(p)\Gamma_5 G(p+k_3)\Gamma_5 G(p+k_3+k_4)$$
$$\times \Gamma_5 G(p-k_1)]d^4p\cdot \mathrm{Tr}(\tau_{a_1}\tau_{a_2}\tau_{a_4}\tau_{a_3}). \quad (27)$$

(In the neutral theory, the factor $\mathrm{Tr}(\tau_{a_1}\tau_{a_2}\tau_{a_4}\tau_{a_3})$ does not appear here, of course.) For all 18 diagrams of the type shown in Fig. 4(a) we get the quantity R_{1a}, where

$$2R_{1a}(k_1,k_2,k_3,k_4) = -\frac{\bar{g}_0^2}{\pi i}\int^{\Lambda_k} R_0(k_1,k_2,l,l')\bar{D}(l)\bar{D}(l')$$
$$\times R_0(-l,\, -l',\, k_3,\, k_4)d^4l \quad (28)$$

and $l' = -l-k_1-k_2 = -l+k_3+k_4$. Evidently, R_0 and R_1 are quantities of the same order (of zero order in $\bar{g}_0^2$), as the factor $\bar{g}_0^2$ in (27), and (28) is "absorbed" by the logarithmically diverging integral. This can easily be confirmed by straightforward calculations. Taking into account only the logarithmic region

$$\eta \leqslant \xi \leqslant L_p$$

in (27) [$\eta = \ln(-k^2/m^2)$, $-k^2$ is the square of the largest meson momentum] and $\eta \leqslant z \leqslant L_k$ in (28), we obtain

$$R_0 = \rho_0\delta_s,$$
$$\rho_0(\bar{Q}) = (16/3)(\bar{Q}^{3/5}-1) + (4\bar{g}_0^2/\pi)\ln(\Lambda_p^2/\Lambda_k^2), \quad (29)$$
$$\delta_s = \delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_1a_3}\delta_{a_2a_4} + \delta_{a_1a_4}\delta_{a_2a_3},$$

$$R_1 = \rho_1\delta_s, \quad \rho_1(\bar{Q}) = -\frac{11}{10}\int_1^{\bar{Q}} \rho_0^2(\bar{q})\bar{d}^2(\bar{q})d\bar{q} \quad (30)$$

for symmetric theory, or analogously,

$$R_0(\bar{Q}) = 24(1-\bar{Q}^{-1/5}) + (6\bar{g}_0^2/\pi)\ln(\Lambda_p^2/\Lambda_k^2), \quad (31)$$

$$R_1(\bar{Q}) = -\frac{3}{10}\int_1^{\bar{Q}} R_0^2(\bar{q})\bar{d}^2(\bar{q})d\bar{q} \quad (32)$$

for neutral theory.

Substituting these values into (26), we can estimate the contribution to (16) from the diagrams of Figs. 1(d), 1(e) containing one or two squares. Consider, for simplicity, the case $\bar{Q} \gg 1$ [i.e., $\bar{g}_0^2(L_k-\eta) \gg 1$], when (26) is maximal. In this case, we can substitute in (26),

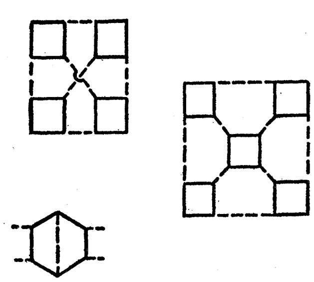

FIG. 6. Examples of "uncontractible" diagrams.

[14] If, in symmetric theory, a_1, a_2, a_3, and a_4 denote the isotopic spin variables of meson lines l, l', $-k$ and $k-l-l'$ in Fig. 1(d), then $A = \tau_{a_1}\tau_{a_2}\tau_{a_4}$ and

$$R_n(l,\, l',\, -k,\, k-l-l') = R_{a_1a_2a_3a_4}{}^{(n)}(l,\, l',\, -k,\, k-l-l')$$

depends on these variables.

in accord with (29), $\rho_0 \approx (16/3)\bar{Q}^{3/5}$, or, according to (31), $R_0 \approx 24+(6\bar{g}_0^2/\pi)\ln(\Lambda_p^2/\Lambda_k^2)=\text{const}$. This yields

$$\gamma_5(5\bar{g}_0^2/3\pi)(1-\bar{Q}^{-4/5}) \approx \gamma_5 5\bar{g}_0^2/3\pi,$$
$$\gamma_5(R_0/24\pi)\bar{g}_0^2(1-\bar{Q}^{-1}) \approx \gamma_5(R_0/24\pi)\bar{g}_0^2,\ (R_0=\text{const}) \tag{33}$$

for the contributions from diagrams of Fig. 1(d), for symmetric and neutral theories respectively (it has been taken into account that in symmetric theory $A\delta_s = 5\tau_{a3}$). The quantities thus obtained are of the order $\bar{g}_0^2$, and vanish if $\bar{g}_0 \to 0$.

Contributions from the diagrams of Fig. 1(e) for double scattering can easily be estimated in a similar manner from (30) or (32). The corresponding quantities are also of the order $\bar{g}_0^2$ and should not be taken into account in the zero-approximation theory. [In symmetric theory, for example, the contribution from Fig. 1(e), for $\bar{Q} \ll 1$, differs from the contribution from Fig. 1(d) only by a numerical factor, since in this case, according to (30),

$$\rho_1(\bar{Q}) \approx -\frac{11}{10}\left(\frac{16}{3}\right)^2 \int_1^{\bar{Q}} \bar{q}^{-2/5} d\bar{q} = -\frac{11}{6}\left(\frac{16}{3}\right)^2 \bar{Q}^{3/5}\Bigg].$$

5. "CONTRACTIBLE" AND "UNCONTRACTIBLE" MESON-MESON SCATTERING DIAGRAMS

The difficulty, however, is that besides the diagrams of Fig. 1(d) and Fig. 1(e), for single and double scattering, there is an infinite set of similar diagrams with a larger number of nucleon squares, to each of which there corresponds a contribution of the same order of magnitude.[15] As a matter of fact there is an infinite number of meson-meson scattering diagrams, each of which corresponds to a contribution of the same order[15] as (29) and (30) or (31) and (32).

These diagrams can be called "contractible." They are composed only of nucleon squares connected by meson lines and are such that if two squares connected by meson lines are successively replaced by one square, one finds that, by gradual simplification, the diagram will reduce to one of the diagrams in Fig. 3. (For example, the diagrams of Fig. 5 are "contractible," whereas those in Fig. 6 are "uncontractible.") If the "contraction" is replaced by the inverse process, then, starting from one of the diagrams of Fig. 3, we arrive at the diagrams of Fig. 4, Fig. 5(b), or 5(c), etc. Substitution of two squares instead of one does not change the order of magnitude of the diagram contribution, as an extra factor $\bar{g}_0^4$ and two diverging integrals (over the meson and nucleon momenta) then appear. The result is, roughly speaking, that the diagram contribution will be multiplied by $\bar{g}_0^4 L_p L_k$, which is of order unity. (If $L_p \to \infty$, $\bar{g}_0^2 L_p \to \pi$, and $\bar{g}_0^4 L_p L_k \to \pi \bar{g}_0^2 L_k$.) The transition from a diagram of Fig. 3 to one of Fig. 4 carried out in (28) is a good example of absorption of the factor $\bar{g}_0^2$ due to divergence of the integral. Similarly, a simple estimate shows that a quantity of higher order in $\bar{g}_0$ corresponds to "uncontractible" diagrams.

If the contribution from the infinite set of all the meson-meson scattering diagrams (i.e., the exact value for the meson-meson scattering amplitude) is denoted by

$$(\bar{g}_0^2/4\pi i)P'(k_1,k_2,k_3,k_4),$$

then, for large meson momenta k_i the P' can be represented by a series of the same form as (1):

$$P' = P(\bar{\kappa}) + \bar{g}_0^2 N(\bar{\kappa}) + \cdots; \quad \bar{\kappa} = \bar{g}_0^2 \ln(\Lambda_k^2/-k^2).$$

The first term of this series is defined by an infinite sum

$$P(k_1,k_2,k_3,k_4) = \sum_{n=0}^{\infty} R_n(k_1,k_2,k_3,k_4), \tag{34}$$

which is taken only over "contractible" diagrams.

Correspondingly, the total contribution to (16) from diagrams of the type shown in Figs. 1(d), 1(e), etc., with an arbitrary number of nucleon loops, is determined by expression (26), in which R_n is replaced by the total sum P. The magnitude of the expression thus obtained can be estimated only if the series (34) can be summed, or if at least it can be shown that the series does not diverge.

In the following, it will be shown that the difficulty connected with the necessity of evaluating the sum (34) (the so-called *párquet* problem) can be circumvented if a special type of limiting process (*b*) (super-two-cutoff case) is considered.

The possibility of evaluating the sum in (34) will be discussed in Sec. 7.

6. "SUPER TWO CUTOFF CASE" (B)

If, as $\Lambda_k \to \infty$, (*b*) is fulfilled, the contribution to (34) from all terms of the sum except R_0 will be infinitesimally small and

$$\lim_{\Lambda_k \to \infty} P = R_0.$$

Indeed, since

$$\bar{Q} = 1 + \frac{5g_0^2}{4\pi} \frac{L_k - \eta}{1+(g_0^2/\pi)(L_p - L_k)},$$

where $\eta \leqslant L_k$, we get for $L_p - L_k \to \infty$,

$$\bar{Q} \to 1 + \frac{5}{4}\left(\frac{L_k - \eta}{L_p - L_k}\right) \to 1$$

and therefore, according to (30) or (32). $R_1 \to 0$.

[15] This was noted by Landau and does not refer to electrodynamics. For the latter (owing to cancellation of the divergences in the total contribution from diagrams of Fig. 3) a quantity R_1, of order e_0^2 with respect to R_0, will correspond to the diagrams of Fig. 4 with two squares. Therefore, the following discussion refers only to meson theories.

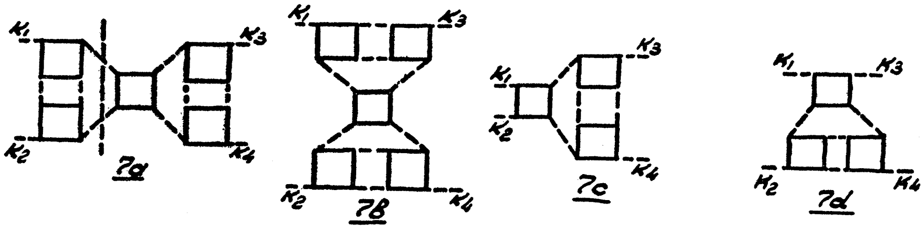

FIG. 7. Diagrams (a) and (c) are reducible, whereas (b) and (d) are irreducible with respect to "separation" of k_1 and k_2 from k_3 and k_4. (In the separation method illustrated in Fig. 7(a) the part contiguous to k_1 and k_2 is irreducible with respect to "separation" of these two lines from the two others.)

Similarly, all other terms of the series (34) vanish, because the quantity $\bar{g}_0^4 L_p L_k$, by which, roughly speaking, the contribution from any diagram is multiplied if one square is replaced by two, itself vanishes.

If $L_k/L_p \ll 1$, R_0 according to (29) or (30), becomes a constant of order unity, and according to (26), for $P=R_0=\text{const}$, the contribution to (16) from all diagrams of the type shown in Figs. 1(d), 1(e), etc. will be proportional to L_k/L_p and vanish for $L_k \to \infty$. [To be more precise, in the limit $L_k \to \infty$, $(L_k/L_p) \to 0$, the contribution to (16) from a diagram with n squares will be proportional to $(L_k/L_p)^n$.]

7. COMPUTATION OF MESON-MESON SCATTERING AMPLITUDE (34)[16]

The sum (34) of all contractible-diagram contributions obeys an integral equation whose form depends only on R_0. In order to deduce this equation we introduce the concept of reducible and irreducible diagrams.

Diagrams which are reducible with respect to "separation" of meson lines k_1 and k_2 from k_3 and k_4 are defined as those which can be divided into at least two parts, connected by only two meson lines. It is assumed, moreover, that the separation is carried out in such a way that lines k_1 and k_2 are connected to one part and k_3, k_4 to the other. [For example, the diagrams of Figs. 4(a), 7(a), and 7(c) are reducible.] Those diagrams which do not possess this property are irreducible with respect to separations of k_1, k_2 from k_3, k_4,[17] [example: diagrams in Fig. 3, Figs. 4(b), 4(c), 7(b), and 7(d)].

For the sake of generality, we shall first consider all (i.e., contractible and uncontractible) meson-meson scattering diagrams. Let P' be the total contribution sum, $R'(k_1,k_2;k_3k_4)$ the sum of all irreducible diagram contributions (in the sense of separation of k_1, k_2 from k_3, k_4), and $F(k_1,k_2;k_3,k_4)$ the sum of all reducible diagram contributions:

$$P'(k_1,k_2,k_3,k_4)=R'(k_1,k_2;k_3,k_4)+F(k_1,k_2;k_3,k_4). \quad (35)$$

P' is symmetric with respect to any transposition of meson lines, whereas R' and F' are invariant if k_1 and k_2, or k_3 and k_4 are transposed, or if k_1, k_2 is replaced by k_3, k_4. P' and R' are related by an integral equation which is similar to the Bethe-Salpeter equation,

$$P'(k_1,k_2,k_3,k_4) = R'(k_1,k_2;k_3,k_4)-\frac{\bar{g}_0^2}{2\pi i}\int R'(k_1,k_2;l,l') \times \bar{D}(l)\bar{D}(l')P'(-l,-l',k_3,k_4)d^4l, \quad (36)$$

where $l'=-l-k_1-k_2$, just as in (27).

For convenience we present here a short deduction of this relation. Consider an arbitrary reducible diagram (with respect to separation of k_1, k_2 from k_3, k_4), and let the site of separation be chosen in such a manner that the part adjacent to k_1 and k_2 is already irreducible (with respect to separation of k_1, k_2 from l, l'). An example of this type of separation is shown in Fig. 7(a). Denote by $\rho_n'(k_1,k_2;l,l')$ and $\sigma_m(-l,-l',k_3,k_4)$ the contributions from both parts of the diagram; we then obtain, evidently, for the total contribution

$$f_{nm}'(k_1,k_2;k_3,k_4)=-\frac{\bar{g}_0^2}{\pi i}\int \rho_n'(k_1,k_2;l,l')\bar{D}(l) \times \bar{D}(l')\sigma_m'(-l,-l',k_3,k_4)d^4l. \quad (37)$$

Summing both sides of this equation over all possible diagrams, i.e., over n and m, we obtain

$$2F'(k_1,k_2;k_3,k_4)=-\frac{\bar{g}_0^2}{\pi i}\int R'(k_1,k_2;l,l') \times \bar{D}(l)\bar{D}(l')P'(-l,-l';k_3,k_4)d^4l, \quad (38)$$

which is equivalent to (36). Equation (38) was obtained by taking into account the fact that when the total set of all possible diagrams in both parts of the reducible diagram are connected by lines l and l', two sets of identical diagrams appear. As a result the factor 2 emerges in the left hand side of (38), just as in (28).

Consider now only contractible diagrams and let P, R, and F denote the corresponding sums only for them. The foregoing considerations can be applied to this case without alteration, and we obtain:

$$P(k_1,k_2,k_3,k_4)=R(k_1,k_2;k_3,k_4)+F(k_1,k_2;k_3,k_4),$$
$$F(k_1,k_2;k_3,k_4)=-\frac{\bar{g}_0^2}{\pi i}\int R(k_1,k_2;l,l')\bar{D}(l)\bar{D}(l') \times P(-l,-l',k_3,k_4)d^4l. \quad (39)$$

[16] The computations made in this section were carried out in collaboration with I. T. Diatlov. See I. T. Diatlove and K. A. Ter-Martirosyan, Soviet Phys. JETP **30**, 416 (1956); Diatlov, Sudakov, and Ter-Martirosyan, Soviet Phys. JETP (to be published); V. V. Sudakov, Soviet Phys. JETP (to be published).

[17] The same diagram can be reducible or irreducible in this sense, depending on how the meson lines approach it. Thus, the diagram in Fig. 4(a) is reducible in the sense of "separation" of k_1, k_2 from k_3, k_4, whereas the diagrams in Figs. 4(b), 4(c) are irreducible.

We may use relations (39) to solve our problem, if we note that any "contractible" diagram is *necessarily reducible* with respect to separation of some pair of meson lines from some other pair (provided that it is not a simple diagram of the type of Fig. 3). Indeed, by carrying out the contraction process, any arbitrarily complex contractible diagram can be transformed into one similar to that shown in Fig. 4, for which this statement is obviously true.

Thus,

$$P(k_1,k_2,k_3,k_4)=R_0(k_1,k_2,k_3,k_4)+F(k_1,k_3;\,k_2,k_4)+F(k_1,k_4;\,k_2,k_3)+F(k_1,k_2;\,k_3,k_4). \quad (40)$$

This results together with (39), yields

$$R(k_1,k_2;\,k_3,k_4)=R_0(k_1,k_2,k_3,k_4)+F(k_1,k_2;\,k_3,k_4)+F(k_1,k_3;\,k_2,k_4)$$

and

$$F(k_1,k_2;\,k_3,k_4)=-\frac{\bar{g}_0^2}{2\pi i}\int[R_0(k_1,k_2,l,l')+F(k_1,l;\,k_2,l')+F(k_1,l';\,k_2,l)]\bar{D}(l)\bar{D}(l')[R_0(l,\,-l',\,k_3,\,k_4)+F(-l,\,k_3;\,-l',\,k_4)+F(-l,\,k_4;\,-l',\,k_3)+F(-l,\,-l';\,k_3,\,k_4)]d^4l. \quad (41)$$

Equation (41) uniquely defines the function F if R_0 is known.

If the meson momenta are large and only the logarithmic region $-l^2\gg-(k_1+k_2)^2$ is taken into account, then (41) can be considerably simplified. An attentive scrutiny of (41) reveals that if the momenta k_1 and k_2 are very large and considerably exceed their sum k_1+k_2, F will be dependent only on the two variables $\zeta=\ln[-(k_1+k_2)^2/m^2]$ and $\eta=\ln(-k^2/m^2)$, where k is the larger of the momenta k_1 and k_2; i.e., in this case[18]

$$F(k_1,k_2;\,k_3,k_4)=\Phi(\eta,\zeta).$$

If, however, all sums k_i+k_j of meson momenta are quantities of the same order [this is exactly the case which is of greatest interest with respect to substitution in (26)] or if there are two large momenta belonging to different pairs—the first to k_1, k_2 and the other to the second pair, then $\eta=\zeta$ and F will depend only on one variable $F=F(\eta)=\Phi(\eta,\eta)$ [by definition η is always $\geqslant\zeta$].

A shortcoming of Eq. (41) is that when F depends on one variable it is coupled in (41) with the value of $F(-l,\,-l';\,k_3,\,k_4)\simeq\Phi(z,\eta)$, depending for $l\to\infty$ on two variables.

[18] It is assumed that the larger of the momenta k_3 and k_4 (momentum k') is of the same order of magnitude as $k_3+k_4=-(k_1+k_2)$; otherwise $F=\Phi(\eta,\zeta,\eta')$, where $\eta'=\ln(-k'^2/m^2)$.

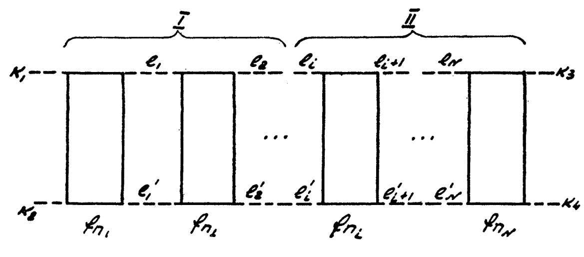

FIG. 8. Division of a reducible diagram into N irreducible diagrams with respect to "separation" of lines l_i and l_i' from l_{i+1} and l_{i+1}'.

Thus, for example, in neutral theory we obtain

$$F(\eta)=-\frac{\bar{g}_0^2}{8\pi}\int_\eta^{L_k}[R_0(z)+2F(z)]\times[R_0(z)+2F(z)+\Phi(z,\eta)]\bar{d}^2(z)dz. \quad (42)$$

$F(\eta)$ therefore can be found only if $\Phi(z,\eta)$ is known. For this reason we are forced to consider in (41) the case when k_1 and k_2 are very large compared with their sum. We then obtain:

$$\Phi(\eta,\zeta)=-\frac{\bar{g}_0^2}{8\pi}[R_0(\eta)+2F(\eta)]\int_\zeta^\eta[R_0(z)+2F(z)+\Phi(z,\zeta)]\bar{d}^2(z)dz-\frac{\bar{g}_0^2}{8\pi}\int_\zeta^\eta[R_0(z)+2F(2)]\times[R_0(z)+2F(z)+\Phi(z,\zeta)]\bar{d}(z)dz \quad (43)$$

and, according to (42), $\Phi(\eta,\eta)=F(\eta)$. The set of Eqs. (42) and (43) has a unique solution which can easily be constructed for $\bar{g}_0^2(L_k-\eta)<1$, or $\bar{g}_0^2(L_k-\eta)>1$. The corresponding calculations are given in the appendix for the case of symmetric theory. The equations corresponding to (42)–(43) in this case are also given in the Appendix.

We shall now demonstrate that if all the momenta sums k_i+k_j are quantities of the same order, then the set (42), (43) is equivalent to a simple equation for the function[19] $P(\eta)=R_0(\eta)+3F(\eta)$.

In this case we deduce anew an equation which is similar to (41). From the very beginning, however, we restrict our considerations only to asymptotic values of the functions and take into account only the logarithmic integration region $\eta\leqslant z\leqslant L_k$.

An arbitrary diagram reducible in the sense of separation of k_1, k_2 from k_3, k_4 consists of a number of diagrams (at least two) connected by meson lines l_i, $l_i'=-l_i+k_1+k_2$ (Fig. 8), each of these diagrams being irreducible with respect to separation of the pair of lines l_i, l_i' from l_{i+1}, l_{i+1}' (see Fig. 8). If the momenta l_i and

[19] In the symmetric theory each of the three functions depends on the isotopic spin variables in an individual manner, i.e., $P(\eta)=R_0(\eta)+F_a(\eta)+F_b(\eta)+F_c(\eta)$, where the subscripts a, b, and c refer to the three diagrams in Fig. 4 [i.e., $F_a=F(k_1,k_2;\,k_3,k_4)$, $F_b=F(k_1,k_3;\,k_2,k_4)$, $F_c=F(k_1,k_4;\,k_2,k_3)$].

l_{i+1} are very large the contribution from the ith diagram in such a chain, f_{n_i}, will depend only on the larger of these momenta. We write it as

$$f_{n_i}(z_i,z_{i+1}),\quad z_i=\ln(-l_i^2/m^2),$$

where f_{n_i} in fact depends only on the larger of the quantities z_i, z_{i+1}. In this case, analogously to (37), we obtain for the total contribution from the diagram in Fig. 8

$$f_{n_0n_1\cdots n_N}(\eta)=\left(\frac{-\bar{g}_0^2}{4\pi}\right)^N\int_\eta^{L_k}dz_1\cdots\int_\eta^{L_k}dz_N \times f_{n_0}(\eta,z_1)\bar{d}^2(z_1)f_{n_1}(z_1,z_2)\bar{d}^2(z_2)\cdots \times\bar{d}^2(z_N)f_{n_N}(z_N,\eta),\quad (44)$$

where N is the number of irreducible diagrams in the chain in Fig. 8, corresponding to the given reducible diagram. The integration region over z_1, z_2, $\cdots$, z_n can be divided into N regions in each of which one of the variables, say z_i, is smaller than the others. Correspondingly, (44) can be represented as the sum of N integrals over these regions:

$$f_{n_0n_1\cdots n_N}(\eta)=\sum_{i=1}^N\frac{-\bar{g}_0^2}{4\pi}\int_\eta^{L_k}f_{n_0n_1\cdots n_{i-1}}(z_i)\bar{d}^2(z_i) \times f_{n_in_{i+1}\cdots n_N}(z_i)dz_i.\quad (45)$$

Here $f_{n_0n_1\cdots n_{i-1}}(z_i)$ is defined in exactly the same manner as in (44), i.e., it is the contribution from that part of the diagram under consideration (part I in Fig. 8) which is contiguous to lines k_1, k_2 and l_i, l_i', and which one would expect if k_1 and k_2 were quantities of the same order as l_i, l_i'. Similarly, $f_{n_in_{i+1}\cdots n_N}(z_i)$ refers to part II in Fig. 8.

By definition,

$$\sum_{n_0}f_{n_0}(z_1)=R(z_1)\quad (46)$$

is the contribution of all irreducible diagrams (with respect to separation of lines k_1, k_2 from l_1, l_1'), and

$$\sum_{i=2}^\infty\sum_{n_0,n_1,\cdots,n_{i-1}}2^{1-i}f_{n_0n_1\cdots n_{i-1}}(z_i)=F(z_i)\quad (47)$$

is the contribution of all reducible diagrams (in this sense).

The factor 2^{i-1} [similar to the factor 2 in (38)] takes into account the fact that on summation over all types of irreducible diagrams in each "contractible" part (of part I in Fig. 8), 2^{i-1} identical types of reducible diagrams appear.

According to (46), (47), and (39),

$$P(z)=R(z)+F(z)=\sum_{i=1}^\infty\sum_{n_0n_1\cdots n_{i-1}}2^{1-i}f_{n_0n_1\cdots n_{i-1}}(z).\ (48)$$

Multiplying Eq. (45) by $1/2^N$ and summing over N and all n_i we obtain, in accordance with (47) and (48),

$$F(\eta)=-\frac{\bar{g}_0^2}{8\pi}\int_\eta^{L_k}P(z)\bar{d}^2(z)P(z)dz.\quad (49)$$

In neutral theory $P(\eta)=R_0(\eta)+3F(\eta)$, and therefore

$$P(\eta)=R_0(\eta)-\frac{3\bar{g}_0^2}{8\pi}\int_\eta^{L_k}P^2(z)\bar{d}^2(z)dz.\quad (50)$$

In the symmetric theory P depends on the isotopic spin variables a_i, $P_{a_1a_2a_3a_4}(\eta)=P(\eta)\delta_s$, where δ_s is defined in (29). In (49) we therefore find (for the function F_a corresponding to diagrams of Fig. 4(a) which are reducible with respect to separation of k_1, k_2 from k_3, k_4),

$$\sum_{\nu,\mu=1}^3P_{a_1a_2\nu\mu}(z)P_{\nu\mu a_3a_4}(z)=[2\delta_s+5\delta_{a_1a_2}\delta_{a_3a_4}]P^2(z).$$

Thus, for the quantity $P(\eta)\delta_s=\rho_0\delta_s+F_a+F_b+F_c$, we get

$$P(\eta)=\rho_0(\eta)-\frac{11\bar{g}_0^2}{8\pi}\int_\eta^{L_k}P^2(z)\bar{d}^2(z)dz.\quad (51)$$

In the approximation considered here [zero order in $\bar{g}_0^2$, for an expansion of type (1)], Eqs. (50) or (51) accurately define $P(\eta)$. These equations can easily be solved after substituting in them expressions (3)–(4) for the functions $\bar{d}$ and (29)–(31) for the functions ρ_0 and R_0. We shall consider the most general case, in which, besides the usual interaction, a direct interaction of the type[20] $\lambda_0\varphi^4$ is included in the Hamiltonian. The only alteration in the formulas given above is that a constant quantity is added to (29) or (31). As a result one should substitute in (50) or (51),

$$\rho_0=(16/3)(x-1)+\bar{b}_0,\quad x=\bar{Q}^{3/5},$$
$$R_0=24(1-x)+\bar{b}_0,\quad x=\bar{Q}^{1/5},\quad (52)$$

respectively, for symmetric and neutral theory, where $\bar{b}_0$ is a constant.[21] Substitution in (50) and (51) yields, after introducing a variable x in accord with (52) and

[20] In the presence of this interaction, the S matrix has the form

$$T\left\{\exp\left[g_0\int\bar{\psi}\gamma_5\tau_\alpha\psi\varphi_\alpha dx+\frac{\lambda_0}{i4!}\int\delta_s\varphi_{a_1}\varphi_{a_2}\varphi_{a_3}\varphi_{a_4}dx\right]\right\}$$

in symmetry theory and

$$T\left\{\exp\left[g_0\int\bar{\psi}\gamma_5\psi\varphi dx+\frac{\lambda_0}{i4!}\int\varphi^4dx\right]\right\}$$

in neutral theory.

[21] This constant is equal to

$$(4/\pi)\bar{g}_0^2\ln(\Lambda_p^2/\Lambda_k^2)+(4\pi\bar{\lambda}_0/\bar{g}_0^2)\quad [\text{or } 4+(4\pi\lambda_0/g_0^2)]$$

in symmetric theory, and to

$$(6/\pi)\bar{g}_0^2\ln(\Lambda_p^2/\Lambda_k^2)+(4\pi\bar{\lambda}_0/\bar{g}_0^2)\quad [\text{or } 6+(4\pi\lambda_0/g_0^2)]$$

in neutral theory. The values in brackets refer to the case $(g_0^2/\pi)\ln(\Lambda_p^2/\Lambda_k^2)\gg1$, for which $(\bar{g}_0^2/\pi)\ln(\Lambda_p^2/\Lambda_k^2)\cong1$; $\bar{\lambda}_0=Q_0^{-1}\lambda_0$.

taking into account (3), (4),

$$P(\bar{b}_0,x)=\frac{16}{3}(x-1)+\bar{b}_0-\frac{11}{6}\int_1^x P^2(\bar{b}_0,x)\frac{dx}{x^2},$$
$$P(\bar{b}_0,x)=24(1-x)+\bar{b}_0-\frac{3}{2}\int_x^1 P^2(\bar{b}_0,x)\frac{dx}{x^2}. \quad (53)$$

Here, as in the following, the first formula refers to symmetric theory and the second to neutral theory. Differentiating (55) with respect to x we obtain

$$\frac{dP(\bar{b}_0,x)}{dx}=\frac{16}{3}-\frac{11}{6}\left(\frac{P(\bar{b}_0,x)}{x}\right)^2,$$
$$\frac{dP(\bar{b}_0,x)}{dx}=-24+\frac{3}{2}\left(\frac{P(\bar{b}_0,x)}{x}\right)^2, \quad (54)$$

where, according to (53), $P(\bar{b}_0,1)=\bar{b}_0$. Solving Eqs. (54) with this boundary condition, we obtain

$$P(\bar{b}_0,x)=\frac{16}{11}x\frac{B-x^{-19/3}}{B+(8/11)x^{-19/3}},\quad x=\bar{Q}^{3/5},$$
$$P(\bar{b}_0,x)=\frac{(145)^{\frac{1}{2}}+1}{3}x\frac{B-x^{\surd 145}}{B+\frac{(145)^{\frac{1}{2}}+1}{(145)^{\frac{1}{2}}-1}x^{\surd 145}},\quad x=\bar{Q}^{-1/5}, \quad (55)$$

where

$$B=(1+\tfrac{1}{2}\bar{b}_0)/(1-\tfrac{11}{16}\bar{b}_0),$$
$$B=\left(1+\frac{(145)^{\frac{1}{2}}+1}{48}\bar{b}_0\right)\Big/\left(1+\frac{(145)^{\frac{1}{2}}-1}{48}\bar{b}_0\right). \quad (56)$$

These formulas show that $P(\bar{b}_0,x)$ is a quantity of the same order of magnitude as $R_0(x)$. Substituting (55) in (26) we find, as in (33), that for $\bar{Q}\gg 1$ the contribution in (16) of all diagrams with any number of nucleon loops is

$$\gamma_5\frac{5\bar{g}_0^2}{11\pi}(1-\bar{Q}^{-4/5})\simeq\gamma_5\frac{5\bar{g}_0^2}{11\pi},$$
$$\gamma_5\frac{(145)^{\frac{1}{2}}+1}{72\pi}\bar{g}_0^2(1-\bar{Q}^{-6/5})\approx\gamma_5\frac{(145)^{\frac{1}{2}}+1}{72\pi}\bar{g}_0^2$$

for symmetric theory and neutral theory, respectively. It is obvious that for $\bar{g}_0^2\to 0$ this contribution vanishes.

8. RENORMALIZATION PROPERTIES OF THE SCATTERING AMPLITUDE $P(b_0,x)$. DEDUCTION OF (54), (55) FROM THE RENORMALIZATION PROPERTIES

In the familiar renormalization procedure,[22] the meson-meson scattering amplitude

$$(g_0^2/4\pi)P(b_0,x)$$

is multiplied after renormalization by

$$Z_3^2=d_c^{-2}(g_c^2,L)=d^2(g_0^2,L).$$

According to (12), $g_c^2=g_0^2/Q_0$, $Q_0=1+(5g_0^2/4\pi)L$, and according to (3) and (4), $Z_3^2=Q_0^{-8/5}$ in symmetric theory and $Z_3^2=Q_0^{-4/5}$ in neutral theory. Thus we obtain

$$Z_3^2\frac{g_0^2}{4\pi}P(b_0,x)=\frac{g_c^2}{4\pi}\frac{P(b_0,x)}{x_0}, \quad (57)$$

where $x_0=Q_0^{3/5}$ in symmetric theory and $x_0=Q_0^{-1/5}$ in neutral theory. Thus after renormalization, $P(b_0,x)$ is replaced in all the formulas by

$$P_c(x)=P(b_0,x)/x_0,$$

which is the meson-meson scattering amplitude after renormalization.[23] As $x=x_0x_c$ [where x_0 and x_c are defined in terms of Q_0 and Q_c as x is in terms of Q], we obtain, in accord with (55),

$$P_c(x)=P(b_c,x_c), \quad (58)$$

where P is a function (55) and b_c is a function of b_0, g_0^2 and L being defined by the formulas

$$(1+\tfrac{1}{2}b_c)/(1-\tfrac{11}{16}b_c)=[(1+\tfrac{1}{2}b_0)/(1-\tfrac{11}{16}b_0)]x_0^{19/3},$$
$$\left(1+\frac{(145)^{\frac{1}{2}}+1}{48}b_c\right)\Big/\left(1+\frac{(145)^{\frac{1}{2}}-1}{48}b_c\right)$$
$$=\left[\left(1+\frac{(145)^{\frac{1}{2}}+1}{48}b_0\right)\Big/\left(1+\frac{(145)^{\frac{1}{2}}-1}{48}b_0\right)\right]x_0^{-\surd 145}.$$

If $P(b_0,x)$ and $P_c(x)$ are replaced by the quantities

$$\mathcal{P}(g_0^2,\lambda_0,L-\xi)=\frac{P(b_0,x)}{b_0}=\frac{g_0^2}{4\pi\lambda_0}P\left(\frac{4\pi\lambda_0}{g_0^2},x\right), \quad (59)$$

$$\mathcal{P}_c(g_c^2,\lambda_c,\xi)=\frac{g_c^2}{4\pi\theta_c}P_c(x)=\frac{g_c^2}{4\pi\lambda_c}\frac{P(b_0,x)}{x^0}, \quad (60)$$

where, according to (59), $\mathcal{P}(g_0^2,\lambda_0,0)=1$ [since $P(b_0,1)=b_0$ for $\xi=L$, $x=1$], one may then rewrite the renormalization relation (57) in a form which is exactly similar to that of (11). $\mathcal{P}$ and $\mathcal{P}_c$ are the vertex parts of the meson-meson scattering diagrams before and after renormalization. According to (57),

$$\lambda_0\mathcal{P}(g_0^2,\lambda_0,L-\xi)=\lambda_c d_c^2(g_c^2,\lambda_c,L)\mathcal{P}_c(g_c^2,\lambda_c,\xi), \quad (61)$$

where Z_3^{-2} is replaced by $d_c^2(L)$. Putting $\xi=L$, we obtain

$$\lambda_0=\lambda_c d_c^2(g_c^2,\lambda_c,L)\mathcal{P}_c(g_c^2,\lambda_c,L). \quad (62)$$

This formula defines the dependence of λ_0 on λ_c, L, and g_c^2, which was expressed above in explicit form. Inserting in (61) the values of λ_c from (62), we obtain,

[22] For the sake of simplicity we consider in this section the case $\Lambda_p=\Lambda_k=\Lambda$, for which $\bar{g}_0^2=g_0^2$, $\bar{\lambda}_0=\lambda_0$, and the quantity $\bar{b}_0$ (see reference 19) equals $4\pi\lambda_0/g_0^2$. To change to the two-cutoff case D, d, g_0^2, λ_0, b_0 should be replaced by $\bar{D}$, $\bar{d}$, $\bar{g}_0^2$, $\bar{\lambda}_0$, and $\bar{b}_0$.

[23] It should be noted that $P_c(x)$, according to (55), remains finite for any value of b_0.

as in (11),

$$\mathcal{P}(g_0^2, \lambda_0, L-\xi)=\mathcal{P}_c(g_c^2,\lambda_c,\xi)/\mathcal{P}_c(g_c^2,\lambda_c,L). \quad (63)$$

Equations (11), (62), and (63) completely determine the renormalization procedure, if one takes into account that α, β, and d in relations (6) depend, in the presence of direct meson-meson interaction, on λ_0; and α_c, β_c, and d_c depend on λ_c.

We now show that Eqs. (54) and (55) for the meson-meson scattering amplitude can be obtained from these equations in a manner similar to that used above to obtain the values (3) and (4) of the functions α, β, and d.

We introduce, similarly to (12), the effective charge of direct meson-meson interaction,

$$\lambda(\xi)=\lambda_0\mathcal{P}(g_0^2, \lambda_0, L-\xi)d^2(g_0^2, \lambda_0, L-\xi) \equiv\lambda_c\mathcal{P}_c(g_c^2,\lambda_c,\xi)d_c^2(g_c^2,\lambda_c,\xi). \quad (64)$$

By a method similar to that used in obtaining (13), we obtain from (11) and (63)

$$\begin{aligned} \alpha'/\alpha&=\alpha_c'/\alpha_c=F_1(g^2,\lambda),\\ \beta'/\beta&=\beta_c'/\beta_c=F_2(g^2,\lambda),\\ d'/d&=d_c'/d_c=F_3(g^2,\lambda),\\ \mathcal{P}'/\mathcal{P}&=\mathcal{P}_c'/\mathcal{P}_c=F_4(g^2,\lambda), \end{aligned} \quad (65)$$

where the functions F_1, F_2, F_3, and F_4 depend only on the effective charges (12) and (64). As an example we present the proof for F_4. From (12) and (64), we obtain

$$\xi=\xi(g_c^2,g^2,\lambda), \quad \lambda_c=\lambda_c(g_c^2,g^2,\lambda);$$

therefore the ratio $\mathcal{P}_c'/\mathcal{P}_c$, which is a function of g_c^2, λ_c, and ξ, can be expressed as a function of g_c^2, g^2, and λ. Thus $\mathcal{P}'/\mathcal{P}$, which is equal to $\mathcal{P}_c'/\mathcal{P}_c$, can be expressed as follows:

$$\mathcal{P}'(g_0^2,\lambda_0, L-\xi)/\mathcal{P}(g_0^2,\lambda_0, L-\xi)=F_4(g_c^2,g^2,\lambda). \quad (66)$$

If the values of g_0^2 and λ_0 are fixed in this equation and L and ξ vary in such a manner that $L-\xi$ remains constant, it will easily be seen [see the similar equation (14)] that the only varying quantity in (66) will be g_c^2. Equality (66), therefore, can be true only if F_4 is in fact independent of g_c^2, i.e., we arrive at the last of Eqs. (65). A similar proof can be applied to the remaining cases. The logarithmic derivatives of the effective charges (12) and (64) yield

$$\begin{aligned} [g^2]'/g^2&=2F_1(g^2,\lambda)+2F_2(g^2,\lambda)+F_3(g^2,\lambda),\\ \lambda'/\lambda&=F_4(g^2,\lambda)+2F_3(g^2,\lambda), \end{aligned} \quad (67)$$

which (if the functions F are known) can be considered as the differential equations for $g^2(\xi)$ and $\lambda(\xi)$. These equations should be solved with the boundary conditions: $g^2(L)=g_0^2$, $\lambda(L)=\lambda_0$. Letting $\xi\rightarrow L$, $\lambda_0\ll1$, $g_0^2\ll1$ (and $\lambda_0\sim g_0^2$) and using perturbation theory, one may determine the functions F_i. For F_1, F_2, and F_3 we obtain the previous expressions (since the effect of direct meson interaction on them is a higher order effect), and hence for $g^2(\xi)$ we obtain the expression (15):

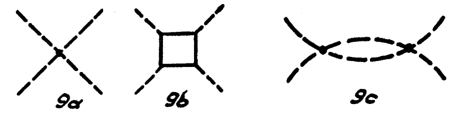

FIG. 9. Essential diagrams in perturbation theory for $\lambda_0\sim g_0^2$.

$$g^2(\xi)=g_0^2/Q(\xi).$$

In order to obtain F_4, one should determine P to first order in g_0^2 and λ_0.

From the diagrams of Fig. 9, we obtain for the cases of symmetric and neutral theory:

$$\lambda_0\mathcal{P}=\lambda_0+\frac{g_0^4}{\pi^2}(L-\xi)-\frac{11}{2}\lambda_0^2(L-\xi),$$

$$\lambda_0\mathcal{P}=\lambda_0+\frac{3g_0^4}{2\pi^2}(L-\xi)-\frac{3}{2}\lambda_0^2(L-\xi).$$

[The three terms in these formulas correspond to diagrams (*a*), (*b*), (*c*) in Fig. 9.]. Hence,

$$F_4=(\mathcal{P}'/\mathcal{P})_{\xi\rightarrow L}=(11/2)\lambda-(g^4/\pi^2\lambda)$$

for symmetric theory and $F_4=\frac{3}{2}\lambda-(3g^4/2\pi^2\lambda)$ for neutral theory. Inserting these values of F_4 and the values of g^2 found previously in (67), we obtain the equations

$$\frac{\lambda'(\xi)}{\lambda(\xi)}=\frac{11}{2}\lambda(\xi)-\frac{g_0^4}{\pi^2\lambda(\xi)Q(\xi)}+\frac{2g_0^2}{\pi Q(\xi)},$$

$$\frac{\lambda'(\xi)}{\lambda(\xi)}=\frac{3}{2}\lambda(\xi)-\frac{3g_0^4}{2\pi^2\lambda(\xi)Q(\xi)}+\frac{g_0^2}{\pi Q(\xi)}.$$

These equations reduce directly to (54) if, in accord with (64), the substitution $\lambda(\xi)=(g_0^2/4\pi)P(b_0,x)\bar{d}^2(x)$ is made and one changes to the independent variable $x=Q^{3/5}$ in the case of symmetric theory and to $x=Q^{-1/5}$ for neutral theory.[24]

9. VANISHING OF MESON CHARGE IN PSEUDOSCALAR THEORY WITH PSEUDOVECTOR COUPLING

As is well known, within the framework of the usual perturbation theory, the theory with pseudovector coupling is not renormalizable.

This, however, cannot be used as an argument against a treatment in which an expansion in powers of g_0^2 is not used.

We shall consider the theory with two-cutoff values.

[24] See also in this connection N. N. Bogolubov and D. V. Shirkoff, Doklady Akad. Nauk S.S.S.R. **103**, 400 (1951); Nuovo Cimento **3**, 845 (1956); D. V. Shirkoff, Doklady Akad. Nauk S.S.S.R. **105**, 972 (1954); J. C. Taylor, Proc. Roy. Soc. (London) **234**, 296 (1956).

We confine our considerations to the limiting process (c),

$$(\Lambda_k^2/m^2)[\ln(\Lambda_p^2/\Lambda_k^2)]^{-1}\ll 1,$$

as this is the simplest case.

We will now show that in this case[25] the Schwinger-Dyson equations

$$\Gamma(p, p-k)=\theta\gamma_5 k=\frac{g_0^2}{m^2\pi i}\int \Gamma G(p-l)\times\Gamma G(p-k-l)\Gamma D(l)d^4l+\cdots, \quad (68)$$

$$\left[p-m-\frac{3g_0^2}{\pi i m^2}\int \theta\gamma_5 k G(p-k)\Gamma D(k)d^4k\right]G(p)=1, \quad (69)$$

$$\left\{k^2-\mu^2+\frac{2g_0^2}{m^2\pi i}\int \mathrm{Sp}[\Gamma G(p)\theta\gamma_5 k G(p-k) -(\Gamma G(p)\theta\gamma_5 k G(p-k))_{k^2=\mu^2}]d^4p\right\}D(k)=1 \quad (70)$$

(where θ is the cut-off function, μ the observable meson mass, and m the "bare" nucleon mass which equals, as shown below, the observable mass), have a simple solution of the form

$$G(p)=(p-m)^{-1},$$

$$\Gamma(p, p-k)=\begin{cases}\gamma_5 k & \text{if the momenta are less than the cutoff values,}\\ 0 & \text{otherwise,}\end{cases} \quad (71)$$

$$D(k)=\left[1+\frac{4g_0^2}{\pi}\ln(\Lambda_p^2/\Lambda_k^2)\right]^{-1}\bar{D}(k),$$
$$\bar{D}(k)=(k^2-\mu^2)^{-1}. \quad (72)$$

First of all it will be proved that (72) is a corollary of (70) and (71).

Inserting (71) in (70) and transforming the integrand in (70), we get

$$\mathrm{Sp}\left[\gamma_5 k\frac{1}{p-m}\gamma_5 k\frac{1}{p-k-m}\right] =\mathrm{Sp}\left[k\frac{1}{p-m}k\frac{1}{p-k-m}\right]-\frac{8m^2k^2}{(p^2-m^2)((p-k)^2-m^2)}.$$

The term

$$\mathrm{Sp}\left[k\frac{1}{p-m}k\frac{1}{p-k-m}\right]=k\frac{1}{p-m}-k\frac{1}{p-k-m}$$

does not make any contribution to the integral in (70).[26]

The second term can be written as

$$\frac{-8m^2k^2}{(p^2-m^2)((p-k)^2-m^2)} =\frac{-8m^2k^2}{(p^2-m^2)^2}-8m^2k^2\frac{2pk-k^2}{(p^2-m^2)^2[(p-k)^2-m^2]}. \quad (73)$$

On substitution of (73) in (70), the second term of (73) gives a convergent integral, whereas the integral of the first term of (73) diverges logarithmically. The part of this divergent integral in the region $\Lambda_k^2<(-p^2)<\Lambda_p^2$ will equal $(k^2-\mu^2)\Pi$, where

$$\Pi=(4g_0^2/\pi)\ln(\Lambda_p^2/\Lambda_k^2).$$

Taking into account in (10) only the largest, logarithmically divergent part of the integral and writing $(1+\Pi)$ in front of the brackets in (70), we get the following expressions for $\bar{D}=(1+\Pi)D(k)$:

$$\bar{D}^{-1}(k)\approx(k^2-\mu^2)[1-(4g_c^2/\pi)\ln(\Lambda_k^2/-k^2)], \quad (74)$$

$$g_c^2=(1+\Pi)^{-1}g_0^2=\frac{g_0^2}{1+(4g_0^2/\pi)\ln(\Lambda_p^2/\Lambda_k^2)}, \quad (75)$$

which is the same as (72) since $g_c^2\ln(\Lambda_p^2/\Lambda_k^2)$, according to (c), is an infinitesimally small quantity. According to (71) and (72), g_c^2 is the renormalized charge. Thus, if (72) is substituted in (68) and (69), g_0^2 will be replaced by g_c^2 and the free-field function $\bar{D}$ is substituted everywhere for D.

Equations (68) and (69) are satisfied in this case, i.e., the integral terms which enter them are infinitesimally small. To prove this we estimate the integral term in (69),

$$M(p)=-\frac{3g_c^2}{m^2\pi i}\int^{\Lambda_k}\gamma_5 k\frac{1}{p-k-m}\gamma_5 k\frac{d^4k}{k^2-\mu^2}.$$

We have

$$M(p)\simeq p\tfrac{3}{4}g_c^2\Lambda_k^2/m^2, \quad (76)$$

to terms of the order of $m^2/-p^2$ and $-p^2/\Lambda_k^2$ relative to unity.

If $-p^2$ is not small compared with Λ_k^2, the magnitude of $M(p)$ will be even smaller.

If, however $p=m$, then $M(m)=\Delta m$ will be the change of mass resulting from interaction with the meson field. In this case, expanding $(p-k-m)^{-1}$ in a series in m, we obtain

$$M(m)=\Delta m\simeq\tfrac{3}{4}mg_c^2\Lambda_k^2/m^2. \quad (77)$$

According to (75) and (c), $g_c^2\Lambda_k^2/m^2\ll 1$, and (76) and (77)

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[25] Only neutral theory will be considered. In this case the S-matrix has the form $T\{\exp[(g_0/m)\int\bar{\psi}\gamma_5(-i\nabla\varphi)\psi dx]\}$, where g_0 is the dimensionless coupling constant and $(-i\nabla)e^{-ikx}=Ke^{-ikx}$.

[26] E. S. Fradkin and Avrorin (to be published). It should be remarked that, if the cut-off procedure is carried out in such a way that

$$\mathrm{Sp}\left[k\frac{1}{p-m}k\frac{1}{p-k-m}\right]$$

yields a nonzero result, the proof that the renormalized charge is zero is even simpler.

are vanishingly small quantities [(77) in particular shows that $\Delta m/m \ll 1$, i.e., the "bare" nucleon mass is practically the same as the observable mass].

In the exact equation (68), besides the term written above, which corresponds to the simplest diagram of Fig. 1(a), one should also take into account other terms which correspond to more complicated diagrams with intersecting meson lines and nucleon loops (Fig. 1).

We first consider the case when the momenta p, $p-k$, k, corresponding to free ends in (68), are small compared with Λ_k.

It is easiest of all to estimate the contribution from the term given in (68),

$$\frac{g_c^2}{m^2\pi i}\int^{\Lambda_k}\gamma_5 l\frac{1}{p-l}\gamma_5 k\frac{1}{p-k-l}\gamma_5 l\frac{d^4l}{l^2}\simeq\gamma_5 k\frac{g_c^2}{4\pi}\frac{\Lambda_k^2}{m^2}.$$

It evidently is vanishingly small compared with $\gamma_i k$.

Consider now an arbitrary diagram of the type of Figs. 1(b) and 1(c) with n intersecting meson lines. Its contribution involves the factor $(g_c^2)^n$; in the integrand we have n functions $\bar{D}=(k^2-\mu^2)^{-1}$, $2n$ functions G, and $2n$ functions Γ, the number of integrations over the momenta $k_1, k_2, \cdots, k_n$ of the mesons being n. The order of magnitude of the integral will not be underestimated if each integration over $-k_i^2$ is carried out independently, starting from $-p^2$ (or $-k^2$, since for $-p^2 \ll \Lambda_k^2$ and $-k^2 \ll \Lambda_k^2$ the lower limit is of no consequence, owing to the quadratic divergence of the integrals). Thus we obtain the following estimate of the contribution I_n from the diagram under consideration,

$$I_n \simeq \gamma_5 k C_n (g_c^2)^n (\Lambda_k^2/m^2)^n, \tag{78}$$

where C_n is a numerical factor. Obviously the magnitude of (78) is arbitrarily small for $g_c^2\Lambda_k^2/m^2 \ll 1$.

Consider now the diagrams of the type of Figs. 1(d), 1(e), etc., which include meson-meson scattering. An elementary "square" of Fig. 3 in the case of pseudovector coupling does not involve any divergences, and yields a contribution equal in order of magnitude to

$$g_c^2(-k^2/m^2)^2 \tag{79}$$

if $(-k^2)\sim(-k_1^2)\sim(-k_2^2)\sim(-k_3^2)\sim-(k_1+k_2+k_3)^2$. Now compare this quantity with the contribution from the diagrams of Fig. 4, containing two squares. Using (71) and (72), we get the following estimate of the contributions:

$$\frac{g_c^8k^4}{m^8}\int^{\Lambda_k} d^4l \sim g_c^4\left(\frac{-k^2}{m^2}\right)^2\left(g_c^2\frac{\Lambda_k^2}{m^2}\right)^2. \tag{80}$$

Thus, in order of magnitude, (80) differs from (79) by a factor $(g_c^2\Lambda_k^2/m^2)^2 \ll 1$, and is vanishingly small. The series of successive meson-meson scattering acts converges rapidly and practically equals (79). In other words, the contribution from all the diagrams with nucleon loops of the type shown in Figs. 1(d) and 1(e), is practically the same as the contribution $\mathcal{G}_{1d}$ from the single diagram in Fig. 1(d); the magnitude of the latter can readily be estimated,

$$\mathcal{G}_{1d}\sim\gamma_5 k C_{1d}\left(g_c^2\frac{\Lambda_k^2}{m^2}\right)^3 \ll \gamma_5 k,$$

where C_{1d} is a constant.

The estimates thus obtained show that all diagrams with intersecting lines and nucleon loops of the type shown in Fig. 1 form a series

$$\gamma_5 k\sum_{n=1}^{\infty} B_n\left(g_c^2\frac{\Lambda_k^2}{m^2}\right)^n \approx \gamma_5 k\sum_{n=1}^{\infty} B_n\left[\frac{\pi}{4}\frac{\Lambda_k^2}{m^2\ln(\Lambda_p^2\Lambda_k^2)}\right]^n, \tag{81}$$

where the B_n are numerical multipliers. This series is evidently asymptotic. However, its sum can be made arbitrarily small relative to $\gamma_5 k$ by virtue of condition (c). Thus Γ really equals $\gamma_5 k$, in agreement with (71).

Up to the present the momenta p and k were considered small compared with Λ_k. If this condition is not obeyed, the difference between Γ and $\gamma_5 k$ will be even smaller than in the case considered above (this is similar to the situation in electrodynamics and in pseudoscalar coupling theory). For example, if

$$-p^2 \gg \Lambda_k^2, \quad -k^2 \ll \Lambda_k^2$$

the quantity $1/p$ (instead of $1/k$) will correspond to the nucleon lines of the diagrams in Figs. 1(b), 1(c), etc. Instead of (78), we obtain the following estimate of the order of the contributions from diagrams with n intersecting meson lines:

$$I_n \simeq \gamma_5 k C_n'\left(\frac{\Lambda_k^2}{-p^2}\right)^n\left(g_c^2\frac{\Lambda_k^2}{m^2}\right)^n,$$

which is even smaller than the quantity in (78).

Thus, by strengthening inequality (c), in the limit $\Lambda_k \to \infty$, we may, with any degree of accuracy, satisfy Eqs. (68), (69), and (70) with functions (71) and (72).

Relation (75), which establishes the connection between the renormalized and "bare" charges, indicates that for $\Lambda_k \to \infty$,

$$g_c^2 \to 0,$$

which is similar to what one finds in electrodynamics and pseudoscalar coupling theory.

10. CONCLUSION

The result $g_c^2 \to 0$ was rigorously proved above for pseudoscalar theory with two different types (a) and (b) of limiting processes $\Lambda_k \to \infty$. In both cases, in the limit $\Lambda_k \to \infty$, the physical interaction disappears; thus under these restricted conditions the result $g_c^2 \to 0$ is independent of the form of the limiting process. It is significant that no ambiguity arises in the theory and after renormalization all the results are independent of the form factors, i.e., of the specific nature of the limiting process (parameters of the type Λ_p and Λ_k which characterize the "diffuseness" completely disappear from the formulas of the renormalized theory; they remain only in the formuls relating the unrenormalized quantities). This indicates that, from the theoretical viewpoint, various types of limiting processes are equivalent; i.e., in the limit of $\Lambda_k \to \infty$, any relationship between Λ_p and Λ_k is permissible.[27]

In other words, physical results for fixed *finite distances* should not depend on the character of the limiting process, i.e., on the relation between Λ_p and Λ_k, which, in the limit, refer to *infinitesimally small distances*.

It can be seen from the foregoing that two cutoffs are necessary to keep, in the range up to Λ_k, the "effective" charge $\bar{g}_0^2$ small, no matter how large the bare charge g_0 might be. A slow, logarithmic dependence of all quantities on the momenta, and the possibility of expanding into a series of the same type as (1), is due to the smallness of $\bar{g}_0^2$. Thus, introduction of two-cutoff values considerably simplifies the problem, as in this case the theory with an arbitrary g_0 directly reduces to the case when the charge is small (if $\Pi \gg 1$) and expansion (1) and formula (7) are valid [or else the problem directly reduces to the case when the interaction is turned off, if (*b*) $\Pi^{-1}\ln(\Lambda_k^2/m^2)\to 0$, or (*c*): $\Pi^{-1}\Lambda_k^2/m^2\to 0$]. Since the results are independent of the character of the limiting process, the result that g_c^2 equals zero can also be expected in a single-cutoff theory in which $\Lambda_p=\Lambda_k$. (According to (7) for a sufficiently small g_0 this is certaintly true.) If, however, g_0^2 is not small in the single-cutoff theory, the functions will depend strongly (nonlogarithmically) on the momenta near the upper limit, and a more refined mathematical technique will be required. Nevertheless, in this case one may also present some general considerations which indicate that the renormalized charge should vanish.

For simplicity consider the case of electrodynamics for which at $e_0^2 \ll 1$ we have[1] $\alpha=\beta=1$,

$$d(k)=[1+(e_0^2/3\pi)\ln(\Lambda^2/-k^2)]^{-1} \approx (3\pi/e_0^2)[\ln(\Lambda^2/-k^2)]^{-1}.$$

The latter equality refers to the case when $(e_0^2/3\pi)\times\ln(\Lambda^2/-k^2)\gg 1$. Inverse proportionality between the function $D=k^{-2}d(k)$ and e_0^2 indicates that in the Lagrangian of the system the part belonging to the free fields can be neglected. [It is not difficult to see that in this case the average of $T(A_\mu(x)A_\nu(y))$ over the physical vacuum, which defines the D function, is inversely proportional to e_0^2.] If, however, the free-field Lagrangian does not play any significant role, at $e_0^2\ll 1$ it is natural to assume that with increasing e_0^2 its role will be even smaller. Therefore $[d(k)]^{-1}$ must also be proportional to e_0^2 for $e_0^2>1$, i.e., $e_0^2 d(k)$ is independent of e_0^2 and always has the form $e_0^2 d(k)\approx 3\pi[\ln(\Lambda^2-k^2)]^{-1}$ (if only $-k^2$ does not approach Λ^2 too closely). Hence for e_c^2 we obtain

$$e_c^2=3\pi[\ln(\Lambda^2/m^2)]^{-1}\to 0,$$

just as in the case when $e_0^2<1$.

It should be noted also that for two cutoffs one may consider firmly proved the validity of the expansion of the various quantities in series of type (1) in powers of an arbitrarily small quantity $\bar{g}_0^2$. Thus, for Γ_5 or g_c^2 we get series of the type

$$\Gamma_5-\gamma_5=\sum_{n=1}^{\infty}(\bar{g}_0^2)^n B_n\left(\bar{g}_0^2\ln\frac{\Lambda_k^2}{-p^2}\right),$$

$$g_c^2=\sum_{n=1}^{\infty}(\bar{g}_0^2)^n G_n\left(\bar{g}_0^2\ln\frac{\Lambda_k^2}{m^2}\right),$$

where, for instance, $G_1(x)=[1+(5x/4\pi)]^{-1}$, etc.

Although these series are apparently asymptotic[28] and not convergent, their sum, nevertheless (as for any asymptotic series) can be approximated, with any degree of accuracy, by the first terms if the latter decrease sufficiently rapidly with increasing n. This was directly demonstrated above for Γ_5. A similar state of affairs also holds for the expansions of other quantities. Therefore, for $\bar{g}_0^2\to 0$ the expansion of the various quantities in series of type (1) is permissible, and does not in any degree undermine the rigorousness of the proof.

Evidently, the vanishing of the renormalized charge is a general difficulty which appears in any theory with point interaction. This difficulty is encountered in electrodynamics, is pseudoscalar and pseudovector meson theories, in meson theories with mixtures of various interactions, in theories with mesons and nucleons of various types which mutually transform into each other without restriction when the interaction has the form $\sum_{\alpha,\beta,\gamma}\bar{\psi}_\alpha\psi_\beta\varphi_\gamma$, or with certain restrictions,[29] and finally, even in the case of a meson field which interacts with itself and possesses an interaction energy of the form $\lambda_0\varphi^4$. The relation between the renormalized constant λ_c and λ_0 in the latter case can be directly determined from (62) and (60), since according to (62)

$$\lambda_c=(g_0^2/4\pi)P(b_0,x_0)d^2(L).$$

[27] This does not signify that the "smearing" function θ can be introduced in an arbitrary way, the only restriction being that $\theta_{\Lambda_k}\to 1$ as $\Lambda_k\to\infty$. Since the unrenormalized theory involves logarithmically and even quadratically divergent integrals, it is important that θ_{Λ_k} approach unity sufficiently rapidly with decreasing momenta. Otherwise some definite general conditions will be violated (e.g., the conditions of the Lehmann theory) which chould be satisfied in any physically reasonable theory. The results obtained in this case will not have any physical sense.

[28] C. A. Hurst, Cambridge Phil. Soc. **48**, 625 (1952); A. Peterman, Helv. Phys. Acta, **26**, 291 (1953).

[29] Theories with mixtures of various interactions and various mesons were studied from this viewpoint by A. D. Galanin, Soviet Phys. JETP (to be published).

This equation, in the limit $g_0^2 \to 0$ (which means that free nucleons and mesons directly interacting with each other are being considered) yields

$$\lambda_c = \frac{\lambda_0}{1+(11/2)\lambda_0 \ln(\Lambda_k^2/m^2)} \quad (82)$$

for the symmetric theory, and

$$\lambda_c = \frac{\lambda_0}{1+\frac{3}{2}\lambda_0 \ln(\Lambda_k^2/m^2)} \quad (83)$$

for neutral theory.

In these formulas λ_0 can only be a positive quantity, as otherwise arbitrarily large values of φ will correspond to the minimum of the energy

$$\frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x_\nu}\right)^2 - \mu^2\varphi^2\right] + \frac{\lambda_0}{4!}\varphi^4(x)$$

and this is obviously absurd.

From formulas (82)–(83) it follows that for point interaction (for $\Lambda_k \to \infty$) the renormalized constant λ_c will also vanish.

The fact that the renormalized charge is zero seems to indicate the existence of a certain operator which, in the case of point interaction, transforms the energy operator $\mathcal{H}$ to the form

$$\bar{\mathcal{H}} = U^{-1}\mathcal{H}U$$

in which it merely represents the sum of free-field energies and the interaction operator disappears. Thus the failure of modern theory lies in the fact that in the limiting case of point interaction it directly leads to the disappearance of any type of physical interaction.

In conclusion, we shall consider the possibility of an experimental proof of inconsistency of the theory. Evidently the important types of experiments will be those in which weakly interacting particles (electrons, photons, and possibly μ-mesons) are involved, as present theory yields quantitative results only for these particles. If the inconsistency of the theory is due to a change in the usual properties of space at small distances $\sim 1/\Lambda$, one should expect that departure of experiment from theory will be observed if the essential momenta are of the order of or exceed $1/\Lambda$. Moreover, the characteristic length should be the same for electrodynamics as for meson theories. Equation (76) shows that Λ^2/m^2 cannot be very large, as then the physical meson charge constant g_c^2 would be small, and this is contrary to experiment. Therefore Λ in order of magnitude equals m, where $1/m = \hbar/M_p c \sim 10^{-14}$ cm. This length corresponds to electrons with an energy of 400–1000 Mev. Thus if the electron energy in electron-electron scattering experiments in the center-of-mass system is of the order indicated above (i.e., 10^4—10^6 Mev in the laboratory system), deviations from theory (from the Møller formula) should be expected. For this reason it would be highly interesting to carry out precise measurements of the cross sections for the Compton effect, Møller scattering, pair production, etc., at these energies.

If μ mesons are also weakly interacting particles (i.e., if the anomalous μ meson scattering described in the literature[30] does not really exist), μ-meson experiments may also be very important. The most promising experiments seem to be those on meson pair creation by γ quanta of same Bev energy, and also experiments on high-energy δ electrons formed by μ particles.

Discussions with L. D. Landau were of great aid to us in obtaining some of the results presented in this paper, and the authors express their sincere thanks to him. We also take the opportunity to express our appreciation to J. I. Diatlov, V. Berestetsky, A. Galanin, and B. Joffe for stimulating discussions.

APPENDIX: LIMITING FORM OF SOLUTION OF EQUATION (41) IN SYMMETRIC THEORY

We shall consider here equations which are similar to (42) and (43) but for the case of symmetric theory. If the momenta k_1 and k_2 are very large so that the largest of them, k, is much greater than k_1+k_2 (i.e., $\eta>\xi$, $\eta=\ln(-k^2/m^2)$, $\xi=\ln[-(k_1+k_2)^2/m^2]$), we seek the solution of (41) in the form:

$$F(k_1,k_2;k_3,k_4) = \Phi(\eta,\xi)\delta_s + \Phi_1(\eta,\xi)\delta_{a_1a_2}\delta_{a_3a_4} \quad \text{(a)}$$

Putting here $\eta=\xi$, $\Phi(\eta,\eta)=F(\eta)$, $\Phi_1(\eta,\eta)=F_1(\eta)$, we then obtain

$$F(k_1,k_2;k_3,k_4)\,|_{\eta=\xi} = F(\eta)\delta_s + F_1(\eta)\delta_{a_1a_2}\delta_{a_3a_4}.$$

According to (40), for the unknown sum

$$P_{a_1a_2a_3a_4}(k_1,k_2,k_3,k_4)$$

of the contributions of all contractible diagrams we have

$$\begin{aligned} P_{a_1a_2a_3a_4}(\eta,\xi) &= P(\eta,\xi)\delta_s + P_1(\eta,\xi)\delta_{a_1a_2}\delta_{a_3a_4}, \\ P_{a_1a_2a_3a_4}(\eta,\eta) &= P(\eta)\delta_s + P_1(\eta)\delta_{a_1a_2}\delta_{a_3a_4}, \end{aligned} \quad \text{(b)}$$

where

$$\begin{aligned} P(\eta,\xi) &= \rho_0(\eta) + 2F(\eta) + \Phi(\eta,\xi), \\ P_1(\eta,\xi) &= \Phi_1(\eta,\xi) - F_1(\eta) \end{aligned} \quad \text{(c)}$$

and

$$\begin{aligned} P\ (\eta) &= P(\eta,\eta) = \rho_0(\eta) + 3F(\eta), \\ P_1(\eta) &= P_1(\eta,\eta) \equiv 0. \end{aligned} \quad \text{(d)}$$

Inserting (a) in (41), considering separately in (41) the integration regions $\xi \leqslant z \leqslant \eta$ and[31] $\eta \leqslant z \leqslant L_k$, and

[30] J. L. Lloyd, and A. W. Wolfendale, Proc. Roy. Soc. (London) **A68**, 1045 (1955); A. I. Alikhanov and G. P. Yeliseev, Izvest. Akad. Nauk U.S.S.R., Ser. fiz. **19**, 732 (1956), and other investigations.

[31] Taking account of the fact that in the first region R_0, $F(k_1,l;k_2,l')$ and $F(k_1,l';k_2,l)$ depend on η, and in the second on z.

equating the coefficients of δ_s and $\delta_{a_1a_2}\delta_{a_3a_4}$, we get

$$\Phi(x,y)=-\tfrac{1}{3}[\rho_0(x)+2F(x)+F_1(x)]\times\int_x^y[\rho_0(\tau)+2F(\tau)+F_1(\tau)+\Phi(\tau,y)]\frac{d\tau}{\tau^2}-\tfrac{1}{3}\int_1^x[\rho_0(\tau)+2F(\tau)+F_1(\tau)]\times[\rho_0(\tau)+2F(\tau)+F_1(\tau)+\Phi(\tau,y)]\frac{d\tau}{\tau^2},\tag{e}$$

$$\Phi_1(x,y)=-\tfrac{1}{3}\int_x^y\{(\tfrac{5}{2})[\rho_0(x)+2F(x)]\times[\rho_0(\tau)+2F(\tau)+\Phi(\tau,y)+\Phi_1(\tau,y)]+F_1(x)[\phi_1(\tau,y)-F_1(\tau)]\}\frac{d\tau}{\tau^2}-\tfrac{1}{3}\int_1^x\{(\tfrac{5}{2})[\rho_0(\tau)+2F(\tau)]\times[\rho_0(\tau)+2F(\tau)+\Phi(\tau,y)+\Phi_1(\tau,y)]+F_1(\tau)[\phi_1(\tau,y)-F_2(\tau)]\}\frac{d\tau}{\tau^2},$$

where η, ξ and z have been replaced by the more convenient variables x, y, and τ, respectively: $x=\bar{Q}^{3/5}(\eta)$, $y=\bar{Q}^{3/5}(\xi)$ and $\tau=\bar{Q}^{3/5}(z)$, and $(g_0^2/4\pi)\bar{d}^2(z)dz=-\tfrac{1}{3}d\tau/\tau^2$. Putting in (c) $x=y$, we obtain two additional equations,

$$F(x)=-\tfrac{1}{3}\int_1^x[\rho_0(\tau)+2F(\tau)+F_1(\tau)]\times[\rho_0(\tau)+2F(\tau)+F_1(\tau)+\Phi(\tau,x)]\frac{d\tau}{\tau^2},\tag{f}$$

$$F_1(x)=-\tfrac{1}{3}\int_1^x\{(5/2)[\rho_0(\tau)+2F(\tau)]\times[\rho_0(\tau)+2F(\tau)+\phi(\tau,x)+\phi_1(\tau,x)]+F_1(\tau)[\phi_1(\tau_1x)-F_1(\tau)]\}\frac{d\tau}{\tau^2},$$

which are equivalent to Eq. (42) of the neutral theory. It should be remarked that Eqs. (53) and (54) obtained directly for $P(x)=\rho_0(x)+3F(x)$ are equivalent to the set of Eqs. (e) and (f) or (42) and (43); i.e., they can apparently be obtained from the latter as their mathematical corollary. However, we were unable to accomplish this.

For simplicity consider the case when $\rho_0(x)$ is defined by formula (52) for $b_0=0$, i.e., $\rho_0(x)=(16/3)(x-1)$. We now prove that in the limiting cases $x-1\ll1$, $y-1\ll1$ (i.e., $\bar{g}_0^2(L-\eta)\ll1$, $\bar{g}_0^2(L-\zeta)\ll1$) and $x>1$, $y>1$ [i.e., $\bar{g}_0^2(L-\eta)\gg1$, $\bar{g}_0^2(L-\zeta)\gg1$], the solution of the set of equations (e) and (f) can be found easily.

The case $x-1\ll1$ and $y-1\ll1$ is equivalent to the usual perturbation theory. The system (e) and (f) can then easily be solved by an iteration procedure, if in the zero approximation $F=F_1=\Phi=\Phi_1=0$ is substituted in the right hand side of these equations. We obtain

$$F(x)=-(1/3)\int_1^x\rho_0^2(\tau)d\tau=-(1/9)(16/3)^2(x-1)^3,$$

$$F_1(x)\approx-(5/18)(16/3)^2(x-1)^3,$$

$$\Phi(x,y)-F(x)=-(1/6)(16/3)^2(x-1)[(y-1)^2-(x-1)^2],$$

$$\Phi_1(x,y)-F_1(x)=-(5/12)(16/3)^2(x-1)[(y-1)^2-(x-1)^2].$$

If these expressions are substituted in the right-hand sides of (e) and (f), one may determine all the functions as series in $(x-1)$ and $(y-1)$.

If $x\gg1$ and $y\gg1$ in (e) and (f), then large values of the integration variable τ are important ($\tau\gg1$). Therefore, neglecting unity compared with τ (or x or y), we insert $\rho_0(\tau)=(16/3)\tau$ in (e) and (f), and we replace, in the second term of (e) and (f), the lower integration limit (unity) by zero. It will now be shown that (e) and (f) have solutions of the form

$$F(x)=Ax,\quad F_1(x)=A_1x,\tag{g}$$

where the values of the constants A and A_1 can be determined by solving (e) and (f), i.e., by substituting (g) in (e) and (f) and differentiating (e) twice with respect to x,

$$\frac{\partial^2\Phi(x,y)}{\partial x^2}=\frac{B}{3x^2}[Bx+\Phi(x,y)],$$

$$\frac{\partial^2\Phi_1(x,y)}{\partial x^2}=\frac{1}{3x^2}\{(5/2)(B-A_1)[(B-A_1)x+\Phi(x,y)+\Phi_1(x,y)]+A_1[\Phi_1(x,y)-A_1x]\},$$

where $B=(16/3)+2A+A_1$. This differential equation should be solved with the boundary conditions

$$\Phi(0,y)=\Phi_1(0,y)=0,$$

$$\left.\frac{\partial\Phi(x,y)}{\partial x}\right|_{x=y}=\left.\frac{\partial\Phi_1(x,y)}{\partial x}\right|_{x=y}=0,$$

which follow from (e) if the expressions (g) for the functions F and F_1 are taken into account.

It is not difficult to see that the solutions are

$$\Phi=-Bx+\frac{B}{\mu}x^{\mu}y^{1-\mu},$$

$$\Phi_1=A_1x-\frac{5B}{3\mu}x^{\mu}y^{1+\mu}+\frac{1}{\nu}[(5/3)B-A_1]x^{\nu}y^{1-\nu},$$

where $\mu>0$ and $\nu>0$ are positive roots of the equations

$$\mu(\mu-1)=\tfrac{1}{3}B,\quad \nu(\nu-1)=\tfrac{1}{3}B+\tfrac{1}{2}(B-A). \qquad \text{(h)}$$

From the condition $\Phi(x,x)=F(x)=Ax$, $\Phi_1(x)=A_1x$, we find

$$A=[(1-\mu)/\mu]B,\quad A_1=-(5/3)[(\nu/\mu)-1]B. \qquad \text{(i)}$$

The last equation together with (h) yields

$$\nu=1+(5/6)(B/\mu),$$

and therefore

$$A_1=-(5/3)[(1/\mu)-1]B-(25/18)(B^2/\mu^2). \qquad \text{(j)}$$

Inserting in the definition of B,

$$B=(16/3)+2A+A_1,$$

the expressions (i) and (j) for the constants A and A_1, we get the following equation for B:

$$B^2+\tfrac{2}{3}\mu^2B-(32/9)\mu^2=0,$$

which should be solved together with Eq. (h) for μ.

The only solution which gives positive values of μ and ν is

$$B=(16/3)(7/11)^2,\quad \mu=49/33,\quad \nu=73/33.$$

According to (i) and (j), we then get

$$A=-(16/3)(16/121),$$
$$B=-(16/3)(40/121).$$

In accord with (g), (c), and (d), we obtain

$$P(x,y)=\begin{cases}(16/3)(x-1)\{1-(88/27)(x-1)^2\\ \quad -(8/3)[(y-1)^2-(x-1)^2]+\cdots\}, & x-1<1,\quad y-1<1,\\ (16/3)x\{1+(5/3)(x^{40/33}y^{-40/33}\\ \quad -x^{16/33}y^{-16/33})+\cdots\}, & x>1,\quad y>1,\end{cases}$$

$$P_1(x,y)=\begin{cases}-(5/12)(16/3)^2(x-1)\\ \quad \times\{[(y-1)^2-(x-1)^2]+\cdots\}, & x-1<1,\quad y-1<1,\\ (16/11)x(x^{16/3}y^{-16/3}-1), & x>1,\quad y>1,\end{cases}$$

and

$$P(x)=\begin{cases}(16/3)(x-1)[1-(88/27)(x-1)^2+\cdots], & x-1<1,\\ (16/3)x,\quad x>1.\end{cases} \qquad \text{(k)}$$

It is easy to verify that for $b_0=0$, $B=1$, formula (55) for $P(x)$ yields in the cases $x-1<1$ and $x>1$ the same limiting values as those in (k). However, we obtained here the expression for $P_{a_1a_2a_3a_4}$ not only for $\eta=\zeta$ but also for the case when $\eta>\zeta$.