formation law is real, whereas in de Broglie waves it is imaginary.

It appears likely that λ-type transformations will be encountered in any geometries with affine connections whenever they are not excluded by the type of symmetry condition characteristic of Christoffel symbols, and further, that λ invariance is likely to be intimately related to ordinary gauge invariance in any theory that purports to contain references to the electromagnetic field.

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Vacuum Polarization and Proton-Proton Scattering in the ^3P State*

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The contribution of the vacuum polarization potential to the ^3P phase shift in proton-proton scattering has been calculated and employed to correct observed phase shifts to obtain the phase shifts associated with the specifically nuclear ^3P potential between two protons. In the energy range from 2 to 5 Mev, about half the observed phase shift can be attributed to vacuum polarization. From the results it is shown that if the potential between the two protons in both the ^1S and ^3P states is represented by a square well of range 2.6×10^{-13} cm, then the ^3P potential is opposite in sign (repulsive) and approximately 13% in magnitude relative to the ^1S potential.

IN two previous papers[1,2] the effect of vacuum polarization on proton-proton scattering in the ^1S state at low energies was examined. On the basis of experimental data, some evidence for the reality of vacuum polarization effects was obtained, and by then introducing a correction for these effects, new values of the constants characterizing the specifically nuclear interaction between two protons in the ^1S state were derived. The changes in these constants—the zero-energy scattering length and the effective range—were small but not negligible.

It was noted in the second paper cited that the correction for vacuum polarization would be relatively much more important in deriving the properties of the specifically nuclear interaction between two protons in the ^3P state for the following two reasons: the vacuum polarization potential has a relatively long range, and the nuclear interaction in this state is much weaker than in the ^1S state. Calculations have now been performed of the contribution of the vacuum polarization potential to the ^3P state phase shift in an appropriate approximation; the results and their implications are discussed in the present paper. In brief summary, the present calculations indicate that approximately one-half of the experimentally observed ^3P phase shift in the energy region from 1 to 5 Mev arises from the vacuum polarization potential, so that the correction for this effect is very important. After correction, the ^3P nuclear interaction is found to be approximately 13% as strong as the ^1S nuclear interaction. The rather anomalous behavior of the ^3P phase shift at higher energies as observed in several measurements is not explained by the vacuum polarization effect, however.

CALCULATION OF THE VACUUM POLARIZATION EFFECT

We shall ignore for the present the possibility of a tensor nuclear interaction between two protons and assume that the nuclear interaction, like the Coulomb and the vacuum polarization potentials, is purely central. One can then easily show that the P-wave phase shift δ_1 is given by the following formula:

$$\sin\delta_1=-\frac{M}{\hbar^2 k}\int_0^\infty Vuv\,dr. \qquad (1)$$

Here M is the proton mass, $k=(ME_L/2\hbar^2)^{\frac{1}{2}}$, where E_L is the energy in the laboratory system, V is the sum of the specifically nuclear potential V_n and the vacuum polarization potential V_{vp}, but not including the Coulomb potential e^2/r, v is the radial P-wave function in the presence of the potential V, while u is the radial P-wave function in the absence of this potential (that is, for a pure Coulomb potential), both u and v being normalized to unit amplitude for large r. Since both the

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[1] L. L. Foldy and E. Eriksen, Phys. Rev. **95**, 1048 (1954).

[2] L. L. Foldy and E. Eriksen, Phys. Rev. **98**, 775 (1955).

specifically nuclear potential and the vacuum polarization potential are weak in the ^3P state, we may approximate u by v in the integrand of (1), and since δ_1 is small, write

$$\delta_1=-\frac{M}{\hbar^2 k}\int_0^\infty Vu^2dr. \tag{2}$$

This result corresponds to treating V, but not the Coulomb potential, in Born approximation, and u is then nothing more than the regular Coulomb function for $l=1$ normalized to unit amplitude for large r. In this approximation, which is presumably adequate for our purposes, the phase shift δ_1 is the sum of the phase shifts $\delta_1{}^n$ due to the specifically nuclear potential and $\delta_1{}^{vp}$ due to the vacuum polarization potential:

$$\delta_1=\delta_1{}^n+\delta_1{}^{vp}, \tag{3}$$

$$\delta_1{}^n=-\frac{M}{\hbar^2 k}\int_0^\infty V_n u^2dr, \tag{4}$$

$$\delta_1{}^{vp}=-\frac{M}{\hbar^2 k}\int_0^\infty V_{vp} u^2dr. \tag{5}$$

Hence by subtracting from the values of δ_1 obtained from the analysis of scattering experiments, the computed values of $\delta_1{}^{vp}$, one obtains values of $\delta_1{}^n$ from which information about the nuclear potential V_n can be directly obtained.

Thus, the problem of vacuum polarization effects on P-wave proton-proton scattering is reduced to the evaluation of the integral (5) with u the regular Coulomb function and V_{vp} given by

$$V_{vp}(r)=\frac{2\alpha e^2}{3\pi r}\int_1^\infty e^{-2\kappa\xi r}\left(1+\frac{1}{2\xi^2}\right)\frac{(\xi^2-1)^{\frac{1}{2}}}{\xi^2}d\xi, \tag{6}$$

with α the fine-structure constant, κ the reciprocal Compton wavelength of the electron, and e the protonic charge.

A first approximation to $\delta_1{}^{vp}$ can be obtained by treating the Coulomb potential in Born approximation as well, in which case one can insert for u in (5) the radial P function for a free particle:

$$u=(\pi kr/2)^{\frac{1}{2}}J_{\frac{3}{2}}(kr). \tag{7}$$

While this approximation might be expected to be crude, the results when compared to more precise calculations (described below) turn out to be remarkably good. At the lowest energy considered (1 Mev) the error in using (7) is less than 4% and decreases at higher energies. On substituting (7) and (6) into (5) and integrating[3] on r, one obtains

$$\delta_1{}^{vp}=-\frac{\alpha e^2 M}{3\pi\hbar^2 k}\int_1^\infty\left(1+\frac{1}{2\xi^2}\right)\times\frac{(\xi^2-1)^{\frac{1}{2}}}{\xi^2}Q_1\left(1+\frac{2\kappa^2\xi^2}{k^2}\right)d\xi, \tag{8}$$

where Q_1 (irregular Legendre function) is given by

$$Q_1(z)=\frac{z}{2}\ln\left(\frac{z+1}{z-1}\right)-1. \tag{9}$$

The integral (8) can be easily evaluated by numerical integration after change of the variable of integration to $s=1/\xi$. However, one of the authors (E.E.) was able to evaluate an equivalent integral exactly with the following result[4]:

$$\delta_1{}^{vp}=-\frac{\alpha\eta}{3\pi}\left\{\frac{38}{9}+\frac{5}{3}\eta^2+\frac{1}{6}(1+\beta^2)^{\frac{1}{2}}(11+5\beta^2)\times\ln\left(\frac{(1+\beta^2)^{\frac{1}{2}}-1}{(1+\beta^2)^{\frac{1}{2}}+1}\right)+\frac{1}{4}\left[\ln\left(\frac{(1+\beta^2)^{\frac{1}{2}}-1}{(1+\beta^2)^{\frac{1}{2}}+1}\right)\right]^2\right\}, \tag{10}$$

where

$$\eta=Me^2/2k\hbar^2,$$
$$\beta=k/\kappa=0.14927\eta.$$

For $\beta\ll 1$, which is valid in the energy range of principal interest ($E>1$ Mev, $\eta<0.16$), (10) may be expanded to yield the formula

$$\delta_1{}^{vp}\simeq-\frac{\alpha\eta}{3\pi}[1.4415-0.1568\eta^2-1.5237\ln\eta+0.0668\eta^2\ln\eta+(\ln\eta)^2]. \tag{11}$$

If one does not make the approximation (7) but inserts the exact regular Coulomb function for u, the problem becomes more complicated. One of the authors (E.E.) obtained an analytic approximation to the integral in this case[4]:

$$\delta_1{}^{vp}\simeq-\frac{\alpha\eta}{3\pi}[1.4415-2.02\eta+1.150\eta^2+0.264\eta^3-1.5237\ln\eta-0.2325\eta^2\ln\eta+(\ln\eta)^2], \tag{12}$$

which is correct to a small fraction of a percent for the range of η under consideration. A check of this formula was made by a direct numerical integration of (5) using tabulated values for the Coulomb function at one energy. As mentioned earlier, for energies above 1 Mev the

[3] G. N. Watson, *A Treatise on the Theory of Bessel Functicns* (Cambridge University Press, Cambridge, 1944), p. 389.

[4] The derivation of formulas (10) and (12) will be published elsewhere by E. Eriksen.

difference between (10) and (12) amounts to only a few percent.

Numerical calculations of the phase shift δ_1^{vp} on the basis of the above formulas indicated that the results over the energy range from 0.5 to 20 Mev could be represented to an accuracy of a few percent by the simple formula

$$\delta_1^{vp}\,(\text{deg}) = -0.0520 + 0.0190 \log_{10} E_L\,(\text{Mev}). \quad (13)$$

ANALYSIS OF EXPERIMENTAL DATA

The available experimental data on P-wave phase shifts which is of sufficient accuracy to obtain information about the ^3P potential consists of results[5] primarily in the energy range from 1.8 to 4.2 Mev, plus a few values at higher energies.[6] The present discussion will be restricted to the phase shifts obtained by Hall and Powell[7] from the analysis of the proton-proton scattering data of Worthington, McGruer, and Findley.[5] These are summarized in Table I. Assuming the validity of the Born approximation and the consequent additivity of nuclear and vacuum polarization contributions to the phase shift, one obtains the nuclear phase shifts δ_1^n given in the same table. It will be noted that approximately half the observed phase shift arises from vacuum polarization. This fact largely invalidates previous analyses[7,8] of this data to obtain information about the specifically nuclear ^3P potential between two protons and indicates that this potential is only about half as strong as previously derived.

An approximate comparison of the ^3P and ^1S nuclear potentials for the proton-proton system can be made by assuming the same potential shape and range for a central interaction in the two situations, and comparing the potential depths obtained by fitting to observations of the respective phase shifts. Assuming again the validity of the Born approximation, one has for the ^3P nuclear phase shift the integral (4). The Coulomb function u can be approximated for small r by the leading term in its expansion in powers of r:

$$u \simeq C_1 (kr)^2, \quad (14)$$

with

$$C_1 = (1+\eta^2)^{\frac{1}{2}} C_0/3 = [2\pi\eta(1+\eta^2)/(e^{2\pi\eta}-1)]^{\frac{1}{2}}/3. \quad (15)$$

Substituting into (4) and taking for V_n a square well of range a and depth V_0, one finds

$$\delta_1^n = -Mk^3 C_1^2 V_0 a^5 / 5\hbar^2. \quad (16)$$

Over the energy range of these experiments (1.8 Mev $< E_L <$ 4.2 Mev, $0.077 < \eta < 0.12$), one finds that C_1^2 varies only between 0.075 and 0.086. Hence δ_1^n is approximately proportional to k^3 or $E_L^{\frac{3}{2}}$. From the values of δ_1^n given in Table I, one finds the weighted mean of $\delta_1^n/E_L^{\frac{3}{2}}$ to be $0.0066°/(\text{Mev})^{\frac{3}{2}}$, and assuming $C_1^2 = 0.08$ this yields

$$V_0 a^5 = 2.1 \times 10^{-63}\ \text{Mev-cm}^5. \quad (17)$$

Taking $a = 2.6 \times 10^{-13}$ cm to correspond to the range observed in the ^1S state, one obtains

$$V_0 = 1.8\ \text{Mev}, \quad (18)$$

to be compared to the value -13.5 Mev for the potential well depth V_0 in the ^1S state. The uncertainty in (18) arises principally from the uncertainty in the experimental values and is of the order of $\pm 50\%$. Thus the ^3P potential is opposite in sign (repulsive) to the ^1S potential and only about 13% of the ^1S potential in magnitude.

The above result is to be interpreted only semiquantitatively since there is probably a substantial tensor contribution to the ^3P potential. The observed ^3P phase shift appears to undergo a change in sign at energies above 10 Mev. This apparently anomalous result, if taken seriously, can be interpreted as indicating that the ^3P potential is essentially repulsive at larger distances and attractive at short distances.[8] Whatever the significance of this behavior, it is clear that vacuum polarization is in no way responsible for it unless the observed low-energy phase shifts are substantially in error.

TABLE I. Scattering phase shifts for protons in the ^3P state.

Lab energy (Mev)	δ_1 (observed)[a]	δ_1^{vp} (calculated)	δ_1^n
1.855	$-0.049° \pm 0.020°$	$-0.047°$	$-0.002° \pm 0.020°$
1.858	$-0.057° \pm 0.024°$	$-0.047°$	$-0.010° \pm 0.024°$
2.425	$-0.075° \pm 0.018°$	$-0.045°$	$-0.030° \pm 0.018°$
3.037	$-0.082° \pm 0.022°$	$-0.043°$	$-0.039° \pm 0.022°$
3.527	$-0.094° \pm 0.023°$	$-0.042°$	$-0.052° \pm 0.023°$
3.899	$-0.109° \pm 0.020°$	$-0.041°$	$-0.068° \pm 0.020°$
4.203	$-0.074° \pm 0.023°$	$-0.040°$	$-0.034° \pm 0.023°$

[a] See reference 7.

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[5] Worthington, McGruer, and Findley, Phys. Rev. **90**, 889 (1953).
[6] Kerman, Kreger, and Kruger, Phys. Rev. **89**, 908 (1953); E. J. Zimmerman and P. G. Kruger, Phys. Rev. **83**, 218 (1951); B. Cork and W. Hartsough, Phys. Rev. **94**, 1300 (1954); J. Rouvina, Phys. Rev. **81**, 593 (1951); J. L. Yntema and M. G. White, Phys. Rev. **95**, 1226 (1954).
[7] H. H. Hall and J. L. Powell, Phys. Rev. **90**, 912 (1953).
[8] A. Keller, Proc. Phys. Soc. (London) **A68**, 930 (1955).