Nucleon Tunneling in $N^{14}+N^{14}$ Reactions*

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The tunneling process is considered as a partial explanation of observations by Reynolds and Zucker on the reactions $N^{14}(N^{14},N^{13})N^{15}$ and $N^{14}(N^{14},C^{13})O^{15}$. The theory of reactions of this type is first formulated in reasonably general terms, employing a classical mechanics treatment of the motion of the colliding nuclei and of the reaction products. The starting point is a set of adiabatic (fixed centers of mass of heavy aggregates) wave functions, which is transformed to a related system of functions insuring a better initial convergence of the iteration procedure. The anisotropy of transfer inherent in the p-nucleon eigenfunctions is taken into account but finally eliminated by the consideration of extreme j-j coupling shell structure eigenfunction assignments, as well as the spin and statistics of the nuclei. The effect of antisymmetrizing the wave function with respect to neutrons and protons in the incomplete shells is worked out. The relation of the angular distribution problem presented by the case of dominant Coulomb effects to a popular form of stripping theory is discussed, with the conclusion that the classical dynamics approximation to the motion of the colliding nuclei is adequate. The dependence of the total cross section on energy and of the differential cross section on angle, when compared with experiment, indicates the presence of another reaction mechanism participating especially at the lower energies.

I. INTRODUCTION

NUCLEAR transfer reactions have been observed by Reynolds and Zucker,[1] Walker, Chackett, and Fremlin,[2] and by Farragi.[3] The first of the three papers just quoted contains an account of quantitative measurements on the reactions $N^{14}(N^{14},N^{13})N^{15}$ and $N^{14}(N^{14},C^{13})O^{15}$. In the other work referred to above, the general nature of the reactions is ascertained or made probable, but the larger energy spread of the incident beam makes comparison with theoretical prediction more difficult. In the present paper special attention is given, therefore, to the two reactions resulting from the bombardment of N^{14} by N^{14}.

It has been pointed out by Breit, Hull, and Gluckstern[4] that the transfer of a nucleon in reactions of this type should give information regarding the nucleon density at the nuclear surface, and the process consisting of the tunneling of nucleons through the region of negative kinetic energy beyond the nuclear surface should be amenable to reasonably precise calculation. The work reported on below has been carried out in order to see to what degree the tunneling process represents the whole situation. The combined evidence of the observations of the variation of the total cross section with energy and of the character of the angular distribution at different energies will be seen to leave little doubt about the insufficiency of the tunneling mechanism as an explanation of the whole phenomenon. The reaction yield does not depend on energy nearly as critically as the tunneling picture suggests, nor does the angular distribution show the expected sharp dip at 90° scattering angle. It has to be concluded therefore that another process is taking place in addition to the leakage of neutrons or protons through the regions of negative kinetic energy. The possibility that virtual state formation is responsible for the discrepancy will be discussed in another publication.[5]

The transfer of a nucleon between the two nuclei is treated in the semiclassical approximation[6] which has been successful in giving an approximate account of Coulomb excitation.[7] The validity of the approximation is understandable as a result of the relatively large mass of the N^{14} nuclei which makes their wavelength short in comparison with the distance of closest approach. The work on Coulomb excitation shows furthermore that the agreement between classical and quantum-mechanical integrals is very much better than simple considerations on the basis of the JWKB method would lead one to expect. There are in fact compensations[8] in the distribution of quantum-mechanical and classical densities which, when combined with the relative smallness of the distance penetrated in the region of negative kinetic energy as compared with the distance of closest approach, contribute to the generality of this phenomenon. This expectation is corroborated by the presence of analytically established connections[9] between the integrals entering the theory of Coulomb excitation for cases of large angular momenta, as well as by the direct numerical comparison[10] which has partly been

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[7] G. Breit and P. B. Daitch, Phys. Rev. **96**, 1447 (1954); Daitch, Lazarus, Hull, Benedict, and Breit, Phys. Rev. **96**, 1449 (1954); L. C. Biedenharn and C. M. Class, Phys. Rev. **98**, 691 (1955).

[8] G. Breit and P. B. Daitch, Proc. Natl. Acad. Sci. **41**, 653 (1955); J. P. Lazarus and S. Sack, Phys. Rev. **100**, 370 (1955).

[9] Gluckstern, Lazarus, and Breit, Phys. Rev. **101**, 175 (1956).

[10] Benedict, Daitch, and Breit, Phys. Rev. **101**, 171 (1956). See also references 6 and 7.

already mentioned. In the Coulomb excitation problem the principal error of the SCT (semiclassical treatment) lies in the employment of classical rather than quantum values of weighting factors, the errors in the radial integrals being of the order of one percent in typical cases. There appears to be no essential reason for expecting the present problem to differ from the case of Coulomb excitation in this respect, since the values of the parameters are no less favorable.

The transfer reactions in $N^{14}+N^{14}$ are characterized by the special circumstance of nearly equal binding energies of the last neutron in N^{14} and N^{15} and of the last proton in N^{14} and O^{15}. The differences in these binding energies are only 0.3 Mev. The treatment of the transfer has to be adapted therefore to the calculation of coupling effects which are peculiar to such cases. The calculations have been made by the following methods: (a) potential-well models for binding the nucleons, employing the method of a previous paper,[11] with identical and nonidentical potential wells, (b) a treatment involving no specializations regarding the character of the forces to which the nucleon is subjected inside the nucleus, but relying instead on the employment of information regarding the radial logarithmic derivative of the wave function at the nuclear surface. The two treatments give nearly the same energy dependence. Treatment (b) will be presented first. Numerical results based on a potential-well model treatment will then be given.

In Sec. II adiabatic functions suitable for the present and allied problems are introduced. Two transfer functions called h_+, h_- are defined for neutrons and the transfer equations are worked out in terms of general properties of the wave functions. Quantities P_+, P_- interpretable as potential barrier penetrabilities are defined in this connection. The special circumstance of approximate equality of binding energy of the transferred neutron and the associated resonance phenomenon are then considered. In Sec. III the application of adiabatic functions to the transfer problem is discussed together with the role of anistropy caused by different barrier penetrabilities of the sublevels of degenerate states. The convergence of a calculation based on adiabatic functions is then examined with the conclusion that for the resonance case it is often poor. Derived functions $\bar{u}$, $\bar{v}$ having more pronounced space localization and insuring more rapid convergence of iterations are introduced. The resultant equations describing transfer are interpreted in terms of the energy matrix in the reference system of the localized functions. These could have been set up at the beginning, but the relations of the method to that of adiabatic functions would not have been clear and the possibility of introducing the transfer functions h_+, h_- would have been more difficult to discuss. In the presentation as it stands, the starting point is concerned with stationary states for fixed positions of the centers of mass of the heavy aggregates and the remainder of the discussion involves straightforward transformations. The reduction to the energy-matrix type of equation such as Eq. (11) involves dropping a small term considered in relation to Eq. (10.4). Complications arising from the rotation of the anisotropy axis with respect to the laboratory system are not discussed because the special values of spins and the extreme j-j coupling nucleon configuration assignments make the transfer function isotropic, as shown in Sec. IV.

Here the spin-orbital combinations are considered in detail and so is the effect of the antisymmetry of the wave function in the neutrons of both nuclei. The relative probabilities of the transfer taking place from incident states of total spin 1 and 0 are also worked out here. It is found that the transfer functions are isotropic even though they would have been anisotropic for the transfer of a single p-neutron bound to a spinless core in the initial and final nuclei. The dependence of the barrier penetrability on the distance between the colliding nuclei turns out to be the same as though the neutron were in an s state, as is seen in Eq. (15.7). The role of the identity of the colliding nitrogen nuclei is taken up next and the effect of antisymmetry on wave functions Ψ_i^U unsymmetrized with respect to the valence nucleons which are contained in the two nitrogen nuclei is considered. This is done in relation to Eq. (16.4). The effect is found to be represented by a factor 2 in the cross section and is related to a similar factor 2 arising in the fictitious case of two C^{13} nuclei considered as having, first, nonidentical, and second, identical C^{12} cores. The section ends with taking into account wave function antisymmetry on the general form of the angular distribution function in terms of that for nonidentical particles.

In Sec. V the angular distribution of the reaction products is calculated in an approximate manner. The characteristic differences between the heavy-particle case of the nitrogen reactions and the light-particle case of (d,p) reactions are first considered. It is shown that the forward pattern of usual stripping theory is of minor importance in the heavy-particle case. On the other hand, there appears a maximum in the backward direction in the case of nonidentical particles. The relationship between the matrix element type of calculation employing ingoing and outgoing wave modifications of the incident and final waves to the semiclassical theory is examined, and the effect of the strong Coulomb repulsion between the heavy particles is brought into evidence in connection with Eqs. (17) to (18). After this justification of the employment of the semiclassical approximation, the relationship of the two methods is considered in more detail for the special case of the Coulomb field in relation to Eqs. (18.3) to (18.8) and from a more general standpoint in Eqs. (19) to (19.4). The semiclassical method is adopted from here on, and

[11] G. Breit, Phys. Rev. **102**, 549 (1956).

the angular distribution is worked out as in Eqs. (20) to (23.1).

In Sec. VI the final reductions for comparison with experiment are made, the variation of yield with energy is worked out, and the angular distribution is calculated. The comparison between neutron and proton transfer is made here on an approximate basis. It is concluded that the tunneling process is insufficient as a complete explanation of the observations.

Notation

The following is a list of the more important symbols in approximate order of occurrence.

x, y, z = coordinates of nucleon (neutron or proton) with respect to *left* nucleus.
$r=(x^2+y^2+z^2)^{\frac{1}{2}}$.
R = distance between heavy particles.
S = surface of nucleus.
(x',y',z') = coordinates of nucleon referred to *right* nucleus.
$r'=(x'^2+y'^2+z'^2)^{\frac{1}{2}}$.
h_- and h_+ = radial parts of nucleon wave function.
u = nucleon wave function outside the nuclei ($u=A_+h_+ +A_-h_-$).
$\mathfrak{l}_{1+}=dh_{1+}/h_{1+}dr$; $\mathfrak{l}_{2-}=dh_{2-}/h_{2-}dr$.
$P_+=h_{1+}/h_{2+}$, $P_-=h_{2-}/h_{1-}$.
α_i is defined in relation to $(\Delta-\alpha_i^2)\psi=0$.
$\alpha=\alpha_i$ if $\alpha_1=\alpha_2=\alpha_3$.
$q=\alpha R$.
$\lambda_1=[\partial(\mathfrak{l}_{1-}-\mathfrak{l}_1)/\partial(\alpha^2)]$, $(\alpha=\alpha_{10})$.
$\lambda_2=[\partial(\mathfrak{l}_{2+}-\mathfrak{l}_2)/\partial(\alpha^2)]$, $(\alpha=\alpha_{20})$.
$\beta_1=-(h_{1+}/h_{1-})(\mathfrak{l}_{1+}-\mathfrak{l}_{1-})/\lambda_1$.
$\beta_2=-(h_{2-}/h_{2+})(\mathfrak{l}_{2-}-\mathfrak{l}_{2+})/\lambda_2$.
$\Delta=|[\frac{1}{4}(\alpha_{10}^2-\alpha_{20}^2)+\beta^2]^{\frac{1}{2}}|$.
$a_+=A_+\beta_2^{-\frac{1}{2}}$, $a_-=A_-\beta_1^{-\frac{1}{2}}$.
$\beta=(\beta_1\beta_2)^{\frac{1}{2}}$.
θ is defined by $\tan\theta=a_+/a_-$.
$\chi=(\alpha_{10}^2-\alpha_{20}^2)/(2\beta)$.
$\gamma_1=(\mathfrak{l}_{1-}-\mathfrak{l}_{1+})/\lambda_1$, $\gamma_2=(\mathfrak{l}_{2+}-\mathfrak{l}_{2-})/\lambda_2$.
ψ_u = wave function corresponding to u.
$q, q_1, \cdots q_n, \cdots q_{n+m}, \cdots$ are general coordinates.
ψ_1, ψ_2 = wave functions for isolated nuclei (ψ_1 for N^{14}, ψ_2 for N^{15}).
Ψ = wave function of the whole system.
φ_1 = wave function of residual nucleus (N^{13}).
φ_2 = wave function of capturing nucleus (N^{14}) before neutron entry.
N, N_1, N_2, N_u, N_v = normalization factors for $\Psi, \psi_1, \psi_2, u, v$.
$\bar{u}=u\mathbf{c}-v\mathbf{s}$, $\mathbf{s}=\sin\theta$.
$\bar{v}=u\mathbf{s}+v\mathbf{c}$, $\mathbf{c}=\cos\theta$.
$\bar{\psi}_u, \bar{\psi}_v$ = functions ψ corresponding to $\bar{u}$ and $\bar{v}$.
$\bar{a}_u, \bar{a}_v$ = coefficients of $\bar{u}$ and $\bar{v}$ appearing in ψ.
$E_u, E_v, \bar{E}_u, \bar{E}_v$ = energy eigenvalues corresponding to $u, v, \bar{u}, \bar{v}$.
I_i, I_f = total spin of incident and product nuclei, respectively.
$\hbar L_i, \hbar L_f$ = relative orbital angular momentum before and after the collision.
$\pi_\pm, \nu_\pm = p^{\frac{1}{2}}$ wave functions of proton and neutron, respectively.
$\chi_m{}^j$ = spin functions of jth nucleus, m indexing the component along the z axis.
$(1)_m, 0_0$ = spin 1 and spin 0 functions for the initial state.
$S_m, \tilde{S}_0$ = spin 1 and spin 0 states of two neutrons.
$\mathfrak{A}_{mn}, \mathfrak{S}_{mn}$ = antisymmetric and symmetric combinations of orbital wave functions.
$\Psi_i{}^U, \Psi_f{}^U$ = unsymmetrized initial and final state wave functions.
$\Psi_i{}^S, \Psi_f{}^S$ = symmetrized initial and final state wave functions.
$\mathfrak{M}$ = factor of exponential in Coulomb wave.
$k=2\pi$ times the wave number of the appropriate particle, as indicated by a subscript.
P subscript refers to N^{13} part of incident particle.
D subscript refers to whole bombarding particle.
T subscript refers to the target (N^{14}) nucleus.
ξ = internal coordinates of N^{13} or target.
$\mathbf{r}_n, s_n$ = spatial and spin coordinates of neutron.
χ = functions describing internal relative motion [in Eq. (17.4)].
θ = scattering angle in c.m. system.
ϵ, w = parameters of the classical orbit for the initial state.
$\eta=e^2/\hbar v$.
$x^c, y^c, x_f{}^c, y_f{}^c$ = coordinates convenient for the description of the classical orbits of initial and final state.
E_c = minimum bombarding energy for contact of nuclei.
$\hbar\omega=\bar{E}_u-\bar{E}_v$.
q_0 = coordinates of nucleon transferred [in (26)–(27)].
$\mathfrak{h}_-$ = radial factor of wave function occurring in (7.1), not necessarily for eigenenergy.
$\langle\psi_1{}^2\rangle_S$ = average value of wave function on S.
$\kappa=2\pi$ times the wave number of particle in the nucleus.
σ_p, σ_n = total cross sections for the transfer of proton and neutron, respectively.

II. ADIABATIC FUNCTIONS

The calculation will be made in terms of adiabatic wave functions corresponding to stationary positions of the heavy particles and a neutron or proton tunneling between them. Recoil actions on the heavy aggregates are neglected. The only adiabatic functions taken into account are those corresponding to the ground states of the colliding nuclei before collision and to the reaction products after the collision. It will be seen that the calculation is very similar to one in terms of the energy matrix referred to unperturbed initial and final states.

The adiabatic functions will be introduced first for a simplified case in which the spins of the particles are not considered. The two heavy fragments, a distance R apart, are exchanging a nucleon, the displacements of which from the centers of the aggregates having centers at O and O' are denoted, respectively, by $\mathbf{r}$ and $\mathbf{r}'$. For large values of R, and neglecting the effect of the

Coulomb field on the nucleon the p-nucleon wave functions outside the left nucleus have the form

$$\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)\left[\frac{1}{\alpha r}+\frac{1}{(\alpha r)^2}\right]e^{-\alpha r}, \tag{1}$$

where α is determined by the binding energy of the nucleon as will be discussed more fully later. The direction of the z axis is from O' to O as in Fig. 1. As O' approaches O, this wave function becomes affected by the proximity of O'. For any R, there are three eigenstates arising from the three functions listed in (1) in the process of adiabatic transformation. For any one of these states, the system of nucleons moving about the fixed centers O and O' has a definite energy. When the neutron is at a point such as n in Fig. 1 in the space between the heavy nuclei, the latter are in their ground states, such as the ground states of N^{13} and N^{14} in the transfer of a neutron from N^{14} to N^{14}, and their energy is well defined. The neutron energy is thus well defined also as the difference between the total energy and the sum of the ground state energies of N^{13} and of N^{14}. The wave equation for the neutron in the space between the heavy nuclei thus has the form

$$(\Delta-\alpha_i^2)\psi=0, \quad (i=1, 2, 3) \tag{1.1}$$

where the α_i have, in general, different values for the states arising from the three functions listed in Eq. (1), the differences in the α_i arising from the energy differences of the three states. Each of these three functions can be analyzed in terms of spherical harmonics on S', and therefore in terms of solutions of Eq. (1.1) corresponding to definite values of the orbital angular moment about O'. Calculation shows that the p parts of such an analysis are, respectively,

$$-3\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{I_{\frac{3}{2}}(\alpha r')}{\alpha R(\alpha r')^{\frac{1}{2}}}\begin{Vmatrix}\left[-\frac{1}{\alpha R}-\frac{1}{(\alpha R)^2}\right]e^{-\alpha R}\left(\frac{x'}{r'}\right)\\ \left[-\frac{1}{\alpha R}-\frac{1}{(\alpha R)^2}\right]e^{-\alpha R}\left(\frac{y'}{r'}\right)\\ \left[1+\frac{2}{\alpha R}+\frac{2}{(\alpha R)^2}\right]e^{-\alpha R}\left(\frac{z'}{r'}\right)\end{Vmatrix}. \tag{1.2}$$

The three p states of O thus become coupled to three p states of O'. There are accordingly six adiabatic states, one pair of such states arising from each pair of states characterized by angular functions x/r, x'/r' in the decoupled condition. It is useful to introduce the coefficients of the angular functions x/r and x'/r' as follows:

$$h_-^x(S)=h_-^y(S)=h_-^z(S)=\left[\frac{1}{\alpha r}+\frac{1}{(\alpha r)^2}\right]\exp(-\alpha r), \tag{1.3}$$

$$h_-^x(S')=h_-^y(S')=3(\pi/2)^{\frac{1}{2}}[I_{\frac{3}{2}}(\alpha r')/(\alpha r')^{\frac{1}{2}}]\times[(1/q^2)+(1/q^3)]e^{-q}, \tag{1.3'}$$

with

$$q=\alpha R, \tag{1.4}$$

$$h_-^z(S')=-3(\pi/2)^{\frac{1}{2}}[I_{\frac{3}{2}}(\alpha r')/(\alpha r')^{\frac{1}{2}}]\times[(1/q)+(2/q^2)+(2/q^3)]e^{-q}, \tag{1.5}$$

and also

$$h_+^x(S')=h_+^y(S')=h_+^z(S')=\left[\frac{1}{\alpha r'}+\frac{1}{(\alpha r')^2}\right]\exp(-\alpha r'), \tag{1.6}$$

$$h_+^x(S)=h_+^y(S)=3(\pi/2)^{\frac{1}{2}}[I_{\frac{3}{2}}(\alpha r)/(\alpha r)^{\frac{1}{2}}]\times[(1/q^2)+(1/q^3)]e^{-q}, \tag{1.7}$$

$$h_+^z(S)=-3(\pi/2)^{\frac{1}{2}}[I_{\frac{3}{2}}(\alpha r)/(\alpha r)^{\frac{1}{2}}]\times[(1/q)+(2/q^2)+(2/q^3)]e^{-q}. \tag{1.8}$$

The designations S, S' show at which surface the expression is convenient to use. The primary definition of the h_- is in Eq. (1.3), that of the h_+ in (1.6). Equations (1.3'), (1.7), (1.8) are not literally correct but are restricted to the coupling of p states.

If the coupling takes place without changing the configuration assignment of the particles inside S and S', then the neutron state between S and S' can be described as a linear combination of h_- and a corresponding h_+. The boundary condition at $r=\infty$ is satisfied by this choice provided the primary meaning of h_- or h_+ is understood. At S and S' the linear combination must now be joined smoothly to the internal functions. The form of the conditions on the linear combinations obtained in this manner is independent of whether one is dealing with h^x, h^y, or h^z and the specification of the h_-, h_+ pair of functions may be omitted therefore for the present.

The combination of radial factors at S and S' may thus be written symbolically in the form

$$u=A_+h_++A_-h_-, \tag{2}$$

and the conditions on the radial logarithmic derivatives $\mathfrak{l}$ at S and S' give

$$A_-/A_+=-(h_{1+}/h_{1-})(\mathfrak{l}_{1+}-\mathfrak{l}_1)/(\mathfrak{l}_{1-}-\mathfrak{l}_1)=-(h_{2+}/h_{2-})(\mathfrak{l}_{2+}-\mathfrak{l}_2)/(\mathfrak{l}_{2-}-\mathfrak{l}_2), \tag{2.1}$$

where

$$h_{1+}=h_+(r), \quad h_{1-}=h_-(r), \quad (\text{at } S) \tag{2.2}$$

$$h_{2+}=h_+(r'), \quad h_{2-}=h_-(r'), \quad (\text{at } S') \tag{2.3}$$

$$\mathfrak{l}_{1+}=dh_{1+}/h_{1+}dr, \cdots, \quad \mathfrak{l}_{2-}=dh_{2-}/h_{2-}dr'. \tag{2.4}$$

The quantities $\mathfrak{l}_1$, $\mathfrak{l}_2$ are, respectively, the logarithmic derivatives of the radial function with respect to r at S and r' at S'. From Eq. (2.1) it follows that

$$(\mathfrak{l}_{1-}-\mathfrak{l}_1)(\mathfrak{l}_{2+}-\mathfrak{l}_2)=P_+P_-(\mathfrak{l}_{1+}-\mathfrak{l}_1)(\mathfrak{l}_{2-}-\mathfrak{l}_2), \tag{3}$$

where

$$P_+=h_{1+}/h_{2+}, \quad P_-=h_{2-}/h_{1-}. \tag{3.1}$$

The quantity P_+ is a measure of barrier penetration for a wave function starting on the right nucleus, while P_- is a similar measure for a function starting at the left nucleus. The logarithmic derivatives l_1 and l_2 are determined by the boundary conditions for the other nucleons and by the energy. It is supposed that the wave function for which two nucleons have left the interior of S and are interacting with the nucleons inside S' is negligible. The reason for doing so is that the penetration factors enter to a higher order for such interactions. It is also supposed that the tunneling effect is sufficiently weak to make it unnecessary to take into account more than first order changes in the logarithmic derivatives with energy. The energy of the coupled states is determined by Eq. (3). If the nuclei are very far apart, $P_+=P_-=0$ and there are then the obvious solutions corresponding to $l_1=l_{1-}$ and $l_2=l_{2+}$. For the first of these the nucleon is attached to S, for the second to S', and the values of α are accordingly those corresponding to a neutron escaping from S with an energy equal to the binding energy of the neutron[12] leaving S.

In the case of the $N^{14}+N^{14}$ transfer reactions the two roots of Eq. (3) are close together, the binding energies of the last neutron being nearly the same in N^{14} and N^{15}. There is therefore a non-negligible resonance effect between the states originating in the conditions $l_1=l_{1-}$ and $l_2=l_{2+}$. The adiabatic wave functions are also affected by this type of resonance and approximations to these wave functions may be worked out as follows. When one drops higher than first-order effects of energy changes on the l, Eq. (3) becomes

$$(\alpha^2-\alpha_{10}^2)(\alpha^2-\alpha_{20}^2)\lambda_1\lambda_2\cong P_+P_-(l_{1+}-l_1)(l_{2-}-l_2), \quad (4)$$

where

$$\lambda_1=[\partial(l_{1-}-l_1)/\partial(\alpha^2)], \quad (\alpha=\alpha_{10}) \quad (4.1)$$

$$\lambda_2=[\partial(l_{2+}-l_2)/\partial(\alpha^2)], \quad (\alpha=\alpha_{20}) \quad (4.2)$$

and α_{10}, α_{20} are determined by

$$l_1-l_{1-}=0, \quad (\alpha=\alpha_{10}) \quad (4.3)$$

$$l_2-l_{2+}=0. \quad (\alpha=\alpha_{20}). \quad (4.4)$$

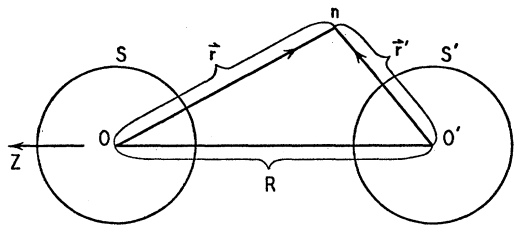

FIG. 1. Sketch illustrating choice of coordinates. Here $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of the neutron n relative to the centers of the nuclei, O and O', respectively. The nuclear surfaces are designated by S and S'. The nuclear centers lie on the z axis, and are separated by a distance R.

[12] The binding energy (= −tightness of binding), is used here as a negative number for a stable nucleus.

Introducing

$$\beta_1=-(h_{1+}/h_{1-})(l_{1+}-l_{1-})/\lambda_1, \quad (4.5)$$

$$\beta_2=-(h_{2-}/h_{2+})(l_{2-}-l_{2+})/\lambda_2, \quad (4.6)$$

and

$$a_+=A_+\beta_2^{-\frac{1}{2}}, \quad a_-=A_-\beta_1^{-\frac{1}{2}}, \quad (4.7)$$

the joining conditions expressed by Eq. (2.1) take the form

$$(\alpha_{20}^2-\alpha^2)a_++\beta a_-=0, \quad \beta a_++(\alpha_{10}^2-\alpha^2)a_-=0, \quad (4.8)$$

with

$$\beta=(\beta_1\beta_2)^{\frac{1}{2}}; \quad \beta^2=(P_+P_-/\lambda_1\lambda_2)(l_{1+}-l_{1-})(l_{2-}-l_{2+}). \quad (4.9)$$

It follows from Eq. (4.8) that

$$(\alpha^2-\alpha_{10}^2)(\alpha^2-\alpha_{20}^2)-\beta^2=0. \quad (5)$$

According to the second of the two forms for β in Eq. (4.9), this quantity is reasonably sensitive to α since it contains the barrier penetrabilities. It is nevertheless useful to solve Eq. (5) for α^2, treating β as a parameter because an approximate value of α can be used in β and the result improved by iteration. The roots of Eq. (5) for α^2 will be called α_1^2 and α_2^2, the choice of subscripts being made such that as β^2 decreases to zero, α_1^2 approaches α_{10}^2 and α_2^2 approaches α_{20}^2. Explicitly,

$$\begin{aligned}\alpha_1^2&=\tfrac{1}{2}(\alpha_{10}^2+\alpha_{20}^2)+\Delta,\\ \alpha_2^2&=\tfrac{1}{2}(\alpha_{10}^2+\alpha_{20}^2)-\Delta,\end{aligned} \quad (\alpha_{10}^2>\alpha_{20}^2) \quad (5.1)$$

while

$$\begin{aligned}\alpha_1^2&=\tfrac{1}{2}(\alpha_{10}^2+\alpha_{20}^2)-\Delta,\\ \alpha_2^2&=\tfrac{1}{2}(\alpha_{10}^2+\alpha_{20}^2)+\Delta.\end{aligned} \quad (\alpha_{20}^2>\alpha_{10}^2) \quad (5.2)$$

Here

$$\Delta=|[\tfrac{1}{4}(\alpha_{10}^2-\alpha_{20}^2)^2+\beta^2]^{\frac{1}{2}}|. \quad (5.3)$$

With these definitions, Eq. (4.8) gives

$$a_+/a_-=\tan\theta, \quad (\alpha=\alpha_1) \quad (5.4)$$

$$a_+/a_-=-\cot\theta, \quad (\alpha=\alpha_2) \quad (5.5)$$

with

$$2\theta=\arctan[2\beta/(\alpha_{10}^2-\alpha_{20}^2)]. \quad (5.6)$$

All conditions of coupling are represented by varying θ in the range $-\pi/4\leq\theta\leq\pi/4$. These ranges correspond to the forms

$$\begin{aligned}\tan\theta&=-\chi+(1+\chi^2)^{\frac{1}{2}}, \quad (\alpha_{10}^2>\alpha_{20}^2)\\ \tan\theta&=-\chi-(1+\chi^2)^{\frac{1}{2}}, \quad (\alpha_{10}^2<\alpha_{20}^2)\end{aligned} \quad (5.7)$$

with

$$\chi=(\alpha_{10}^2-\alpha_{20}^2)/(2\beta). \quad (5.8)$$

The angle θ is the angle of rotation of the principal axes for α^2.

Without approximations the ratio of the radial wave functions at the nuclear surfaces, $u(r')/u(r)$, may be represented by

$$\frac{u(r')}{u(r)}\frac{l_{1+}-l_{1-}}{l_{2+}-l_{2-}}=P_-\frac{l_{1+}-l_1}{l_{2+}-l_2}=\frac{1}{P_+}\frac{l_{1-}-l_1}{l_{2-}-l_2}. \quad (6)$$

Since $\mathfrak{l}_{1-}-\mathfrak{l}_1$ involves P_+P_- when $\mathfrak{l}_1\cong\mathfrak{l}_{1-}$, the last of these forms shows that in this case $u(r')\ll u(r)$ for weak penetration. In the approximation of Eqs. (4) to (5),

$$\frac{u(r')}{u(r)}=\left(\frac{\gamma_1P_-}{\gamma_2P_+}\right)^{\frac{1}{2}}\frac{\gamma_2\tan\theta+\beta}{\beta\tan\theta+\gamma_1},\quad(\alpha=\alpha_1)\qquad(6.1)$$

$$\frac{u(r')}{u(r)}=\left(\frac{\gamma_1P_-}{\gamma_2P_+}\right)^{\frac{1}{2}}\frac{\gamma_2-\beta\tan\theta}{\beta-\gamma_1\tan\theta},\quad(\alpha=\alpha_2)\qquad(6.2)$$

with

$$\gamma_1=(\mathfrak{l}_{1-}-\mathfrak{l}_{1+})/\lambda_1,\quad\gamma_2=(\mathfrak{l}_{2+}-\mathfrak{l}_{2-})/\lambda_2.\qquad(6.3)$$

For $\theta=\pi/4$, the root α_1 gives a symmetric function in the sense that $a_+=a_-$, while α_2 gives an antisymmetric u. For $\theta=-\pi/4$, the roles of the symmetric and antisymmetric functions with respect to α_1 and α_2 are reversed.

The normalization of the wave function is determined by the integrals over the configuration space of all the particles. It is impractical to calculate it on account of the difficulty of the many-body problem. However, it is possible to introduce the integrals for isolated nuclei as parameters and to express the normalization integral of the adiabatic wave function in terms of the normalization integrals of separated nuclei. The latter are of the same type as those which enter the theory of reactions caused by neutron bombardment, but the energy is so different in the present case that a truly quantitative use of data on neutron-induced reactions is also very difficult. In the space outside both S and S' and for $\alpha=\alpha_1$,

$$u=N_u[(\gamma_2/P_+)^{\frac{1}{2}}(h_+/h_{2+})\sin\theta+(\gamma_1/P_-)^{\frac{1}{2}}(h_-/h_{1-})\cos\theta].\qquad(6.4)$$

The corresponding formula for $\alpha=\alpha_2$ is obtained by the replacements $u\to v$, $N_u\to N_v$, $\theta\to\frac{1}{2}\pi+\theta$. It is important to use here the primary meaning of h_- and h_+, *viz.* that of Eqs. (1.3) and (1.6) rather than the values which are convenient for joining purposes only, such as in Eq. (1.3′). The wave function corresponding to u is

$$\psi_u=N_u[(\gamma_2/P_+)^{\frac{1}{2}}(x'/r')(h_+/h_{2+})\sin\theta+(\gamma_1/P_-)^{\frac{1}{2}}(x/r)(h_-/h_{1-})\cos\theta],\qquad(6.5)$$

a specialization to one of the three directions of the p state having been made. The part of the above expression containing h_-/h_{1-} would join smoothly to the internal function inside S if it were not for the changed energy. For reasonably large R this change is small and may then be neglected. For isolated nuclei the wave functions will be denoted by ψ_1 and ψ_2, the function ψ_1 referring to the nucleus containing the neutron before capture (N^{14}) and ψ_2 to the nucleus which the neutron enters after capture (N^{15}). The coordinates of the neutron will be denoted collectively by q, the coordinates of other nucleons in the first nucleus by q_1, q_2, $\cdots$, q_n, and the coordinates of other nucleons in the second nucleus by q_{n+1}, q_{n+2}, $\cdots$, q_{n+m}. The wave functions ψ_1 and ψ_2 will be normalized by

$$\int\psi_1^2dqdq_1dq_2\cdots dq_n=\int\psi_2^2dqdq_{n+1}dq_{n+2}\cdots dq_{n+m}=1.\qquad(7)$$

Considering the wave functions with angular dependence x/r and x'/r', and extending them beyond the nuclear radii by continuation employing the Schroedinger equation for each nucleus, the functions ψ_1 and ψ_2 assume the forms

$$\psi_1=N_1(x/r)[h_-(r)/h_{1-}]\varphi_1(q_1,\cdots,q_n),\quad(r>a_1)\qquad(7.1)$$

$$\psi_2=N_2(x'/r')[h_+(r')/h_{2+}]\varphi_2(q_{n+1},\cdots,q_{n+m}),\quad(r'>a_2)\qquad(7.2)$$

the first of which applies when the neutron has left the first nucleus, the second when the second nucleus has been dissociated. The nuclear radii for nuclei 1 and 2 are denoted by a_1 and a_2, respectively. The wave function of the residual nucleus (N^{13}) left by the neutron is called φ_1, while φ_2 is the function of the capturing nucleus (N^{14}) before it has been entered by the neutron. The wave function of the whole system is

$$\Psi=\Psi(q;q_1,q_2,\cdots q_n;q_{n+1},\cdots,q_{n+m}),\qquad(7.3)$$

and it has to be normalized by

$$\int\Psi^2dqdq_1\cdots dq_ndq_{n+1}\cdots dq_{n+m}=1.\qquad(7.4)$$

Since the nuclear surfaces S and S' do not intersect, the configuration space can be divided into three parts according to the three possibilities:

$$\text{(I) } r<a_1,\quad\text{(II) } r'<a_2,\quad\text{(III) } r>a_1,\ r'>a_2.$$

Region (I) corresponds to the condition before escape, II to that after capture, III to the tunneling stage. In accordance with Eq. (6.5), one has for region III,

$$\Psi=\psi\varphi_1\varphi_2,\quad\text{(in III)}\qquad(7.5)$$

where by ψ is meant the expression in Eq. (6.5). The two parts of ψ corresponding to the two terms in brackets in Eq. (6.5) give rise to two parts of Ψ which can be joined smoothly to constant multiples of $\psi_1\varphi_2$ and of $\psi_2\varphi_1$, respectively, making use of Eqs. (7.1), (7.2). One has thus

$$\begin{aligned}\Psi&=(N/N_1)(\gamma_1/P_-)^{\frac{1}{2}}\cos\theta\psi_1\varphi_2,\quad(r<a_1)\\ \Psi&=(N/N_2)(\gamma_2/P_+)^{\frac{1}{2}}\sin\theta\psi_2\varphi_1.\quad(r'<a_2)\end{aligned}\qquad(7.6)$$

The normalization condition on N is accordingly obtainable by direct substitution of Ψ into the normalization integral by means of Eqs. (7.5) and (7.6). In this calculation, there occurs the integral of

$$(x/r)^2\varphi_1^2[h_-(r)/h_{1-}]^2$$

over the region outside both S and S'. Since the main interest is in appreciable nuclear separations R and since r is the distance from the center of S to the neutron, this integral may be approximated by extending the integration over all of the space outside S. Similarly the integral containing $(x'/r')^2$ can be approximated by extending the integration over all of the space outside S'. These approximations result in the occurrence of the same combinations as occur in integrating ψ_1^2 and ψ_2^2 over their configuration space. The energy of the adiabatic state is only slightly different from the energies of the isolated nuclei, and therefore only a small error is made by replacing the combinations by unity. The normalization condition thus becomes

$$1/N^2=(\gamma_1/N_1^2P_-)\cos^2\theta+(\gamma_2/N_2^2P_+)\sin^2\theta +2(\gamma_1\gamma_2/P_+P_-)^{\frac{1}{2}}\sin\theta\cos\theta \times\int_{\mathrm{III}}(xx'h_+h_-/rr'h_{2+}h_{1-})dq, \quad (8)$$

where subscript III on the integral means that it is extended over $r>a_1$ and $r'>a_2$, the space outside both S and S'. The normalization conditions on ψ_1 and ψ_2 give

$$N_1^2=\left[1-\int_{\mathrm{I}}\psi_1^2dqdq_1\cdots dq_n\right]\Big/\int(xh_-/rh_{1-})^2dq, \quad (8.1)$$

$$N_2^2=\left[1-\int_{\mathrm{II}}\psi_2^2dqdq_{n+1}\cdots dq_{n+m}\right]\Big/ \int(x'h_+/r'h_{2+})^2dq. \quad (8.2)$$

Here the subscripts I and II refer to the regions defined just before Eq. (7.5), and the integrals in the denominators of Eqs. (8.1) and (8.2) are taken over the space outside S and S' respectively. Since the integral in Eq. (8) vanishes for large internuclear distances and since its value depends on the existence of an overlap of h_+ and h_-, the first two terms are usually a good approximation to the value of $1/N^2$.

III. TRANSFER IN TERMS OF ADIABATIC AND FIXED FUNCTIONS

So far the adiabatic functions have been considered without bringing in the changes which occur as a result of the relative motion of the nuclei. The following special circumstances enter. (a) The particle transferred is in a $p_{\frac{1}{2}}$ state. The functions h_-, h_+ of the preceding section differ for the three possible p states. There is thus an anisotropy of transmission depending on the orientation of the axis of angular dependence with respect to the internuclear line. (b) On account of the anisotropy, it is necessary to distinguish between the adiabatic wave functions in the laboratory coordinate system and in the coordinate system whose z axis rotates with the internuclear line. If the transfer were that of a spinless neutron uncoupled to other nucleons, the distinction between the two coordinate systems would have been important. (c) The probable nuclear configurations of N^{13} and N^{15} on the j-j coupling model re-establish isotropy of the transfer as will be seen presently.

For these reasons, the details of distinctions between stationary and rotating systems for anisotropic transfer are not needed for the immediate application. It is necessary, however, to discuss the solution of the equations resulting from expressing the wave function in terms of adiabatic functions. According to Eqs. (5.4) and (5.5), there are two adiabatic functions arising out of the two states being coupled by tunneling. The radial factor for one of them is as in Eq. (6.4). That for the second is obtainable by changing $\theta\rightarrow\frac{1}{2}\pi+\theta$ and is therefore

$$v=N_v[(\gamma_2/P_+)^{\frac{1}{2}}(h_+/h_{2+})\cos\theta -(\gamma_1/P_-)^{\frac{1}{2}}(h_-/h_{1-})\sin\theta]. \quad (6.4')$$

This expression is *symbolic only*, the angular dependence being omitted. The corresponding wave function in the space between the nuclei is

$$\psi_v=N_v[(\gamma_2/P_+)^{\frac{1}{2}}(x/r)(h_+/h_{2+})\cos\theta -(\gamma_1/P_-)^{\frac{1}{2}}(x'/r')(h_-/h_{1-})\sin\theta]. \quad (6.5')$$

The difference in the energy of ψ_u and ψ_v produces only a slight effect on the space dependence of h_+/h_{2+} and h_-/h_{1-} and will be neglected. It will be assumed that for any R one may approximate

$$\psi=a_u\psi_u+a_v\psi_v, \quad (9)$$

i.e., the entrance of other states is assumed to be unimportant. Substitution of ψ into the wave equation gives two simultaneous equations connecting a_u, a_v, da_u/dt, and da_v/dt. Before the collision, while the neutron is in nucleus 1 and while $\theta=0$, $a_v=0$. One would be tempted to solve the equations on the assumption that $|a_v/a_u|\ll1$ throughout the collision. Such an assumption is not justifiable, however, in the general case. An obvious exception to it is formed in the case of complete degeneracy for which $\alpha_{10}=\alpha_{20}$ so that $\theta=\pi/4$. In this case the adiabatic functions ψ_u and ψ_v do not correspond to localization of the neutron in one or another nucleus. For any other case, however, $R\rightarrow\infty$ gives $\beta=0$ and localization results according to Eq. (5.6). Nevertheless, if the nuclei are close enough to make

$$|\chi|\ll1, \quad (9.1)$$

the adiabatic functions become essentially like those for exact degeneracy and the neutron is not even approximately localized in one of the nuclei. This condition can be obtained even if β is very small by making $|\alpha_{10}^2-\alpha_{20}^2|$ small enough. If $|a_u|$ were $\cong1$ during the collision, one would obtain, therefore, a practically 50% prob-

ability of transfer close to the perihelion which would decrease again at the end of the orbit. This is obviously not the case, and the assumption $|a_u|\cong 1$ cannot be generally applicable therefore. Accordingly $|a_v|$ is not in general $\ll 1$, and successive approximations arranged with $a_v=0$ as the starting approximation do not necessarily converge rapidly. These circumstances enter only if there is approximate resonance and the important parameter is χ. Estimates show that in the case of the $N^{14}+N^{14}$ reaction, $|a_v/a_u|\ll 1$ for $E=10$ Mev but that this approximation is not applicable for $E=15$ Mev. For these reasons, the successive approximations will be arranged in terms of such linear combinations of adiabatic functions as to make them correspond to the nucleons being mostly in one or the other nucleus throughout the collision. This part of the calculation is appreciably simplified if it is assumed that

$$\gamma_1=\gamma_2, \quad N_u=N_v. \tag{9.2}$$

Since in the application one would have difficulty in assigning values to differences of these quantities, the more complicated general formulas will be omitted. Instead of u and v, the functions

$$\bar{u}=u\mathbf{c}-v\mathbf{s}, \quad \bar{v}=u\mathbf{s}+v\mathbf{c}, \tag{9.3}$$

with

$$\mathbf{s}=\sin\theta, \quad \mathbf{c}=\cos\theta \tag{9.4}$$

will be used. The $\bar{u}$ and $\bar{v}$ correspond to the neutron being very nearly in the left and right nuclei, respectively. The functions ψ corresponding to $\bar{u}$ and $\bar{v}$ will be called $\bar{\psi}_u$ and $\bar{\psi}_v$, respectively. Instead of (9), one has

$$\psi=\bar{a}_u\bar{\psi}_u+\bar{a}_v\bar{\psi}_v, \tag{10}$$

so that

$$\bar{a}_u=a_u\mathbf{c}-a_v\mathbf{s}, \quad \bar{a}_v=a_u\mathbf{s}+a_v\mathbf{c}. \tag{10.1}$$

Substitution in the wave equation gives

$$\begin{aligned} &d\bar{a}_u/dt+\dot{\theta}\bar{a}_v+\dot{R}\left(u,\frac{\partial v}{\partial R}\right)\bar{a}_v+\frac{i}{\hbar}(E_u-E_v)\mathbf{sc}\bar{a}_v \\ &\qquad +\frac{i}{\hbar}(E_u\mathbf{c}^2+E_v\mathbf{s}^2)\bar{a}_u=0, \\ &d\bar{a}_v/dt-\dot{\theta}\bar{a}_u-\dot{R}\left(u,\frac{\partial v}{\partial R}\right)\bar{a}_u+\frac{i}{\hbar}(E_u\mathbf{s}^2+E_v\mathbf{c}^2)\bar{a}_v \\ &\qquad +\frac{i}{\hbar}(E_u-E_v)\mathbf{sc}\bar{a}_u=0, \end{aligned} \tag{10.2}$$

where E_u and E_v are the energy eigenvalues corresponding to the adiabatic functions u and v, respectively. The orthogonality of u and v has been made use of in obtaining (10.2) to set $(u, \partial v/\partial R)=-(\partial u/\partial R, v)$. Both u and v have been taken to be real since the p functions can always be chosen in this manner. From (9.3) one obtains

$$\partial v/\partial\theta=-u, \quad (u, \partial v/\partial\theta)=-1, \tag{10.3}$$

and hence,

$$\dot{\theta}+\dot{R}\left(u,\frac{\partial v}{\partial R}\right)=\left(u,\dot{R}\frac{\partial v}{\partial R}-\dot{\theta}\frac{\partial v}{\partial\theta}\right)=\left(u,\dot{R}\frac{Dv}{DR}\right), \tag{10.4}$$

where Dv/DR takes into account variations of the adiabatic function taking place apart from θ. Since this leaves only the slight changes in $\bar{u}$ and $\bar{v}$ caused by energy dependence, these terms are very small and will be neglected. One obtains thus from (10.2)

$$\begin{aligned} &d\bar{a}_u/dt+\frac{i}{\hbar}[\langle E\rangle+\tfrac{1}{2}(E_u-E_v)\cos 2\theta]\bar{a}_u \\ &\qquad +\frac{i}{2\hbar}(E_u-E_v)\sin 2\theta\bar{a}_v=0, \\ &d\bar{a}_v/dt+\frac{i}{\hbar}[\langle E\rangle-\tfrac{1}{2}(E_u-E_v)\cos 2\theta]\bar{a}_v \\ &\qquad +\frac{i}{2\hbar}(E_u-E_v)\sin 2\theta\bar{a}_u=0, \end{aligned} \tag{10.5}$$

with

$$\langle E\rangle=(E_u+E_v)/2. \tag{10.6}$$

Employment of Eqs. (5) to (5.3) in (10.5) gives

$$\begin{aligned} &\left(\frac{d}{dt}+\frac{i}{\hbar}\bar{E}_u\right)\bar{a}_u-\frac{i\hbar}{2M}\beta\bar{a}_v=0, \\ &\left(\frac{d}{dt}+\frac{i}{\hbar}\bar{E}_v\right)\bar{a}_v-\frac{i\hbar}{2M}\beta\bar{a}_u=0, \end{aligned} \tag{11}$$

with

$$\bar{E}_u=-(\hbar^2/2M)\alpha_{10}^2, \quad \bar{E}_v=-(\hbar^2/2M)\alpha_{20}^2, \tag{11.1}$$

where β is as in Eq. (4.9). Equation (11) suggests the interpretation that the energy matrix in the reference system of $\bar{u}$ and $\bar{v}$ has the diagonal elements $\bar{E}_u$ and $\bar{E}_v$, its nondiagonal element being $-(\hbar^2/2M)\beta$. Equation (11) is simpler to use than corresponding relations in terms of adiabatic functions. The reason for introducing the latter is that they give a derivation of β and also that it appeared useful to demonstrate that a consistent use of adiabatic functions combined with consideration of convergence leads to the consideration of the energy matrix with respect to unperturbed functions in separate nuclei. It should be pointed out that the choice of the $\bar{\psi}_u$, $\bar{\psi}_v$ in preference to ψ_u, ψ_v has been made without a real proof regarding the functions $\bar{\psi}_u$, $\bar{\psi}_v$ being the best for securing rapid convergence, the considerations being largely intuitive in this connection. The usefulness of the adiabatic functions in deriving β has been that considerations regarding joining of functions are natural in a calculation of energy levels for stationary nuclei and that these considerations simplify the calculation of β.

In extending Eq. (11) to the case of spin coupling and coupling to other nucleons in the two nuclei, it will suffice to consider the modifications in the energy matrix caused by the introduction of these changes in the wave function. It may be noted here that a treatment in terms of adiabatic wave functions constructed from the integral equation, as proposed previously,[11] leads to the same final results. This treatment will be discussed in a separate publication.[13]

IV. EFFECT OF SPIN AND OTHER COUPLINGS

Since N^{14} obeys Einstein-Bose statistics, the initial states have total nuclear spin $I_i=0$ or 2 for even L_i, and spin 1 for odd L_i. Here $L_i\hbar$ denotes the angular momentum of relative motion of the nitrogens before the collision. Denoting similarly the relative orbital angular momentum after the collision by $L_f\hbar$ and the total spin of product nuclei by I_f, combined requirements of conservation of total angular momentum $J\hbar$ and of parity restrict the possibilities to those listed in Table I. The general selection rules are seen to allow changes of relative orbital angular momentum L by $\Delta L=0, \pm 2$. The possibilities are more severely restricted if a specific model for the nuclei is used. Referring to protons by the letter π, to neutrons by ν, to p single-particle functions by p, and designating the total angular momentum in units $\hbar$ by a subscript, it is convenient to introduce single-particle functions

$$(p^{\pi}{}_{\frac{1}{2}})_+=\pi_+, \quad (p^{\pi}{}_{\frac{1}{2}})_-=\pi_-, \quad (p^{\nu}{}_{\frac{1}{2}})_+=\nu_+, \quad (p^{\nu}{}_{\frac{1}{2}})_-=\nu_-. \tag{12}$$

Here subscripts + and − refer to the sign of the projection on the axis of quantization of the angular momentum of a particle in a $p_{\frac{1}{2}}$ state. The p-shell neutron configurations of N^{13}, N^{14}, and N^{15} will be taken to be $(p_{\frac{3}{2}})^4$, $(p_{\frac{3}{2}})^4p_{\frac{1}{2}}$, and $(p_{\frac{3}{2}})^4(p_{\frac{1}{2}})^2$, respectively, and a similar assignment of proton wave functions will be used. This assumes that transfer takes place to the ground state. It appears very probable[14] that pure j-j coupling does not apply to these nuclei. The qualitative features of the angular distribution curve, *viz.*, the strong peaks in the forward and backward directions, should not be affected by intermediate coupling since they are caused primarily by the presence of the factor $\exp[-2\alpha(R-2a)]$ which is present for all tunneling processes, and do not depend much on the powers of $1/R$ which are affected by assumptions regarding the coupling. Disregarding particles in closed shells, the incident conditions of the two nuclei are described by the functions

$$(\chi^i{}_1,\chi^i{}_0,\chi^i{}_{-1})=(\pi^i{}_+\nu^i{}_+, [\pi^i{}_+\nu^i{}_-+\pi^i{}_-\nu^i{}_+]/\sqrt{2}, \pi^i{}_-\nu^i{}_-), \quad (j=1,2) \tag{12.1}$$

[13] M. E. Ebel (to be published).

[14] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953); B. Jancovici and I. Talmi, Phys. Rev. **95**, 289 (1954).

TABLE I. Values of L_f allowed by conservation of J and of parity.

I_i \ I_f	Even L_i: 0	1	J	I_i \ I_f	Odd L_i: 0	1	J
0	L_i	L_i	L_f	1	L_i		L_f
2	$L_i\pm2$, L_i		L_f	1		L_i, L_i-2	L_f+1
		L_i, L_i-2	L_f+1			L_i	L_f
		$L_i\pm2$, L_i	L_f			L_i+2, L_i	L_f-1
		L_i+2, L_i	L_f-1				

corresponding to spin 1 for each N^{14}. In the final state the neutrons in N^{15} form a closed shell and, referring to N^{15} as the second nucleus, the N^{13} contains the proton π^1 while N^{15} contains π^2, ν^1, and ν^2. For fixed centers of mass of the nitrogens, there are four final state functions

$$(\pi^1{}_+\pi^2{}_+, [\pi^1{}_+\pi^2{}_-+\pi^1{}_-\pi^2{}_+]/\sqrt{2}, \pi^1{}_-\pi^2{}_-) \times[\tilde{\nu}^1{}_+\nu^2{}_--\tilde{\nu}^1{}_-\nu^2{}_+]/\sqrt{2}=(1^f{}_1, 1^f{}_0, 1^f{}_{-1}), \tag{12.2}$$

$$[\pi^1{}_+\pi^2{}_--\pi^1{}_-\pi^2{}_+](\tilde{\nu}^1{}_+\nu^2{}_--\tilde{\nu}^1{}_-\nu^2{}_+)/2=0^f{}_0, \tag{12.3}$$

the first three giving total spin 1, the last spin 0, and spin projection being indicated by a subscript. Since the functions $\nu^1{}_+$, $\nu^1{}_-$ refer to different situations before and after collision, the functions after collision are denoted by $\tilde{\nu}$. Since the nitrogen nuclei are heavy in comparison with the neutron, the exchange of linear momentum when the neutron is leaving or is absorbed by a nucleus is relatively small and the orbital angular momentum of the heavy particles is approximately unchanged. Formally the same may be seen from the fact that the Hamiltonian of three mass points with masses m_3, $\ll M_1$, M_2 and potential energy $V_1(r_{13})+V(r_{23})$ commutes with the relative angular momentum operator of M_1 and M_2. On account of the presence of $\mathbf{R}_2-\mathbf{R}_1$ in the physically important operators representing relative motion of M_1+m_3 with respect to M_2 and of M_1 with respect to M_2+m_3, these operators do not exactly commute with the Hamiltonian and some exchange of orbital angular momentum is expected for the more distant collisions. In these cases the large lever arm increases the importance of the small exchange of linear momentum. Since the tunneling effect decreases rapidly with distance, this angular momentum transfer will be neglected. The conservation of total angular momentum now implies that the sum of nuclear spins be conserved because it commutes with the Hamiltonian in the same approximation. It is unnecessary to consider therefore the states with initial nuclear spin 2 because this spin cannot be reproduced by the final available values of 0 and 1. It may be noted, however, that at low bombarding energies the approximation used is more questionable, distant collisions being then relatively more important, and that the arguments given do not exclude the possibility of an increased importance of incident states with total spin 2 at low energies. Estimates indicate, however, that for 10-Mev bom-

barding energy such effects are not serious. The incident-state spin functions are

$$(1)_1=(\chi^1{}_1\chi^2{}_0-\chi^1{}_0\chi^2{}_1)/\sqrt{2},\quad (1)_0=(\chi^1{}_1\chi^2{}_{-1}-\chi^1{}_{-1}\chi^2{}_1)/\sqrt{2},$$
$$(1)_{-1}=(\chi^1{}_0\chi^2{}_{-1}-\chi^1{}_{-1}\chi^2{}_0)/\sqrt{2}, \quad (12.4)$$

$$(0)_0=(\chi^1{}_1\chi^2{}_{-1}-\chi^1{}_0\chi^2{}_0+\chi^1{}_{-1}\chi^2{}_1)/\sqrt{3}, \quad (12.5)$$

the values of the spin and its projection being indicated in parentheses and as a projection, respectively.

The assignment of configurations to the particles does not imply that the wave function is described by such combinations as in Eqs. (12.2) and (12.3). Since it is not practical, however, to calculate corrections to these functions arising from interactions within the same nucleus, these functions will be used in a literal sense. Interactions between wave functions of the same configuration have no off-diagonal elements between the functions describing space degeneracy. Central forces spoil j-j coupling and the self consistency of the model, which is admittedly imperfect in this respect.

It is seen from Eq. (1.2) that the coupling of the orbital functions by tunneling action is anisotropic, the wave function with a lobe along the internuclear line OZ being transferred differently from the two other functions. The anisotropy complicates the consideration of transfer, which will be made, therefore, in two steps. In the first, the effect of equal coefficients of x'/r', y'/r', and z'/r' will be considered, in the second the effect of coupling through z alone will be taken into account, and the total effect will be obtained through the superposition of the two effects. The transfer depends on the off-diagonal energy matrix element. The form of Eqs. (12.2) and (12.3) shows, therefore, that one needs only the antisymmetric part of the wave function obtained by considering the transfer of the orbital functions.

For the isotropic type of transfer, one obtains for the antisymmetric parts

$$\text{A.P.}\quad (1)_1=\pi^1{}_+\pi^2{}_+(\nu^1{}_+\nu^2{}_- - \nu^1{}_-\nu^2{}_+)/2,$$
$$\text{A.P.}\quad (1)_0=[(\pi^1{}_+\pi^2{}_- + \pi^1{}_-\pi^2{}_+)/\sqrt{2}]\times(\nu^1{}_+\nu^2{}_- - \nu^1{}_-\nu^2{}_+)/2, \quad (12.6)$$
$$\text{A.P.}\quad (1)_{-1}=\pi^1{}_-\pi^2{}_-(\nu^1{}_+\nu^2{}_- - \nu^1{}_-\nu^2{}_+)/2,$$
$$\text{A.P.}\quad (0)_0=(\sqrt{3}/4)(\pi^1{}_+\pi^2{}_- - \pi^1{}_-\pi^2{}_+)\times(\nu^1{}_+\nu^2{}_- - \nu^1{}_-\nu^2{}_+). \quad (12.7)$$

The off-diagonal energy matrix element between the initial and final states contains firstly factors arising from the combinations of $\bar{\nu}^1{}_+$ with $\nu^1{}_+$, $\bar{\nu}^1{}_-$ with $\nu^2{}_-$, which are the same, and secondly factors arising from the scalar products of the antisymmetric parts of the $(1)_i$ and of 0_0 with the $1'_j$ and $0'_0$. The form of the functions listed for the three functions of spin 1 in Eq. (12.6) and for the function of spin 0 in Eq. (12.7) shows that these off-diagonal elements vanish except for the combinations of $(1)_i$ with $1'_i$ and of $(0)_i$ with $0'_0$. On comparing (12.6) and (12.7) with (12.2) and (12.3), the coefficients multiplying the factors coming from the ν and $\bar{\nu}$ energy elements are seen to be

$$(1/\sqrt{2},\ 1/\sqrt{2},\ 1/\sqrt{2},\ \sqrt{3}/2). \quad (12.8)$$

The orbital neutron functions may be written as

$$(u_1,u_0,u_{-1})=(3/4\pi)^{\frac{1}{2}}\times[-(x+iy)/\sqrt{2}r,\ z/r,\ (x-iy)/\sqrt{2}r]R(r), \quad (13)$$

with the normalization

$$\int_0^\infty R^2r^2dr=1. \quad (13.1)$$

In terms of these,

$$\nu^1{}_+=(u_0\alpha-\sqrt{2}u_1\beta)/\sqrt{3},\quad \nu^1{}_-=(\sqrt{2}u_{-1}\alpha-u_0\beta)/\sqrt{3}, \quad (13.2)$$

where α and β are the usual spin functions. On account of anisotropy, the energy matrix elements of u_0 in one nucleus with the u_0 of the other differ from those for u_1 and u_{-1}. This effect may be represented by the changes

$$\nu^1{}_+\rightarrow\nu^1{}_+ + Au^1{}_0\alpha^1,\quad \nu^1{}_-\rightarrow\nu^1{}_- - Au^1{}_0\beta^1, \quad (13.3)$$

in the $p_{\frac{1}{2}}$ states of the transferred nucleon, which cause corresponding changes in the $\chi^1{}_i$, *viz.*,

$$(\chi^1{}_1,\chi^1{}_0,\chi^1{}_{-1})\rightarrow(\chi^1{}_1,\chi^1{}_0,\chi^1{}_{-1}) + Au^1{}_0(\alpha^1\pi^1{}_+,\ [-\beta^1\pi^1{}_+ + \alpha^1\pi^1{}_-]/\sqrt{2},\ -\beta^1\pi^1{}_-). \quad (13.4)$$

These changes give rise to changes in the $(1)_j$ and $(0)_0$ functions which have the following parts antisymmetric in the two neutrons:

$$\text{A.P.}\quad [\Delta_A(1)_1]=[A/2(6)^{\frac{1}{2}}]\times\{S^\pi{}_1\Xi+\tilde{S}^\pi{}_0(S_0\mathfrak{A}_{1,0}-\tilde{S}_0\mathfrak{S}_{0,1})\},$$
$$\text{A.P.}\quad [\Delta_A(1)_0]=[A/2(6)^{\frac{1}{2}}]\times\{S^\pi{}_0\Xi+\tilde{S}^\pi{}_0(S_1\mathfrak{A}_{0,-1}+S_{-1}\mathfrak{A}_{0,1})\}, \quad (14)$$
$$\text{A.P.}\quad [\Delta_A(1)_{-1}]=[A/2(6)^{\frac{1}{2}}]\times\{S^\pi{}_{-1}\Xi+\tilde{S}^\pi{}_0(S_0\mathfrak{A}_{-1,0}+\tilde{S}_0\mathfrak{S}_{0,-1})\},$$

where

$$(S_1,S_0,S_{-1})=(\alpha^1\alpha^2,\ [\alpha^1\beta^2+\alpha^2\beta^1]/\sqrt{2},\ \beta^1\beta^2), \quad (14.1)$$
$$\tilde{S}_0=(\alpha^1\beta^2-\alpha^2\beta^1)/\sqrt{2} \quad (14.2)$$

are spin states of the two neutrons corresponding to resultant spins of one and zero, respectively,

$$(S^\pi{}_1,S^\pi{}_0,S^\pi{}_{-1})=(\pi^1{}_+\pi^2{}_+,\ [\pi^1{}_+\pi^2{}_- + \pi^1{}_-\pi^2{}_+]/\sqrt{2},\ \pi^1{}_-\pi^2{}_-) \quad (14.3)$$

are functions of angular momentum 1 arising by vector coupling of the $p_{\frac{1}{2}}$ protons,

$$\tilde{S}^\pi{}_0=(\pi^1{}_+\pi^2{}_- - \pi^1{}_-\pi^2{}_+)/\sqrt{2} \quad (14.4)$$

is similarly the function of angular momentum zero,

$$\Xi=S_1\mathfrak{A}_{0,-1}-\tilde{S}_0\mathfrak{S}_{0,0}+S_{-1}\mathfrak{A}_{1,0} \quad (14.5)$$

with

$$\mathfrak{A}_{mn}=u^1{}_m u^2{}_n-u^1{}_n u^2{}_m, \quad \mathfrak{S}_{mn}=u^1{}_m u^2{}_n+u^1{}_n u^2{}_m, \quad (14.6)$$

the superscripts denoting the two neutrons, as previously. It is also found that

$$\text{A.P. } \Delta_A(0)_0=(A/12)\{\tilde{S}^{\pi}{}_0\Xi-S^{\pi}{}_1(S_0\mathfrak{A}_{-1,0}+\tilde{S}_0\mathfrak{S}_{0-1}) +S^{\pi}{}_0(S_1\mathfrak{A}_{0,-1}+S_{-1}\mathfrak{A}_{0,1}) -S^{\pi}{}_{-1}(S_0\mathfrak{A}_{1,0}-\tilde{S}_0\mathfrak{S}_{0,1})\}. \quad (14.7)$$

The neutron function of the final state can be expressed as

$$\tilde{S}^{\nu}{}_0=(\nu^1{}_+\nu^2{}_--\nu^1{}_-\nu^2{}_+)/\sqrt{2}=[\mathfrak{A}_{0,-1}S_1+\mathfrak{A}_{-1,1}S_0 +\mathfrak{A}_{1,0}S_{-1}+(\mathfrak{S}_{1,-1}-\tfrac{1}{2}\mathfrak{S}_{0,0})\tilde{S}_0]/3. \quad (14.8)$$

It readily follows from Eqs. (14) · · · (14.8) that

$$(\tilde{S}^{\nu}{}_0, \text{ A.P. } \Delta_A(1)_j)=[A/(6)^{\frac{1}{2}}]S^{\pi}{}_j, \quad (j=1, 2, 3) \quad (15)$$

$$(\tilde{S}^{\nu}{}_0, \text{ A.P. } \Delta_A(0)_0)=(A/2)\tilde{S}^{\nu}{}_0. \quad (15.1)$$

The changes in the functions $u^1{}_j$ corresponding to Eq. (13.3) are, according to Eq. (13.2),

$$(u^1{}_1, u_0{}^1, u^1{}_{-1}) \rightarrow (u^1{}_1, (1+\sqrt{3}A)u^1{}_0, u^1{}_{-1}), \quad (15.2)$$

the isotropic part being now included. According to Eqs. (15) and (15.1), the inclusion of the anisotropic part thus changes the coefficients listed in (12.8) through the addition of

$$A(1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, \tfrac{1}{2}). \quad (15.3)$$

Combining the two, there results the set of coefficients

$$(1/\sqrt{2})[1+(\sqrt{3}A)/3](1,1,1,\tfrac{1}{2}\sqrt{6}), \quad (15.4)$$

where the result is expressed in terms of $\sqrt{3}A$ since this combination enters Eq. (15.2). It is seen that the effect of the anisotropy does not change the ratios of the matrix elements between the initial and final states. In the notation of Eqs. (1.7), (1.8) the isotropic part corresponds to the factors $[1/q^2+1/q^3]e^{-q}$, while $1+\sqrt{3}A$ corresponds to the factor

$$1+\sqrt{3}A \rightarrow \left[\frac{1}{q}-\frac{2}{q^2}-\frac{2}{q^3}\right]e^{-q} =\left[\frac{1}{q^2}+\frac{1}{q^3}\right]e^{-q}-\left[\frac{1}{q}+\frac{3}{q^2}+\frac{3}{q^3}\right]e^{-q}. \quad (15.5)$$

On the same scale, $1+(\sqrt{3}A)/3$ of Eq. (15.4) corresponds therefore to

$$1+\sqrt{3}A/3 \rightarrow -e^{-q}/(3q), \quad (15.6)$$

and the net effect is as though

$$\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)\left[\frac{1}{(\alpha r)}+\frac{1}{(\alpha r)^2}\right]e^{-\alpha r} \rightarrow -\left(\frac{\pi}{2}\right)^{\frac{1}{2}}[I_{\frac{3}{2}}(\alpha r')/(\alpha r')^{\frac{1}{2}}] \times\left(\frac{e^{-q}}{q}\right)\left(\frac{x'}{r'}, \frac{y'}{r'}, \frac{z'}{r'}\right), \quad (15.7)$$

instead of Eq. (1.2). The special circumstances regarding nuclear spins and configuration assignments thus resulted in making the dependence on R which enters q such as applies to an s state. It will also be noted that, according to Eqs. (12.6), (12.8), (15), and (15.1), the part of the energy matrix responsible for the coupling of the states of one nucleus to the other has elements only for the $1'_j$ coupled to the $(1)_j$, with the same j, and for $0'_0$ to 0_0. For the first three, one deals with a constant times the unit matrix as is seen from Eqs. (12.8) and (15). As the nuclei pass each other the coordinate system of Fig. 1 is rotated, and the energy matrix has been considered above with respect to this rotating system. On account of the simplifications in its form, however, it remains unchanged on transformation to the coordinates of the center-of-mass system since, for both submatrices referring to angular momenta 0 and 1, the matrix undergoes a similarity transformation and, being in both cases a constant times a unit matrix, it remains unchanged. The complexities of anisotropic transfer combined with the rotation of the axis of symmetry thus do not enter the present problem.

The colliding nitrogen nuclei can be represented by wave functions of the following types:

$$\begin{aligned}&\chi^1{}_m\chi^2{}_m[\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)+\psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)]/\sqrt{2},\\ &[(\chi^1{}_m\chi^2{}_\mu+\chi^1{}_\mu\chi^2{}_m)/\sqrt{2}]\\ &\quad\times[\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)+\psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)]/\sqrt{2}, \quad \mu\neq m, \quad (16)\\ &(1)_i[\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)-\psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)]/\sqrt{2},\end{aligned}$$

where the $\psi(\mathbf{r})$ refer to the space behavior of the centers of mass of the colliding nitrogen nuclei. In the interests of simplicity of notation, the symbols ψ_1 and ψ_2 are used here in a different sense from that employed in Eqs. (7), (7.1), and (7.2), the chance of confusion of the two meanings being small. When one takes

$$\int|\psi_1(\mathbf{r})|^2d\mathbf{r}=\int|\psi_2(\mathbf{r})|^2d\mathbf{r}=1,$$

the wave functions represent one nitrogen nucleus in the location of ψ_1 colliding with another nitrogen nucleus in the location of ψ_2. The collision of two unpolarized sets of nuclei is describable as a statistical mixture with equal probabilities of the above set of 9 functions. By forming linear combinations of the first six functions, the spin functions can be made to give angular momenta 2 and 0 and the system is describable as a statistical mixture with equal probabilities of 5, 3, and 1 states with total spins 2, 1, and 0, respectively. The space factors multiplying the spin functions are the same as correspond to the collision of two spinless identical particles with symmetric and antisymmetric space functions. The five states of spin 2 cause no reaction within the approximations of the present paper.

The calculation of transfer for identical particles can be reduced to one for nonidentical ones. It turns out to

be incorrect, however, to neglect the fact that the colliding nuclei contain particles in which the wave function is antisymmetric. For states of total nuclear spin 1, the initial unsymmetrized state may be written as

$$\Psi_i{}^U=(1)_j(1,1,1;2,2,2)$$
$$=[S_j(\pi_1(1)\pi_2(2))\tilde{S}_0(\nu_1(1)\nu_2(2))$$
$$+\tilde{S}_0(\pi_1(1)\pi_2(2))S_j(\nu_1(1)\nu_2(2))]$$
$$\times\psi(\mathbf{r}(C_1)-\mathbf{r}(C_2))/\sqrt{2}, \quad (16.1)$$

where in the $(1)_j(1,1,1;2,2,2)$ the entries in parentheses before the semicolon give in order, from left to right, the indices of the π, ν, and C (here C stands for the C^{12} core) which are grouped in one nucleus, while the entries after the semicolon similarly give the indices of π, ν, and C grouped in the other nucleus. The factor $\psi(\mathbf{r}(C_1)-\mathbf{r}(C_2))$ represents the relative motion of the two C^{12} nuclei which corresponds to the factors $\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)$ in Eq. (16). On the right side of (16.1), the index of C^{12} to which the π or ν is attached stands in parentheses of that π or ν, while the π and ν themselves are identified by subscripts. The function of Eq. (16.1) has to be made antisymmetric in π_1 and π_2, as well as in ν_1 and ν_2, but it must be made symmetric in C_1 and C_2. Antisymmetrizing in ν_1 and ν_2 is accomplished by forming

$$[1_j(1,1,1;2,2,2)-1_j(1,2,1;2,1,2)]/\sqrt{2}.$$

In a similar manner this function can be antisymmetrized in π_1 and π_2 and then symmetrized in C_1 and C_2. The result is

$$\tfrac{1}{4}\{[S_j(\pi_1(1)\pi_2(2))-S_j(\pi_2(1)\pi_1(2))]$$
$$\times[\tilde{S}_0(\nu_1(1)\nu_2(2))-\tilde{S}_0(\nu_2(1)\nu_1(2))]$$
$$+[\tilde{S}_0(\pi_1(1)\pi_2(2))-\tilde{S}_0(\pi_2(1)\pi_1(2))]$$
$$\times[S_j(\nu_1(1)\nu_2(2))-S_j(\nu_2(1)\nu_1(2))]\}$$
$$\times[\psi(\mathbf{r}(C_1)-\mathbf{r}(C_2))-\psi(\mathbf{r}(C_2)-\mathbf{r}(C_1))]=\Psi^S{}_i. \quad (16.2)$$

The unsymmetrized final state, i.e., the result of solving the wave equation starting with an unsymmetrized initial state as the unperturbed solution and subtracting the initial state, is of the form

$$\Psi_f{}^U=S_j(\pi_1(1)\pi_2(2))\tilde{S}_0(\nu_1(2)\nu_2(2))\varphi(\mathbf{r}(C_1)-\mathbf{r}(C_2)). \quad (16.3)$$

Here the factors multiplying φ are normalized to unity in the same way as ψ in (16.1) is multiplied by an internal motion-spin factor which is normalized to unity. The factor φ gives the relative motion of C_1 with respect to C_2 after collision in the same convention regarding sense of direction as ψ gives it before collision. The final-state function in (16.3) is already antisymmetric in ν_1 and ν_2. For this reason, there is one less power of $1/\sqrt{2}$ entering the preparation of the properly symmetrized final state than in the corresponding step for the initial function. The final symmetrized function is

$$\Psi_f{}^S=\tfrac{1}{2}[S_j(\pi_1(1)\pi_2(2))-S_j(\pi_2(1)\pi_1(1))]$$
$$\times[\tilde{S}_0(\nu_1(2)\nu_2(2))\varphi(\mathbf{r}(C_1)-\mathbf{r}(C_2))$$
$$-\tilde{S}_0(\nu_1(1)\nu_2(1))\varphi(\mathbf{r}(C_2)-\mathbf{r}(C_1))]. \quad (16.4)$$

The consideration of the matrix element of an interaction energy taken to be symmetric in ν_1 and ν_2, π_1 and π_2, C_1 and C_2 shows that $\Psi_i{}^S$ and $\Psi_f{}^S$ can be replaced by effective values

$$(\Psi_i{}^S)_{\text{eff}}=S_j(\pi_1(1)\pi_2(2))\tilde{S}_0(\nu_1(1)\nu_2(2))$$
$$\times\psi(\mathbf{r}(C_1)-\mathbf{r}(C_2)), \quad (16.5)$$

$$(\Psi_f{}^S)_{\text{eff}}=S_j(\pi_1(1)\pi_2(2))[\tilde{S}_0(\nu_1(2)\nu_2(2))$$
$$\times\varphi(\mathbf{r}(C_1)-\mathbf{r}(C_2))-\tilde{S}_0(\nu_1(1)\nu_2(1))$$
$$\times\varphi(\mathbf{r}(C_2)-\mathbf{r}(C_1))]. \quad (16.6)$$

In this reduction, the additional assumption is made that all interactions with the two protons π_1 and π_2 may be disregarded in the calculation of neutron transfer. This assumption is made in order to simplify the right-hand sides of Eqs. (16.5) and (16.6), making it possible to omit the second term in square brackets in Eq. (16.1). It is not essential to the correctness of the final comparison of probabilities with symmetrized and unsymmetrized functions.

The wave functions ψ and φ used in the present calculation of the matrix element have to be normalized to unity in the fundamental volume in order to make $\Psi_i{}^U$, $\Psi_f{}^U$, $\Psi_i{}^S$, and $\Psi_f{}^S$ suitable eigenfunctions for the calculation of matrix elements. The plane waves which these functions approach asymptotically give the direction in which C_1 moves with respect to C_2 in the case of the unsymmetrized function which applies to nonidentical particles. The symmetrized functions do not identify the C_1 and C_2, as has been clear in connection with Eq. (16). Nevertheless the $\Psi_f{}^S$ of Eq. (16.4) describes the final state as one in which the C_2 carries an extra neutron if its $\mathbf{r}$ occurs with a minus sign in φ and C_1 carries an extra, though different, neutron if its $\mathbf{r}$ occurs with a minus sign in φ. Since the function φ corresponds in the present consideration to a modified plane wave, both parts of the quantity in its second set of square brackets describe a N^{13} receding from a N^{15} in the same direction, which is the same as the corresponding direction for the unsymmetrized function. In the symmetrized case, the transition matrix element is a sum of two parts resulting from the use of the two terms in the square brackets of Eq. (16.6). The first of these has the same structure as the matrix element between $\Psi_f{}^U$ and $\Psi_i{}^U$. The second has the form which corresponds to the ejection of N^{13} at an angle which is the supplement of the angle for the first term. If the first term corresponds to a large-angle collision with a neutron leaving C_1, the second corresponds to a small-angle collision with a neutron leaving C_2. Both result

in the occurrence of N^{13} at a large angle, in the first case by stripping of a nucleus that suffered large-angle scattering, in the second by the stripping of the other nucleus which suffered a small-angle collision. These types of matrix elements combine linearly according to Eq. (16.6), and there is therefore an interference effect between the large- and small-angle scattering in the case of identical particles. Since the calculations below are concerned with tunneling action only and since transfer at small angles is very small in the applications, further consideration of the interference term appears to be unnecessary for the immediate purpose.

Disregarding the interference term and making φ correspond to large-angle collisions, comparison of Eqs. (16.5) and (16.6) with Eqs. (16.1) and (16.3) shows the presence of an extra factor $\sqrt{2}$ in the former. The transition probability for identical particles which have been considered, in comparison with the states $\Psi_i{}^U$ and $\Psi_f{}^U$ of the nonidentical particles, is therefore twice as large. The origin of this difference may be described as follows. Whenever the symmetrization or antisymmetrization in a pair of particles calls for the introduction of a $1/\sqrt{2}$, in the initial and final state function, as is the case for π_1 and π_2, there is brought in a factor $\frac{1}{2}$ in the transition matrix element which is compensated by the doubling of this element resulting from contributions of the two terms having the original as well as the reversed order of the arguments. The antisymmetrization in ν_1 and ν_2 does not call, however, for an extra $1/\sqrt{2}$ in the final state since in this state the function is antisymmetric in ν_1 and ν_2. On the other hand, the removal of $-\tilde{S}_0[\nu_2(1)\nu_1(2)]$ and doubling of $\tilde{S}_0[\nu_1(1)\nu_2(2)]$ is justified by (ν_1,ν_2) interchange; the interchange of C_1 and C_2 then justifies the replacement

$$\tilde{S}_0(\nu_1(1)\nu_2(2))[\psi(\mathbf{r}(C_1)-\mathbf{r}(C_2))-\psi(\mathbf{r}(C_2)-\mathbf{r}(C_1))]$$
$$\rightarrow[\tilde{S}_0(\nu_1(1)\nu_2(2))+\tilde{S}_0(\nu_1(2)\nu_2(1))]\psi(\mathbf{r}(C_1)-\mathbf{r}(C_2)).$$

A second consideration of ν_1, ν_2 interchange then shows that the removal of $\tilde{S}_0(\nu_1(2)\nu_2(1))$ can be compensated for by the introduction of a factor 2 in the matrix element. The step from the symmetrized functions to the equivalent forms of (16.5) and (16.6) thus involves one factor 2 from (π_1,π_2) and two factors 2 from the combined consideration of (C_1,C_2) and (ν_1,ν_2) interchanges, resulting in a factor 8, while the normalization factors are $\frac{1}{2}$ each for π_1, π_2 and C_1, C_2 and $1/\sqrt{2}$ from ν_1, ν_2, giving a total factor $1/(4\sqrt{2})$ from normalization and a net factor $\sqrt{2}$. The latter corresponds to the factor 2 in intensity mentioned previously. This factor is with reference to probability of transfer calculated for nonidentical neutrons on nonidentical carbon nuclei. It is the result of interference of final states obtained from C_1 loosening ν_1 with those in which C_1 loosens ν_2, and of states in which C_2 loosens ν_2 with states in which it loosens ν_1.

This fact is readily verified by considering a fictitious case of fixed nonidentical C nuclei each having a neutron attached to it at time $t=0$. If initially the neutrons are in the state $[\nu^1{}_+(1)\nu^2{}_-(2)-\nu^1{}_-(1)\nu^2{}_+(2)]/\sqrt{2}$, the transfer to the state $[\nu^1{}_+(2)\nu^2{}_-(2)-\nu^1{}_-(2)\nu^2{}_+(2)]/\sqrt{2}$ takes place at half the rate of that for a wave function antisymmetric in the neutrons and normalized to unity. In both cases there is a neutron at $t=0$ at one of the centers of force, but the antisymmetrized wave function produces the final state in two ways which interfere constructively. The probability of transfer to the state of interest is therefore twice as large for the symmetrized function. Equations (16.5) and (16.6) show also that the integrand of the matrix element contains the combination

$$\psi^*(\mathbf{r}(C_1)-\mathbf{r}(C_2))\varphi(\mathbf{r}(C_2)-\mathbf{r}(C_1)),$$

in addition to the combination containing C_1 and C_2 in the same order in ψ^* and φ. As a result, the scattering amplitude contains the combination $f(\theta)-f(\pi-\theta)$, where $f(\theta)$ is the angle-dependent part of the scattering amplitude in direction θ for nonidentical particles. In addition to the final state of total nuclear spin 1, there is the state of total spin 0 which can result in a transfer. A similar consideration gives for it also a factor 2 on account of neutron identity, but the sign in the combination $f(\theta)+f(\pi-\theta)$ occurs in place of that for the state with total spin 1. The combined effect of particle identity is then to replace the $|f(\theta)|^2+|f(\pi-\theta)|^2$ from Eq. (15.4), with allowance for observation of recoils but without consideration of particle identity, as follows:

$$|f(\theta)|^2+|f(\pi-\theta)|^2\rightarrow 2\{(2/9)\times 3|f(\theta)-f(\pi-\theta)|^2$$
$$+\tfrac{1}{3}|f(\theta)+f(\pi-\theta)|^2\}=2\{|f(\theta)|^2+|f(\pi-\theta)|^2$$
$$-\tfrac{1}{3}[f^*(\theta)f(\pi-\theta)+f(\theta)f^*(\pi-\theta)]\}. \quad (16.7)$$

The factors inside the curly braces arise as follows: In the incident wave, the relative probabilities of the four relative spin orientation substates are equal. According to Eq. (15.4), therefore, the three substates with $I=1$ and the state with $I=0$ occur in the final wave with relative probabilities having ratios $1:1:1:\frac{3}{2}$, respectively. Assuming that the reaction occurs, the *a posteriori* probability that a given substate with $I=1$ should occur is therefore $1/(1+1+1+\frac{3}{2})=2/9$. Since there are three such states, the $|f(\theta)-f(\pi-\theta)|^2$ occurs with the coefficient $(2/9)\times 3$. The relative probability of formation of $I=0$ is, on the other hand, $(3/2)\times(2/9)=\frac{1}{3}$, which is, accordingly, the coefficient of the combination $|f(\theta)+f(\pi-\theta)|^2$.

The factor 2 in front of the curly braces takes account of the effect of neutron identity derived in Eqs. (16.1) to (16.6). The other factors inside the curly braces have been adjusted as though one dealt with elastic scattering. The reasons for doing so are as follows. The cancellation of $(1/\sqrt{2})^2$, resulting from the $1/\sqrt{2}$ which occurs with the space functions in Eq. (16), takes place as in the familiar elastic scattering problem

when one sums over the possibility of finding either of the two particles in the same final direction. Since N^{13} and N^{15} are distinguishable, this doubling of the final probability does not take place. On the other hand, the neutron can be transferred in either direction and so the factor 2 appears again, provided it is agreed to make comparisons with calculations in which the neutron transfer is counted in one direction only, i.e., the transfer is counted from $\bar{u}$ to $\bar{v}$ only.

For nonidentical particles, the ratio of cross sections at $\theta=\pi/2$ and $\theta=0$ is

$$2|f(\pi/2)/f(0)|^2, \quad \text{(nonidentical)}$$

since $|f(\pi)|\ll|f(0)|$ in the applications. On the other hand, according to (16.7) the corresponding ratio for identical particles is

$$(4/3)|f(\pi/2)/f(0)|^2. \quad \text{(identical)}$$

The ratio of the two values is $\frac{2}{3}$.

In order to show the relationship to elastic scattering formulas, the quantity in curly braces in Eq. (16.7) was written omitting the factor consisting of a sum of statistical weights of the initial spin states multiplied by the probability of finding the final spin state in the initial wave function. This factor will be explicitly stated later.

Instead of going through (16.7), one could calculate directly the statistical weights of $I=0$ and $I=1$ as $\frac{1}{9}$ and $\frac{1}{3}$, the spin content factors from (15.4) as $\frac{3}{4}$ and $\frac{1}{2}$; the resultant products are $\frac{1}{12}$ and $\frac{1}{6}$ in place of $\frac{1}{3}$ and $\frac{2}{3}$ occurring inside the curly braces of (16.7), and amount to a common factor $\frac{1}{4}$. Multiplication of the right side of (16.7) by $\frac{1}{4}$ gives, therefore, the quantity by which $|f(\theta)|^2+|f(\pi-\theta)|^2$ should be replaced in order to take into account spin, particle identity, and associated statistics.

V. ANGULAR DISTRIBUTION

The angular distribution caused by tunneling action is very different from that expected according to the theory of the (d,p) stripping reactions.[15] In the latter there is a large probability for the proton to keep on moving in approximately the original direction, which is, however, affected by the recoil of the neutron in a relatively minor way. In doing so it gives rise to the angular distribution pattern of the proton. The effect of the neutron on the angular distribution of the proton depends on the orbital angular momentum of the neutron after it is captured. This process is somewhat similar to the diffraction of a wave by a sphere, with the difference that the nucleus formed after neutron capture singles out a particular orbital angular momentum l as well as the fact that the deuteron is broken up in the process. It will be noted that the maxima in approximately the forward direction, dealt with in the theory of the (d,p) reactions, correspond to distant small-angle collisions. The wave function representing the relative motion of the proton and neutron in the deuteron affects the result and contains in it a kind of tunneling effect. It will be noted, however, that the exponential factor representing the decay of the probability of finding the neutron away from the proton does not appear in the final result. The reason for this is that the Coulomb repulsion between the proton and the bombarded nucleus is neglected. All distances of the proton being equally probable, the exponential decay of the neutron probability matters only in the way it combines with an oscillatory factor in the transition matrix element which is associated with the change in the proton momentum. Accordingly, the "deuteron factor" of Bhatia *et al.* varies relatively mildly with angle, leaving the diffraction effect as the principal one.

The tunneling action in a reaction like that between N^{14} and N^{14} is, on the other hand, strongly selective in favor of close and, therefore, large-angle collisions. The tunneling factor makes the probability of small-angle events negligible and the main mechanism of the usual stripping theory is absent. The N^{13} may appear, however, in the forward or backward direction as a result of the close collisions, the two possibilities corresponding to pickup and stripping events. The distinctions between the heavy particle and deuteron cases will now be considered more concretely. In doing so, the target nucleus will be supposed to be different from the bombarding N^{14} so as to be able to distinguish between pickup and recoil events.

The N^{13} part of the projectile will be referred to by the letter P, this nucleus being a generalization of the proton in the (d,p) case. The whole bombarding nucleus will similarly be referred to by D. All distances will be taken with respect to the center of mass. The consideration below is along the lines of the Bhatia *et al.*[16] paper for the (d,p) reaction. The initial wave function of relative motion has the form of

$$\Psi_{Di}=\mathfrak{M}(r_D'-(\mathbf{k}_D\cdot\mathbf{r}_D')/k_D,\ k_D)\exp(i\mathbf{k}_D\cdot\mathbf{r}_D'). \quad (17)$$

Here $\mathbf{r}_D'=\mathbf{r}_D-\mathbf{r}_T$, with $\mathbf{r}_D$ and $\mathbf{r}_T$ standing respectively for the coordinates of D and of the target T. The wave number $\mathbf{k}_D/2\pi$ refers to the relative motion of D and T. The factor multiplying the exponential is written for the special case of the Coulomb field. The notation $\mathfrak{M}$ is used in order to indicate the connection with the confluent hypergeometric function, frequently denoted

[15] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951); Bhatia, Huang, Huby, and Newns, Phil. Mag. **43**, 485 (1952); P. B. Daitch and J. B. French, Phys. Rev. **87**, 900 (1952); F. L. Friedman and W. Tobocman, Phys. Rev. **92**, 93 (1953); W. Tobocman, Phys. Rev. **94**, 1655 (1954); W. Tobocman and M. H. Kalos, Phys. Rev. **97**, 132 (1955). In the latter two references the effect of a strong Coulomb field is considered. Here, as well as in the experimental paper of W. W. Pratt, Phys. Rev. **97**, 131 (1955), the proton maximum shifts toward 180° for cases of strong Coulomb effects.

[16] Bhatia, Huang, Huby, and Newns, reference 15.

by M, by means of which $\mathfrak{M}$ can be represented.[17] The special form of this factor will be seen, however, to be immaterial for the qualitative part of the present discussion. In the final state, the relative motion of the stripped projectile with respect to the final nucleus is represented by

$$\Psi_{Pf}=\mathfrak{M}^*(r_P'+(\mathbf{k}_P\cdot\mathbf{r}_P')/k_P,\, k_P)\exp(i\mathbf{k}_P\cdot\mathbf{r}_P'). \quad (17.1)$$

Here

$$\mathbf{r}_P'=\mathbf{r}_P-(M_n/M_f)\mathbf{r}_n-\mathbf{r}_T \quad (17.2)$$

is the displacement vector from the center of mass of the residual nucleus with mass M_f to the particle P, and $\mathbf{k}_P'/2\pi$ is the wave number of relative motion of the reaction products. In Eq. (17), the factor $\mathfrak{M}$ will be arranged to give an outgoing wave modification of the plane wave; in Eq. (18.1), the form of the factor $\mathfrak{M}^*$ will secure the desired[18] occurrence of the ingoing wave modification for the final state. In the transition matrix element there occurs the factor

$$\exp\{-i\mathbf{k}_P\cdot\mathbf{r}_P'+i\mathbf{k}_D\cdot\mathbf{r}_D'\}$$
$$=\exp\{-i(\mathbf{k}_P-M_P\mathbf{k}_D/M_D)\cdot(\mathbf{r}_P-\mathbf{r}_n)$$
$$+i(\mathbf{k}_D-M_i\mathbf{k}_P/M_f)\cdot\mathbf{r}_n\}, \quad (17.3)$$

where M_P and M_D are, respectively, the masses of P and D while M_i and M_f stand for the masses of the target nucleus and the nucleus formed by adding a neutron to M_i. The $\mathbf{r}_n$ in the above phase factor becomes $\mathbf{R}$, the vector distance of n from the center of the target nucleus when one introduces[16] an interaction energy on the surface of a sphere of radius R. There appears, accordingly, a $J_{l+\frac{1}{2}}(|\mathbf{k}_D-M_i\mathbf{k}_P/M_f|R)$ diffraction factor as a result of integration over the neutron coordinate. On the other hand, the first part of the right-hand side of Eq. (17.3) contains $\mathbf{r}_P-\mathbf{R}$, the vector from a point on the interaction shell to the N^{13} of the nucleus to be stripped. In the discussion of Bhatia *et al.*, this factor is multiplied by the internal deuteron function and gives rise to the so-called deuteron factor. The integration is then carried out including the whole range of values of $|\mathbf{r}_P-\mathbf{R}|$ from 0 to ∞. Since small values of $|\mathbf{r}_P-\mathbf{R}|$ are improbable on account of the strong Coulomb effects, this procedure obviously needs modification. In the matrix element there occurs the integral sum

$$I=\sum\int\chi_f^*(\mathbf{r}_n,s_n,\xi_T)\chi_P^*(\xi_T)\Psi_{Pf}^*(\mathbf{k}_P,\mathbf{r}_P')$$
$$\times V(\mathbf{r}_n,s_n,\xi_T)\chi_T(\xi_T)\chi_D(\mathbf{r}_n-\mathbf{r}_P,\,s_n,\xi_T)$$
$$\times\Psi_{Di}(\mathbf{k}_D,\mathbf{r}_D')d\mathbf{r}_n d\mathbf{r}_P d\xi_T d\xi_P. \quad (17.4)$$

Here the χ stands for functions describing internal relative motion, while the Ψ are as in Eqs. (17) and (17.1); T stands for target, n for neutron, ξ for the internal coordinates, and the summation is understood to be over the neutron spin coordinates s_n. The function χ_D contains barrier penetration effects, while the interaction potential V is supposed to have appreciable values only for $r_n\cong R$. Since in the present problem both Ψ_{Pf} and Ψ_{Di} have very small values when $\mathbf{r}_P$ is close to zero, the χ_D begins to count only when r_P is larger than the value allowed by the factors $\mathfrak{M}$ of Eqs. (17) and (17.1), which will be seen to correspond to the distance of closest approach in the classical treatment of the relative motion of the colliding particles. The factor χ_D under the integral thus introduces the value of χ_D for $\mathbf{R}-(\mathbf{r}_P)_{\text{cl. app.}}$ which is seen to be in qualitative agreement with the value used in the SCT calculations. The further correspondence of the transition matrix element method to the semiclassical one will be discussed presently. Before doing so, however, the change in the role of the diffraction maximum has to be discussed.

The consideration of χ_D has shown that the distant collisions, which correspond quantum-mechanically to large values of the relative orbital angular momentum L, are not important. If they were, the diffraction maximum would appear because the functions $\mathfrak{M}$ would not be important, in consequence the Ψ_{Di} and Ψ_{Pf} could be approximated by plane waves, and there would appear the usual Bessel function of the argument $\cong|\mathbf{k}_D-\mathbf{k}_P|R$; this approximation corresponds to R being multiplied by an effective k which is the absolute value of the vector difference between the initial and final heavy-particle $\mathbf{k}$. For a bombarding energy of 10 Mev, $1/k_D=7.7\times10^{-14}$ cm, and for $R=3.4\times10^{-13}$ cm the argument of the Bessel function is $8.8\sin(\theta/2)$, where θ is the scattering angle in the c.m. system. For $\theta=42°$ this quantity becomes nearly π, and for smaller values of θ it has values which do not vary too critically with R in their effect on the value of the Bessel function. The reason for this is the relative smallness of $|\mathbf{k}_P-\mathbf{k}_D|$. If, on the other hand, one attempts to use the Bessel function for large θ, a much greater sensitivity to R results. For $\theta=\pi$, one has approximately

$$\Delta[|\mathbf{k}_P-\mathbf{k}_D|R]\cong-\tfrac{1}{4}(\Delta\theta)^2(k_PR).$$

For $\Delta\theta=\frac{1}{2}$, the change in the argument is, accordingly, -0.31. The Bessel function factor is not significant therefore unless R can be defined to the fractional accuracy $0.31/4.4=0.06$. Such a close specification of the radius of the equivalent interaction shell is highly improbable. Since the shell doubtless has a larger thickness than the amount arrived at above, the employment of the Bessel function with a fixed R for the close collisions would be meaningless. The value of $\Delta\theta$ used in the estimate is comparable with the whole width of the angular distribution curve calculated on

[17] A. Sommerfeld, Ann. Physik **11**, 257 (1931).

[18] A. Sommerfeld, *Atombau und Spektrallinien* (F. Vieweg and Son, Braunschweig, 1939), Vol. 2, pp. 457 and 502; G. Breit and H. A. Bethe, Phys. Rev. **93**, 888 (1954); see the last-named reference for a review of other work.

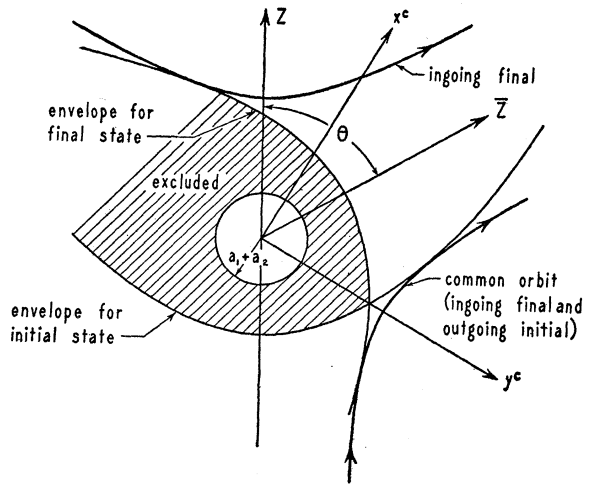

FIG. 2. Sketch showing orbits for the scattering. Here Z is the direction of incidence, and $\bar{Z}$ is the direction of scattering, θ being the scattering angle. The envelopes for the initial and final states for orbits of a given energy are the parabolas illustrated. The excluded region between the envelopes is shaded and includes the region of nuclear interaction bounded by the circle of radius a_1+a_2. The coordinates x^c and y^c convenient for describing the initial state are also shown.

a semiclassical basis for tunneling and one may assume, therefore, that except for cases in which high accuracy is needed, the variation of the Bessel function factor may be neglected.

The connection between the matrix element type of calculation and the semiclassical treatment may be seen by employing a classical dynamics approximation to the wave function. Representing any wave function as

$$\psi=\exp(iS/\hbar), \quad S=S^0+(\hbar/i)S^1+\cdots \tag{18}$$

one may interpret[19]

$$\exp(2S^1)$$

as the density of classical systems represented by S^0. The comparison of the quantum and classical results, neglecting terms in the expansion of S following S^1, will be discussed here. This approximation amounts to neglecting barrier-penetration effects of the heavy particles in the present application. When the incident wave is represented by means of Eq. (18), the trajectories obtained by classical dynamics form a set of hyperbolas which are tangent to a paraboloid of revolution. The interior of the paraboloid is excluded according to classical dynamics. The final state in the matrix calculation is the ingoing wave modification of a plane wave, and its associated classical orbits are also tangent to a paraboloid, which is not the same as the first paraboloid. The orbits of Ψ_{Di} and of Ψ_{Pf} do not coincide in the general case. If, however, one neglects the energy difference between the initial and final states, one of the orbits is the same, as is illustrated in Fig. 2. The common orbit is the orbit of the semiclassical theory approximation and the direction of the plane wave propagation vector of the final state is the final direction of the common orbit if this orbit is considered as an orbit of the initial state. For the incident state, it is convenient to introduce axes x^c and y^c related to the orbit as in Fig. 2. The classical orbit can be represented parametrically by

$$\begin{aligned} x^c&=a'(\epsilon+\cosh w), \quad y^c=a'(\epsilon^2-1)^{\frac{1}{2}}\sinh w,\\ r&=a'(1+\epsilon\cosh w), \quad t=(a'/v)(w+\epsilon\sinh w), \end{aligned} \tag{18.1}$$

where ϵ is the eccentricity, a' one-half the distance of closest approach, and v the relative velocity at infinite distance r. Referring to the figure, w varies from $-\infty$ to ∞ as the point moves up. In the approximation of Eq. (18) including S^0 and S^1 only, one finds, by a straightforward application of classical dynamics with S^0 as the action integral,

$$\begin{aligned} \psi_1=&[\epsilon e^{-w}/(\epsilon e^{-w}-e^w/\epsilon)]^{\frac{1}{2}}\\ &\times\exp\{(i\mu va'/\hbar)[-1-w+e^w/\epsilon+\ln\epsilon]+ikz\},\\ &(-\infty<w<w_0); \end{aligned} \tag{18.2}$$

$$\begin{aligned} \psi_2=&[\epsilon e^{-w}/(e^w/\epsilon-\epsilon e^{-w})]^{\frac{1}{2}}\\ &\times\exp\{(i\mu va'/\hbar)[-1-w+e^w/\epsilon+\ln\epsilon]+ikz\},\\ &(w_0<w<\infty). \end{aligned} \tag{18.3}$$

Here w_0 is the value of w for contact with the paraboloid, and μ the reduced mass for the collision. The function ψ_1 represents the condition for approach, ψ_2 for recession. Removing from ψ_1 and ψ_2 the factor

$$\exp(-ika'-i\eta\ln\eta),$$

one obtains in the case of ψ_1 directly the asymptotic form of the quantum-mechanical plane wave part of the wave function; and in the case of ψ_2 the outgoing part of the hypergeometric function solution is reproduced provided the Coulomb phase shift σ_0 is approximated by its asymptotic value for large η, *viz.*,

$$\sigma_0\sim\pi/2+\eta\ln\eta-\eta.$$

Since η is large in the present problem and since the factor of absolute value 1 is immaterial for the conclusions, the above representation is seen to contain the characteristic features of the quantum ψ. The coordinate z occurring in Eqs. (18.2) and (18.3) can be expressed as

$$z=a'(-1+\epsilon\sinh w-e^w/\epsilon). \tag{18.4}$$

For the final state there is another set of axes x_f^c, y_f^c, z_f and another orbit parameter w_f. The z_f axis is the direction of the final plane wave and x_f^c, y_f^c are obtained from x^c, y^c by a rotation, with the convention that they coincide with x^c, y^c for the common orbit. One obtains the representation of orbits in the final

[19] W. Pauli, *Handbuch der Physik* (Verlag. Julius Springer, Berlin, 1933), Vol. 24, 1, p. 166 ff.

state by

$$x_f^c=a'(\epsilon+\cosh w_f),\quad y_f^c=-a'(\epsilon^2-1)^{\frac{1}{2}}\sinh w_f,$$
$$r=a'(1+\epsilon\cosh w_f). \tag{18.5}$$

One has a convenient interchange of the forms of the two parabolic coordinates in terms of the parameters w_f and w, and one obtains the ingoing modification of the plane wave from the outgoing one without a separate calculation. For the common orbit, comparison of (18.5) with (18.1) shows that

$$w_f=-w. \tag{18.6}$$

Upon making this change, it is found that for the common orbit the final state is represented by

$$\varphi_1=[\epsilon e^w/(\epsilon e^w-e^{-w}/\epsilon)]^{\frac{1}{2}}$$
$$\times\exp\{-i(\mu va'/\hbar)[-1+w+e^{-w}/\epsilon+\ln\epsilon]+ikz_f\},$$
$$(w_{0f}<w<\infty); \tag{18.7}$$
$$\varphi_2=[\epsilon e^w/(e^{-w}/\epsilon-\epsilon e^w)]^{\frac{1}{2}}$$
$$\times\exp\{-i(\mu va'/\hbar)[-1+w+e^{-w}/\epsilon+\ln\epsilon]+ikz_f\},$$
$$(-\infty<w<w_{0f}). \tag{18.8}$$

Here w_{0f} is the value of w for contact with the second paraboloid. In Eqs. (18.2) and (18.3), the factors occurring in addition to $\exp(ikz)$ give an approximation to $\mathfrak{M}$ of Eq. (17); in Eqs. (18.7) and (18.8), the factors occurring in addition to $\exp(ikz_f)$ represent the $\mathfrak{M}^*$ of (17.1). If one were to disregard the difference in the phase of the initial and final states which arises on account of these representations of $\mathfrak{M}$ and $\mathfrak{M}^*$, one would be dealing with a combination $\exp[ik(z-z_f)]$ which gives rise to the diffraction maximum of ordinary stripping theory. Neglect of the additional phases is not justifiable, however. In fact, the phase of $\psi\varphi^*$ for points on the common orbit is obtainable from Eqs. (18.2), (18.3), (18.7), and (18.8) as

$$\text{Phase}(\psi\varphi^*)=2ka'[\ln\epsilon-2], \tag{19}$$

use having been made of

$$z_f=a'(1+\epsilon\sinh w+e^{-w}/\epsilon). \tag{19.1}$$

The fact that the phases agree within a constant is not peculiar to the present problem. Each phase can be obtained as

$$\int\sum p_jdq_j+\text{const}, \tag{19.2}$$

the integral being taken along the orbit. While an appeal to Eq. (19.2) would have been sufficient from a logical viewpoint, the results of the calculation verifying this conclusion in detail have been given above because they show some of the conditions needed for making this application of classical dynamics valid. Thus it shows that a large value of η is essential and that the agreement between classical and quantum formulas for Rutherford scattering is associated with an agreement of relative phases of the incoming plane wave and the outgoing spherical wave, which gives an added reason for believing results of the semiclassical approximation.

In addition to the phase being the same on the common orbit, the changes in phase vanish in first order in its vicinity, as is seen from an expansion of the difference of the two S^0:

$$S^0-S^{0\prime}=C+\sum_j(p_j-p_j')\Delta q_j+\cdots. \tag{19.3}$$

Here C is the constant phase difference referred to in Eq. (19), while Δq_j is the displacement of a point from a point on the common orbit. The term p_j-p_j' is obtained by employing the relation $p_j=\partial S^0/\partial q_j$. For the common orbit,

$$p_j=p_j', \tag{19.4}$$

and hence the first-order effects vanish in much the same way as in a proof of Hamilton's principle. The common orbit is, therefore, surrounded by a volume within which only second-order variations of the phase are important. This volume contributes especially well to the value of the matrix element. In the limit of very short wavelengths one obtains, as will be shown elsewhere, a description by the matrix element which degenerates exactly into the classical orbit description. But even in the present case it is seen that the main contributions come from the vicinity of the classical orbit, and that this circumstance is inseparable from the disappearance of precisely the phase difference which is responsible for the appearance of the diffraction pattern. The latter is caused by parts of coordinate space which have little to do with the common orbit as is seen in the case of undistorted plane waves, for which there is no common orbit except for the trivial case of no deflection. This type of process is strongly modified, however, by the expulsion of the wave function from the interior of the paraboloids. It is thus seen that in the heavy-particle case, the essentials of the angular distribution should be obtainable by considering motion in orbits on a classical mechanics basis.

Employing the parametrization of Eq. (18.1), the transfer probability amplitude is obtained by integrating β of Eq. (4.8) over the orbit and is, therefore, proportional to

$$\oint(e^{-\alpha R}/R)dt=(2/v)\int_0^\infty\exp\{-\alpha a'(1+\epsilon\cosh w)\}dw. \tag{20}$$

Here the disappearance of higher powers of $1/q$ which had led to the derivation of Eq. (15.7) is used, and the r of Eq. (18.1) is replaced by R of Eq. (1.2), the two symbols being used in the same sense. From Eq. (20), it follows that

$$\oint(e^{-\alpha R}/R)dt=(2/v)K_0(\alpha a'\epsilon)\exp(-\alpha a'), \tag{20.1}$$

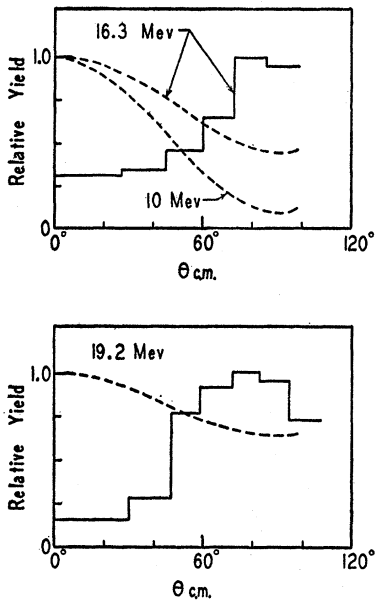

FIG. 3. Relative angular distribution of N^{13} nuclei from the reaction $N^{14}+N^{14}$ at various bombarding energies. The solid histograms present the results of experimental measurements, while the predictions from tunneling theory are indicated by dashed lines. The theoretical curves are plotted neglecting particle identity but including the effect of recoils.

where K stands for the Bessel function of an imaginary argument of the second kind. The asymptotic expansion

$$K_0(x)\sim\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}e^{-x}\left(1-\frac{1}{8x}+\frac{9}{128x^2}\cdots\right) \quad (20.2)$$

gives the approximation

$$\oint(e^{-\alpha R}/R)dt\cong[(2\pi)^{\frac{1}{2}}/v](\alpha a'\epsilon)^{-\frac{1}{2}}\times\exp[-\alpha a'-\alpha a'\epsilon], \quad (20.3)$$

which can also be derived directly from Eq. (20) by approximating the integral as the integral of a Gauss function. The angular dependence enters here through ϵ only, and for intensities it may be represented by

$$[\exp(-2\alpha\epsilon a')]/(\alpha\epsilon a'). \quad (20.4)$$

The eccentricity ϵ is connected with the impact parameter p by

$$\epsilon^2=1+(p^2/a'^2), \quad (20.5)$$

from which it follows that

$$\epsilon d\epsilon=pdp/a'^2. \quad (20.6)$$

When one considers all collisions in an annular region between p and $p+dp$, the probability of transfer is seen to be proportional to

$$\exp(-2\alpha\epsilon a')d\epsilon. \quad (21)$$

In the applications of these formulas which are made below, the terms in Eq. (20.2) which have been neglected in obtaining (20.3) are not important and the approximation of Eq. (21) will therefore be used. The scattering angle in the system of the center of mass will be referred to as θ. It is related to ϵ by

$$\epsilon^2\sin^2(\theta/2)=1, \quad (21.1)$$

which gives

$$\epsilon d\epsilon=-\tfrac{1}{2}\sin^{-4}(\theta/2)\sin\theta d\theta, \quad (21.2)$$

the factors having been arranged so as to bring the solid angle subtended by final directions into evidence. The factor in front of the solid angle is seen to be proportional to the Rutherford scattering per unit solid angle. The relative number of transfers per unit solid angle of final directions follows from Eqs. (21) and (21.2). Without the use of the asymptotic form of K_0 it is

$$[1/\sin^4(\theta/2)][K_0(\alpha a'/\sin(\theta/2))]^2, \quad (23)$$

while with it the effect is proportional to

$$[1/\sin^3(\theta/2)]\exp[-2\alpha a'/\sin(\theta/2)]. \quad (23.1)$$

The angular distribution function is seen to be independent of the value of the assumed nuclear radius. This circumstance follows directly from the fact that R occurs in Eq. (15.7) only through q and that the other quantities in (15.7) are independent of the orbit. Qualitatively, the absence of the nuclear radius b in the angular distribution formula may be considered to follow from the fact that a decrease in the assumed value of the nuclear radius adds the same amount to the distance through which the neutron must penetrate, independently of the value of the impact parameter. By means of Eq. (23.1) there is obtained the angular distribution plotted in Fig. 3. Comparison with the experimental curves shows no similarity. The fact that the theoretical result does not involve b appears to be significant in this comparison. The theoretical curves are plotted neglecting particle identity but including the effect of recoils. According to Eq. (16.7), the theoretical curves should be lowered by a factor 2/3 at $\theta=90°$ as a result of particle identity. The disagreement between tunneling theory and experiment is essentially unchanged by the inclusion of this factor.

VI. COMPARISON WITH EXPERIMENT

The considerations of nucleon transfer made here have assumed that the colliding N^{14} nuclei do not come into contact. If the nuclear radius is written as

$$a=r_0A^{1/3}$$

then this condition demands that the laboratory system bombarding energy be less than E_c, where E_c is given for the $N^{14}+N^{14}$ collision by

$$E_c=(29.2/r_0)\times(10^{-13}\text{ Mev-cm}).$$

For $r_0=1.4\times10^{-13}$ cm, corresponding to a radius 3.38×10^{-13} cm, this energy is 20.9 Mev. The experimental cross-section curve as a function of energy begins to level off above about 16 Mev, which apparently indicates the onset of compound nucleus forma-

tion. The comparison with experiment will be limited to the energy region below E_c.

The total cross section for neutron transfer may be obtained quite readily from the various formulas now available. The procedure is to calculate β from the second of the forms (4.9), and to use this result in Eq. (11) to obtain the probability of transfer for a particular orbit of the heavy particles. This result is then summed over all possible orbits. The effects of spin are shown in Eq. (15.4), and of the identity of the particles in Eq. (16.7).

As has been demonstrated in Eq. (15.7), the factor $1+A/\sqrt{3}$ in Eq. (15.4) has the effect only of replacing the h_{2-} of Eqs. (1.3) and (1.5), and the h_{1+} of Eqs. (1.7) and (1.8), by effective h's:

$$h_{2-}=-(\pi/2)^{\frac{1}{2}}[I_{\frac{3}{2}}(\alpha a_2)/(\alpha a_2)^{\frac{1}{2}}](e^{-q}/q),$$
$$h_{1+}=-(\pi/2)^{\frac{1}{2}}[I_{\frac{3}{2}}(\alpha a_1)/(\alpha a_1)^{\frac{1}{2}}](e^{-q}/q). \quad (24)$$

Using these expressions in Eq. (3.1) and then substituting in Eq. (4.9), one obtains in succession

$$\Gamma_{1-}/\alpha=-(2/z)-z/(1+z), \qquad z=\alpha a_1$$
$$\Gamma_{1+}/\alpha=-(2/z)+(z\sinh z)/(z\cosh z-\sinh z),$$
$$\gamma_1\lambda_1=\frac{\alpha z^2 e^z}{(1+z)[\sinh z-z\cosh z]},$$

and similarly for $\gamma_2\lambda_2$. Also,

$$P_+=(a_2^2/a_1^2)\frac{\sinh z-z\cosh z}{(1+z)q}\exp[-q+\alpha a_2]$$

and

$$\beta=(1/\lambda_1\lambda_2)^{\frac{1}{2}}[\alpha a_1/(1+\alpha a_1)][\alpha a_2/(1+\alpha a_2)](1/R)\times\exp\{-\alpha(R-a_1-a_2)\}. \quad (24.1)$$

Here the nuclear radii are denoted by a_1 and a_2, and R is the distance between the centers of the nuclei. The factor $(1/\lambda_1\lambda_2)^{\frac{1}{2}}$, which depends only upon the properties of the isolated nuclei, is the only place in which the internal wave function of a nucleon in the nucleus enters. Thus a separation between properties of the nucleus and of the dynamics of the collision has been achieved. The expression for β is not exact, since powers of $\alpha^2-\alpha_{10}^2$ or $\alpha^2-\alpha_{20}^2$ higher than the first have been neglected. The exact expression is of the same form, but with λ_1 and λ_2 replaced by similar derivatives evaluated at an energy intermediate to the energy of the nucleon in the isolated nucleus and the actual energy. Since λ_1 and λ_2 are found to be relatively insensitive to changes in energy of this magnitude, the use of the approximate form of β introduces no appreciable error.

With β determined, Eq. (11) may be solved to find the probability of transfer. If the system is chosen to be in a state characterized by

$$|\bar{a}_u|=1, \quad \bar{a}_v=0 \text{ at } t=-\infty,$$

a first-order calculation of the probability of transfer yields

$$|\bar{a}_v(+\infty)|^2=(\hbar/2M)^2\left|\int_{-\infty}^{\infty}e^{-i\omega t}\beta dt\right|^2, \quad (24.2)$$

in which $\hbar\omega=\bar{E}_u-\bar{E}_v$ is the energy difference of the two states $\bar{u}$ and $\bar{v}$. The factor $e^{-i\omega t}$ contributes only $\cos\omega t$ to the integrand, the $\sin\omega t$ part vanishing on integration since β is an even function of t. Numerical estimates indicate that in the case of $N^{14}+N^{14}$ collisions, the replacement of $\cos\omega t$ by 1 leads to an overestimate of the total cross section by only about 2%; accordingly, this simplification will be made here. The particular classical orbit chosen for the motion of the system enters through the relation between R and t. These quantities are conveniently expressed in terms of ϵ and w by Eq. (18.1). If the variation in α due to the change in β over the orbit is neglected, the substitution of Eq. (24.1) in Eq. (24.2) and the subsequent integration over orbits leads to the result

$$|\bar{a}_v(+\infty)|^2$$
$$=\left(\frac{\hbar}{Mv}\right)^2\frac{1}{\lambda_1\lambda_2}\left(\frac{\alpha a_1}{1+\alpha a_1}\right)^2\left(\frac{\alpha a_2}{1+\alpha a_2}\right)^2$$
$$\times\exp\{-2\alpha(a'-a_1-a_2)\}[K_0(\alpha a'\epsilon)]^2$$
$$\cong\frac{\pi}{2}\left(\frac{\hbar}{Mv}\right)^2\frac{1}{\lambda_1\lambda_2}\left(\frac{\alpha a_1}{1+\alpha a_1}\right)^2\left(\frac{\alpha a_2}{1+\alpha a_2}\right)^2(1/\alpha a'\epsilon)$$
$$\times\exp\{2\alpha(a_1+a_2-a'-a'\epsilon)\}. \quad (25)$$

The second of these forms is readily obtained directly from the orbital integral without the use of Bessel functions of imaginary argument.

The change in ϵ corresponding to a change of $\hbar$ in the relative orbital angular momentum is small compared with $1/(\alpha a')$ for close collisions. The total cross section may be obtained, therefore, by integrating Eq. (25) over all orbits and multiplying by the proper factors to take account of effects of spin and of particle identity. These factors have been discussed previously; the consideration of spin gives a factor

$$\tfrac{1}{3}\times(1/\sqrt{2})^2+\tfrac{1}{9}\times(\sqrt{3}/2)^2=\tfrac{1}{4},$$

the $\frac{1}{3}$ and $\frac{1}{9}$ being the statistical weights of triplet and singlet states, the $1/\sqrt{2}$ and $\sqrt{3}/2$ being the amplitudes with which these spin functions are contained in the initial state function. Thus, as has already been seen right after Eq. (16.7), the right-hand side of that equation should be multiplied by $\frac{1}{4}$ to give the number of N^{13} nuclei per unit solid angle. Neglecting the relatively small interference term in (16.7) and applying the factor $\frac{1}{4}$, there is left $\frac{1}{2}|f(\theta)|^2+\frac{1}{2}|f(\pi-\theta)|^2$ which is equivalent to $|f(\theta)|^2$ for total cross-section calculation. The latter quantity is, however, the intensity per unit solid angle which would be obtained if one used β of

Eq. (24.1) and calculated transfer between nonidentical nuclei taking into account only the $\bar{u}\rightarrow\bar{v}$ process, i.e., neglecting the possibility of right to left as well as left to right transfers. This calculation is just what has been done in Eq. (24.2). Remembering that the annular area $2\pi pdp$, with p standing temporarily for the impact parameter, may be written as $2\pi a'^2\epsilon d\epsilon$, the total cross section for neutron transfer becomes

$$\sigma_n=2\pi a'^2\int_1^\infty |\bar{a}_v|^2\epsilon d\epsilon, \tag{25.1}$$

which gives for a final result

$$\sigma_n=\frac{\pi^2}{2}\left(\frac{\hbar}{Mv}\right)^2\left(\frac{1}{\alpha^2\lambda_1\lambda_2}\right)\left(\frac{\alpha a_1}{1+\alpha a_1}\right)^2\left(\frac{\alpha a_2}{1+\alpha a_2}\right)^2 \times\exp\{-2\alpha(2a'-a_1-a_2)\}. \tag{25.2}$$

This form is readily obatined from Eq. (25) by employing the second of the two forms for $|a_v(+\infty)|^2$. It is also possible to do the integration over orbits exactly, employing

$$\int_x^\infty K_0^2(x)xdx=\tfrac{1}{2}x^2[K_1^2(x)-K_0^2(x)].$$

The quantity $K_1^2(\alpha a')-K_0^2(\alpha a')$ replaces then the factor $[\pi/2(\alpha a')^2]\exp(-2\alpha a')$ present in Eq. (25.2). The quantity $1/\lambda_1$ of Eq. (4.1) may be expressed in terms of the mean value of ψ_1^2 on the nuclear surface as follows. Denoting q of Eq. (7) temporarily as q_0, all coordinates collectively as q, and designating the value of E as an additional argument of ψ_1, one has from the wave equation

$$-\frac{\hbar^2}{2M}\sum_0^n \mathrm{div}_j[\psi_1(q,E')\nabla_j\psi_1(q,E) -\psi_1(q,E)\nabla_j\psi_1(q,E')]=(E-E')\psi_{1,E}\psi_{1,E'}. \tag{26}$$

For simplicity, the designation of the spins and the occurrence of spin operators are not indicated here. The equation will be integrated over the portion of configuration space restricted by $r<a_1$ but not subject to other restrictions. By Green's theorem, the terms with $j=1, 2, \cdots, n$ in Eq. (26) give no contribution since ψ_1 vanishes at infinitely distant points of the three-dimensional spaces $q_1, q_2, \cdots, q_n$. The left side thus gives

$$-\frac{\hbar^2}{2M}\int dq_1dq_2\cdots dq_n\int[\psi_1(q,E')\partial\psi_1(q,E)/\partial r_0 -\psi_1(q,E)\partial\psi_1(q,E')/\partial r_0]dS, \tag{26.1}$$

where dS is the element of area on the spherical surface $r_0=a_1$. Since at $r_0\geq a_1$ the transferable particle is not interacting with the parent nucleus, the function ψ_1 may be factored as in Eq. (7.1). On account of the employment of energies differing from that of the bound state, the function h_- cannot be used, however. In fact, imposing the boundary conditions on $q_1, q_2, \cdots, q_n$ defines the radial factor occurring in (7.1), and except at the eigenenergy this factor will not vanish at $r_0=\infty$ as h_- does. The radial function thus defined will be referred to as $\mathfrak{h}_-(r_0)$, and from here on the subscript 0 of r_0 will be dropped. Employing the normalization

$$\int\varphi_1^2dq_1dq_2\cdots dq_n=1, \tag{26.2}$$

one obtains for the term displayed in Eq. (26.1) a value in terms of a difference of the logarithmic derivative of $\mathfrak{h}_-$ which, when introduced in Eq. (26), gives, on going to the limit $E=E'$,

$$\partial[\partial\mathfrak{h}_-/\mathfrak{h}_-\partial r]=-(2M/\hbar^2)(3/N_1^2)(1/4\pi a_1^2) \times\int_{r<a_1}\psi_1^2(q,E)dq_0dq_1\cdots dq_n. \tag{26.3}$$

Introducing the mean over the hypersurface corresponding to $r=a_1$ in the $3(n+1)$-dimensional space, *viz.*,

$$\langle\psi_1^2\rangle_S=(N_1^2/3)\int\varphi_1^2(q_1,\cdots q_n)dq_1dq_2\cdots dq_n=N_1^2/3, \tag{26.3'}$$

and observing that

$$\partial/\partial(\alpha^2)=-(\hbar^2/2M)\partial/\partial E,$$

there results

$$\frac{\partial\mathfrak{l}_1}{\partial(\alpha^2)}=\frac{\int_{r<a_1}\psi_1^2(q,E)dq_0dq_1\cdots dq_n}{4\pi a_1^2\langle\psi_1^2\rangle_S}, \tag{26.4}$$

with

$$\mathfrak{l}_1=\partial\mathfrak{h}_-/\mathfrak{h}_-\partial r \tag{26.5}$$

which corresponds to the meaning of $\mathfrak{l}_1$ in Eq. (4.1). On the other hand, one obtains similarly, making use of the radial equation for h_-,

$$\frac{\partial\mathfrak{l}_{1-}}{\partial(\alpha^2)}=\frac{-\int_{r>a_1}\psi_1^2(q,E)dq_0dq_1\cdots dq_n}{4\pi a_1^2\langle\psi_1^2\rangle_S}. \tag{26.6}$$

From Eqs. (26.4), (26.6), and (4.1) there follows[20]

$$-1/\lambda_1=[4\pi a_1^2\langle\psi_1^2\rangle_S]\Big/\int\psi_1^2dq_0dq_1\cdots dq_n, \tag{27}$$

[20] The integral containing ψ_1^2 is extended over all space and is very similar to the analogous integral occurring in the calculation of partial widths of well-defined resonance levels as in G. Breit and F. L. Yost, Phys. Rev. **47**, 508 (1935); **48**, 203 (1935); G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936); G. Breit,

which is the probability of finding the particle within a spherical shell of unit thickness at the nuclear surface. In the case of a potential well model, this relation becomes

$$-1/\lambda_1=[a_1R(a_1)]^2\Big/\int_0^\infty R^2(r)r^2dr, \quad (27.1)$$

with R designating the radial function. The exponential factor is just the usual probability for penetration through a barrier of length $2a'-a_1-a_2$, the minimum distance between nuclei. Finally, $\hbar/Mv$ is the wavelength of a nucleon moving along with the nucleus at a velocity v. The factors $[\alpha a_1/(1+\alpha a_2)]^2$ and $[\alpha a_2/(1+\alpha a_2)]^2$ are peculiar to the transfer of nucleons in p-states and do not occur in analogous calculations for the transfer of an s nucleon.

Estimates of the neutron cross section have been made at laboratory system bombarding energies of 10 and 15 Mev, using a square well potential to determine λ_1 and λ_2. These energies are below the point at which the experimental cross-section curve begins to turn over. For such a potential, the parameter λ appearing in the cross-section formulas is readily found from Eq. (27.1). The result is

$$\lambda_1=-(a_1/2)[(\kappa^2+\alpha^2)/\kappa^2](1+\alpha a_1)^{-2}\times(3+3\alpha a_1+\alpha^2a_1^2), \quad (27.2)$$

where κ, 2π times the wave number of the particle in the well, is determined by the matching condition

$$\kappa^2/(\kappa a_1\cot\kappa a_1-1)=\alpha^2/(1+\alpha a_1). \quad (27.3)$$

The radius of each of the nuclear wells was chosen to be equal to 3.38×10^{-13} cm, which corresponds to a constant $r_0=1.4\times10^{-13}$ cm for $A=14$ in the usual $A^{1/3}$ formula. The depths of the wells were chosen so as to give the proper neutron binding energies, and turn out to be 33.6 and 33.0 Mev for N^{14} and N^{15}, respectively. The value of α used is that appropriate to an isolated nucleus, i.e.,

$$\alpha=\alpha_{10}=7.28\times10^{12}\ \text{cm}^{-1},$$

if 1 designates the nucleus giving up the neutron. If the maximum value of β encountered, the value at the perihelion of the $\epsilon=0$ orbit, is used to calculate a new α from Eq. (5.1), there results for 15-Mev bombardment

$$\alpha=7.31\times10^{12}\ \text{cm}^{-1}.$$

This value, if used in Eq. (25.2), decreases the cross section by less than 1%, and hence the neglect of the variation of α over the orbit is justified. The change is much smaller at 10 Mev. The smallness of this effect may be understood by recalling that the only way for the shift in α to be large is for, β itself to be large, implying that the penetration factor e^{-q}/q is large and $R=q/\alpha$ is small. However, the change in α will not in turn affect β appreciably just because R is itself small, so that the change in q is not appreciable.

TABLE II. Neutron transfer cross section σ_n in cm² for the $N^{14}+N^{14}$ reaction.

	$E'=10$ Mev	$E'=15$ Mev
Theory (using $a=3.38\times10^{-13}$ cm)	1.29×10^{-30}	8.40×10^{-28}
Theory (using $a=4.0\times10^{-13}$ cm)	5.86×10^{-30}	3.79×10^{-27}
Experiment	5.0×10^{-29}	9.0×10^{-28}

The results of the calculation are given in Table II, together with corresponding values using a larger nuclear radius $a=4.0\times10^{13}$ cm, which has been suggested for the N^{14} nucleus. Experimental values[13] of the cross section are also given. It will be noted that the absolute values of the cross section are not completely out of line with experiment, but that the ratio of the cross section at 15 Mev to that at 10 Mev is about 36 times as large as that observed. This discrepancy, which is unaffected by the use of the larger nuclear radius, appears to be particularly significant since according to the considerations outlined here, the energy dependence should be contained only in $1/v^2$ and in a', the distance of closest approach. It would seem difficult to explain this difference without introducing an additional process which would be relatively more important at lower energies. In all of the work it was assumed that transfer takes place to the ground state. Any transfer to excited states would give a dependence on incident energy which is even more rapid than the already too-strong dependence obtained under this assumption.

The treatment thus far has assumed that a neutron is transferred between the heavy nuclei. If, on the other hand, a proton is transferred, then special circumstances enter because of the Coulomb interaction between proton and nucleus, and the preceding treatment must be modified. Thus, for example, the wave function of the proton in the region between the two nuclei no longer satisfies Eq. (1.1), but instead must be a solution of the Schrödinger equation in the field of two point charges. The simple functions of Eq. (1) may no longer be used; in fact, the energy eigenfunctions now will not correspond to definite values of the orbital angular momentum about O'. It is true that it still would be possible to define adiabatic functions which approach functions of definite angular momentum about O' as the separation of the nuclei increases. These functions could be analyzed in terms of a set of similar functions centered about O, and the method of the energy matrix outlined previously could then be applied.

Phys. Rev. **58**, 1068 (1940). This connection is a natural one since the escape probability of a particle into free space must be related to the probability of its transfer into another distant nucleus. The quantities entering in the present paper are similar in form to the reduced widths of E. P. Wigner, Phys. Rev. **70**, 606 (1946) and E. P. Wigner and L. E. Eisenbud, Phys. Rev. **72**, 29 (1947), but differ from the latter through a different meaning of the word level and the related employment of a different region of integration.

In practice, however, it would be difficult to construct these functions.

An estimate of the relative cross sections for proton and neutron transfer can be obtained from a consideration of the tunneling probabilities on a simplified one-dimensional model. For this, the transfer is considered to occur only along the line joining the centers of the two nuclei; hence, the barrier to penetration may be replaced by a one-dimensional barrier of width equal to the distance between the nuclear surfaces along this line. The previous considerations in connection with Eq. (25.2) show that the quantity of significance is the probability of finding the neutron originally associated with one nucleus at the surface of the other nucleus, i.e., the probability for a neutron penetrating the barrier between the two nuclei. The ratio of proton to neutron cross sections is then just the ratio of the penetration probabilities for proton and neutron. These probabilities may be calculated for the simple one-dimensional model proposed by the JWKB method; the result is proportional to the integral over orbits of factors

$$\exp\left\{-2\int(2M/\hbar^2)^{\frac{1}{2}}(V-E)^{\frac{1}{2}}dr\right\},$$

in which the integral extends over the distance between nuclear surfaces and the integrand is to be evaluated for the potential V and nucleon energy E applicable for proton or neutron transfer. The effect of the transfer of p nucleons may be considered by including in V the centrifugal barrier. The integral of such factors over the orbits would in general be rather complicated; however, use may be made of the fact that almost all of the transfer occurs at the perihelion of the orbit. Then the ratio of the proton cross section to the neutron cross section may be approximated by

$$\sigma_p/\sigma_n=\exp\left\{-2\int_a^{2a'-a}(2M/\hbar^2)^{\frac{1}{2}}\times[(V_p-E_p)^{\frac{1}{2}}-(V_n-E_n)^{\frac{1}{2}}]dr\right\}. \quad (28)$$

The quantity V_n-E_n appearing here is just the negative of the binding energy for the neutron in N^{14}. The relationship between this approximation and the previous treatment is clear, since the neutron factor may be written as

$$\exp\left\{-2\int_a^{2a'-a}(2M/\hbar^2)^{\frac{1}{2}}(V_n-E_n)^{\frac{1}{2}}dr\right\}=\exp\{-\alpha(2a'-2a)\}, \quad (28.1)$$

which gives the dominant dependence of the cross section for transfer upon the energy of the state involved. The various other factors, involving the probability for the nucleon being outside the nuclear surface, may be assumed to be approximately the same for neutron and proton, if the transfer of a p nucleon is involved in both cases. The various factors due to the spins of the nucleons and nuclei are exactly the same, of course.

For proton transfer, the potential V_p must include also the Coulomb part of the potential produced by each nucleus. One additional circumstance must be kept in mind for the proton case. The energy E_p appearing in the integrand is not the energy of the level in the isolated nucleus, but rather the energy of this level in the Coulomb field of the other nucleus. The raising of this level is, to first order,

$$\Delta E_p=(Z'e^2)/(2a'), \quad (28.2)$$

where Z' is the charge of the nucleus receiving the proton.

If one uses $a=3.38\times10^{-13}$ cm as the nuclear radius, the logarithm of the ratio of proton to neutron transfer cross sections at 10-Mev bombarding energy is

$$\log_{10}(\sigma_p/\sigma_n)=0.194,$$

while experimental measurements indicate that this ratio is about 0.294. This result is not too surprising, since the crude estimate above has neglected the three-dimensional nature of the barrier and the possibility of transfer at points other than the perihelion of the orbit. The height of the Coulomb barrier decreases in directions away from the line of centers, and also as the separation of the nuclei increases. The above estimate has thus been made with the maximum Coulomb barrier present, and more refined estimates should enhance the probability of proton transfer. It may be mentioned that these results are relatively insensitive to the value of the nuclear radius, since the total Coulomb barrier is almost of constant height.

At 15-Mev bombarding energy, corresponding to near contact of the nuclei, the neutron and proton cross sections are estimated to be about the same, in substantial agreement with experiment. The variation in the Coulomb barrier with path of transfer and separation of the nuclei might be expected to be of less importance for this case.

The effects upon the transfer cross section of possible static distortions of the nuclei during the collision have been estimated, and found not to change the cross sections significantly. The considerations, which follow those given elsewhere,[4] are based upon a liquid drop model. The change in the length of the neutron barrier due to a distortion of the nucleus proper is found to reduce the total neutron cross section by a factor varying from $\frac{1}{6}$ to $\frac{1}{2}$ at 15 Mev, and from $\frac{1}{3}$ to $\frac{3}{4}$ at 10 Mev, depending upon the amount of distortion assumed. The ratio of the cross section at 15 Mev to that at 10 Mev could be reduced at most to $\frac{1}{2}$ its previous value. The

change in the height of the Coulomb barrier, and the distortion of the proton wave function, give essentially negligible changes in the ratio of proton to neutron cross sections at 10 Mev.

It is thus seen that the dependence of the total cross section on energy, as well as the dependence of the differential cross section on angle, indicate the presence at low energies of an additional process, possibly virtual state formation.

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Alpha-Particle Model of C^{12}

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The α-particle model for C^{12} has been re-examined. In addition to correlating the 0^+, 2^+, and 0^+ states at 0, 4.43, and 7.65 Mev, respectively, two possible identifications are given for the 9.61-Mev level: 1^- or 2^+. These levels completely determine the model, and the position and character of all levels up to 15 Mev are given. The main defect of the model is its prediction of a 3^- state at 5.53 Mev which has never been observed. The separation of the α particles in C^{12} is 3.7×10^{-13} cm and the mean zero-point kinetic energy per vibrational degree of freedom is about 2 Mev.

WHEN the α-particle model was first discussed, it was impossible to evaluate in detail its predictions of level schemes for light nuclei because of insufficient experimental information. This situation is now greatly improved. Dennison,[1] for example, has correlated a considerable number of states in O^{16} with this model. To determine whether the agreement is restricted to just this nucleus, the α-particle model for C^{12} has been re-examined. The physical basis of the α-particle model will not be discussed here,[2] although it is certainly open to question, nor will its position in the over-all theory of nuclear structure be evaluated.

In the α-particle model of C^{12} the equilibrium configuration is an equilateral triangle of side s with the α particles at the vertices. Only small displacements from equilibrium are considered and it is assumed that rotation and vibration are separable. The potential energy is

$$V=\tfrac{1}{2}\alpha(Q_1^2+Q_2^2+Q_3^2)+\beta(Q_1Q_2+Q_1Q_3+Q_2Q_3), \quad (1)$$

where the internal coordinates Q_1, Q_2, and Q_3 are length changes of the sides of the triangle. The constants α and β will be determined from the observed energy level spectrum. The frequencies of the familiar normal vibrations are

$$\omega_1^2=3(\alpha+2\beta)/M_\alpha, \quad \omega_2^2=\tfrac{3}{2}(\alpha-\beta)/M_\alpha, \quad (2)$$

where the subscripts specify the degeneracy and M_α is the α-particle mass. The rotational motion is that of a symmetric top ($I_1=I_2=\frac{1}{2}I_3=\frac{1}{2}M_\alpha s^2$). Only those quantum states are allowed which satisfy Bose statistics for the α particles. Wheeler[3] has listed the number of allowed states as a function of n_1 and n_2, the occupation numbers of the vibrational modes, J, the total angular momentum, and K, its projection on the figure axis. The parity[4] of a level is determined solely by the rotational wave function and is $(-)^K$. Since $|K|\leq J$, 0^- states do not occur. Finally, the excitation energy is

$$E=[J(J+1)-\tfrac{1}{2}K^2]\Delta+n_1\delta_1+n_2\delta_2, \quad (3)$$

with $\Delta=\hbar^2/2I_1$, $\delta_1=\hbar\omega_1$, and $\delta_2=\hbar\omega_2$. As Wheeler pointed out, the requirement of Bose statistics eliminates a considerable number of states, particularly low-lying ones. Thus 1^+ states involve a minimum excitation of the degenerate mode ω_2 of three quanta, and the first state of this type will not be found until the excitation energy is above 20 Mev. Table I gives the eigenvalues for the allowed states of low excitation. The non-degenerate mode ω_1 is not included since its symmetry (even) and parity (even) are independent of n_1. Hence, additional states are obtained from those of Table I by exciting this mode by amounts $n_1\delta_1$, where n_1 is any integer. The present simple description of the α-particle model states of C^{12} is, of course, restricted to low excitation. Above 7.4 Mev the virtual nature of the levels

[1] D. M. Dennison, Phys. Rev. **96**, 378 (1954).

[2] A. Herzenberg [Nuovo cimento **10**, 986 and 1008 (1955)] has recently restudied some of the fundamental problems.

[3] J. A. Wheeler, Phys. Rev. **52**, 1083 (1937).

[4] Professor L. Rosenfeld has kindly informed us that his list of "parities," Table 13.21 in *Nuclear Forces* (North Holland Publishing Company, Amsterdam, 1948), gives the behavior of the wave function under reflections in a side of the equilateral triangle.