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Diffusion of Charged Particles across a Magnetic Field

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In this paper we study the diffusion of charged particles across a magnetic field, treating collision-induced steps of the guiding centers as a random walk process. Particular attention is paid to the vanishing of the first-order flux due to like-particle collisions.

I. INTRODUCTION

THE problem we consider here is the diffusion of an ionized plasma across a magnetic field.[1] In the absence of collisions the center of the Larmor orbit (called the guiding center) of a charged particle remains fixed on the same field line; collisions allow the guiding center to step about in a random way and, therefore, lead to diffusion. Ions may diffuse by colliding either with electrons or with each other. It might appear that the latter process is most important since ions can exchange momentum more easily with each other than with electrons. However, it turns out that collisions between like particles actually produce no flux in first order (proportional to the gradient of the particle density), because of a peculiar cancellation between those terms in the flux proportional to the first moment and the second moment of the random step. The first nonvanishing term in the flux due to like-particle collisions is proportional to the third derivative of the particle density. The net result is that, if the density does not vary much over an ion Larmor radius, ion-electron collisions predominate; otherwise ion-ion collisions predominate.

II. DISTRIBUTION FUNCTION

In particular, we shall consider the problem of a plasma in which the density depends on the x coordinate alone, in a uniform magnetic field B in the z direction. We consider the plasma density to be small enough that its magnetic effects can be neglected—i.e., we do not have to consider gradients of B due to the plasma. We shall assume that the field is strong enough so that many Larmor revolutions are made between collisions. We shall also assume that electron currents can flow (through end-plates, for example) in such a way as to keep the plasma neutral at all points.

The first problem that arises is what to take for the zeroth order distribution function $f(x,\mathbf{v})$. Since collisions are regarded as a perturbation, it is reasonable to require that f be a solution of the steady-state Liouville equation neglecting collisions:

$$v_x\frac{\partial f}{\partial x}+\omega\left\{v_y\frac{\partial f}{\partial v_x}-v_x\frac{\partial f}{\partial v_y}\right\}=0. \quad (1)$$

Here ω is the Larmor frequency eB/mc, with e the charge and m the mass of the particle. The general solution of (1) is

$$f=f(x+v_y/\omega,\ v_z,\ v_x^2+v_y^2). \quad (2)$$

A simple interpretation of (2) is afforded by the fact that $X\equiv x+v_y/\omega$ is the x coordinate of the guiding center of a particle whose coordinates in phase space are x, $\mathbf{v}$. Hence (2) merely indicates that the distribution function can be an arbitrary superposition of Larmor orbits; the function f is not specified further. The situation here is quite different from that in ordinary kinetic theory, where the zeroth order distribution must be the Maxwell-Boltzmann distribution.

Since we wish to study spatial diffusion rather than thermal diffusion, it seems reasonable to require further

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[1] This problem has in part been discussed previously by L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1955), by a different method from the one used here.

that f be of the separated form

$$f=N\left(x+\frac{v_y}{\omega}\right)g(v_z,v^2),$$

where $N(X)$ is the density of guiding centers in space and g is a normalized velocity distribution. The functional form of g is, of course, arbitrary, and in different physical situations different g's may be appropriate. We shall consider that particular g which gives an exact solution to the full Boltzmann equation for constant N, namely

$$f=N\left(x+\frac{v_y}{\omega}\right)\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left(-\frac{mv^2}{2kT}\right). \qquad (3)$$

This appears to be the most reasonable analog of the usual zeroth order distribution function of kinetic theory,

$$N(X)\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left(-\frac{mv^2}{2kT}\right).$$

We take a distribution of the type (3) for each type of charged particle in the plasma, all with the same temperature, and assume that there are no neutral particles.

III. CALCULATION OF THE FLUX

Consider now two types of particles, not necessarily different, which we denote by subscripts 1 and 2. We shall calculate the flux $F_1(X)$ of guiding centers of type 1 due to collisions of the particles 1 and 2. To this end we use the well-known stochastic expression for the flux[2]

$$F_1=N_1(X)\langle\Delta X_1\rangle-\frac{1}{2}\frac{\partial}{\partial X}[N_1(X)\langle(\Delta X_1)^2\rangle]. \qquad (4)$$

Third and higher moments of the guiding center step ΔX_1 are neglected, for a reason to be discussed in Sec. IV. Equation (4), coupled with the conservation equation

$$\partial N_1/\partial t=-\partial F_1/\partial X, \qquad (5)$$

determines the behavior of the guiding-center density $N_1(X)$ in time.

The step ΔX_1 of the guiding center is related to the change in the y-component of the velocity by

$$\Delta X_1=\Delta v_{1y}/\omega_1. \qquad (6)$$

The averages indicated in (4) include an average over the velocities of particle 1 and of the particles 2 with which it can collide, and an average over the scattering angles in the collision.

For a guiding center 1 at X, the associated particle 1 is at $x=X-(v_{1y}/\omega_1)$. The density of particles 2 at x is the same as the density of guiding centers 2 at $x+(v_{2y}/\omega_2)=X+[(v_{2y}/\omega_2)-(v_{1y}/\omega_1)]$. Hence the probability per unit time that a particle 1 with guiding center at X will be involved in a collision, with scattering into solid angle $d\Omega$ is

$$P(v_1,v_2,\Omega)d^3v_1d^3v_2d\Omega=d^3v_1d^3v_2\left(\frac{m_1}{2\pi kT}\cdot\frac{m_2}{2\pi kT}\right)^{\frac{3}{2}}$$
$$\times\exp\left(-\frac{m_1v_1^2}{2kT}\right)N_2(X+v_{2y}/\omega_2-v_{1y}/\omega_1)$$
$$\times\exp\left(-\frac{m_2v_2^2}{2kT}\right)v\sigma(\Omega)d\Omega. \qquad (7)$$

Here v is the relative velocity, $|\mathbf{v}_1-\mathbf{v}_2|$, and $\sigma(\Omega)$ is the differential scattering cross section. The occurrence of the relative velocity suggests that one transform to center-of-mass variables

$$M=m_1+m_2, \qquad m=m_1m_2/M,$$
$$\mathbf{V}=(m_1\mathbf{v}_1+m_2\mathbf{v}_2)/M, \qquad \mathbf{v}=\mathbf{v}_1-\mathbf{v}_2. \qquad (8)$$

Equation (7) then becomes

$$P(v,V,\Omega)d^3vd^3Vd\Omega=d^3vd^3V\left(\frac{m}{2\pi kT}\cdot\frac{M}{2\pi kT}\right)^{\frac{3}{2}}$$
$$\times\exp(-mv^2/2kT)\exp(-MV^2/2kT)$$
$$\times N_2(X+\delta)v\sigma(\Omega)d\Omega, \qquad (9)$$

where

$$\delta\equiv\frac{c}{B}\left\{\left(\frac{m_2}{e_2}-\frac{m_1}{e_1}\right)V_y-m\left(\frac{1}{e_2}+\frac{1}{e_1}\right)v_y\right\}. \qquad (10)$$

In order to perform the integrals, we expand

$$N_2(X+\delta)=N_2(X)+N_2'\delta+\frac{1}{2!}N_2''\delta^2+\frac{1}{3!}N_2'''\delta^3 \qquad (11)$$

and drop higher terms. We also express ΔX_1 in terms of v, V, and the scattering angles θ and ϕ in the center-of-mass system; from (6), (8), and an elementary trigonometrical calculation, we find

$$\Delta X_1=(m/m_1)(v/\omega_1)[\sin\theta\cos\phi\sin\chi-(1-\cos\theta)\cos\chi], \qquad (12)$$

where χ is the angle between $\mathbf{v}$ and the y direction, i.e., $v_y=v\cos\chi$.

The two averages that occur in (4) are then

$$\langle\Delta X_1\rangle=\int d^3vd^3Vd\Omega P(v,V,\Omega)\Delta X_1,$$
$$\langle(\Delta X_1)^2\rangle=\int d^3vd^3Vd\Omega P(v,V,\Omega)(\Delta X_1)^2. \qquad (13)$$

[2] S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943). See Eq. (126) and the preceding discussion.

Since $\sigma(\Omega)$ depends only on θ and the magnitude v, we may do directly the integrals over $\mathbf{V}$, over the angles of $\mathbf{v}$, and over $\phi(0-2\pi)$. Using (9), (10), (11), and (12) in (13), we find that many terms obviously integrate to zero, and that

$$\langle\Delta X_1\rangle=\frac{8\pi^2}{3}\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\left(\frac{mc}{e_1B}\right)^2\left\{\left(1+\frac{e_1}{e_2}\right)I_1\frac{\partial N_2}{\partial X}+\frac{1}{10}\left(\frac{mc}{e_1B}\right)^2\left(1+\frac{e_1}{e_2}\right)^3I_2\frac{\partial^3N_2}{\partial X^3}\right\},\quad(14)$$

$$\tfrac{1}{2}\langle(\Delta X_1)^2\rangle=\frac{8\pi^2}{3}\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\left(\frac{mc}{e_1B}\right)^2\left\{I_1N_2+\frac{1}{20}\left(\frac{mc}{e_1B}\right)^2\left(1+\frac{e_1}{e_2}\right)^2I_3\frac{\partial^2N_2}{\partial X^2}\right\},\quad(15)$$

where the I_n are integrals over the scattering cross section:

$$I_1\equiv\int_0^\infty dv\int_0^\pi d\theta\sigma(v,\theta)v^5\exp\left(-\frac{mv^2}{2kT}\right)(1-\cos\theta)\sin\theta,$$

$$I_2\equiv\int_0^\infty dv\int_0^\pi d\theta\sigma(v,\theta)v^7\exp\left(-\frac{mv^2}{2kT}\right)\times(1-\cos\theta)\sin\theta,\quad(16)$$

$$I_3\equiv\int_0^\infty dv\int_0^\pi d\theta\sigma(v,\theta)v^7\exp\left(-\frac{mv^2}{2kT}\right)\times[\sin^2\theta+3(1-\cos\theta)^2]\sin\theta.$$

In those terms in (14) and (15) which involve the second and third derivatives, we have kept only the part which does not vanish when particles 1 and 2 are identical.

If we put (14) and (15) in the expression (4) for the flux, and keep only terms involving the first derivative, we find

$$F_1=\frac{8\pi^2}{3}\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\left(\frac{mc}{e_1B}\right)^2I_1\left\{\left(1+\frac{e_1}{e_2}\right)N_1\frac{\partial N_2}{\partial X}-\frac{\partial}{\partial X}(N_1N_2)\right\}.\quad(17)$$

It can be seen that this expression vanishes when the set of particles 2 are the same as the set of particles 1. This result is independent of the form of the differential scattering cross section. The first moment of the step produces a flux in the direction of the gradient (opposite to the usual result) which just cancels the flux due to the second moment. The reason for the counter-flux from the first moment can be seen with the aid of Fig. 1. The circle represents a Larmor orbit with guiding center at the point O. Suppose the density N_2 is greater at a than at b. Then more collisions will take place at a than at b. For a collision at a, the guiding center moves, on the average, on the arc of a circle about a. Hence more guiding centers will move in the direction of the gradient than against it.

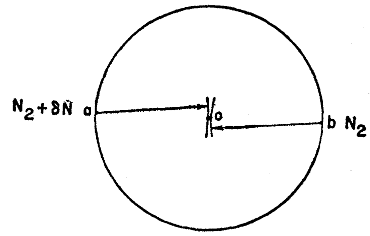

FIG. 1. Diagram illustrating reason for counter-flux from first moment.

The appropriate scattering cross section is the Rutherford formula,

$$\sigma(v,\theta)=\frac{e_1^2e_2^2}{4m^2v^4}\left(\sin\frac{\theta}{2}\right)^{-4}.$$

With this expression inserted in (16), one can easily do the indicated integrals, and one finds

$$I_1=\frac{2kT}{m}\left(\frac{e_1e_2}{m}\right)^2\ln\left(\frac{2}{\theta_0}\right),$$

$$I_2=\left(\frac{2kT}{m}\right)^2\left(\frac{e_1e_2}{m}\right)^2\ln\left(\frac{2}{\theta_0}\right),\quad(18)$$

$$I_3=2I_2.$$

Here θ_0 is the minimum angle of scattering, which occurs in a collision in which the impact parameter is equal to the Debye shielding radius

$$\frac{\theta_0}{2}=\frac{e_1e_2}{mv^2}\left(\frac{4\pi Ne^2}{kT}\right)^{\frac{1}{2}}.\quad(19)$$

In (18) we have dropped terms small compared to the logarithm.

With these results, (17) finally gives the flux of particles 1 due to collisions with particles 2,

$$F_1(2)=\frac{4}{3}\left(\frac{2\pi mc^2}{kT}\right)^{\frac{1}{2}}\left(\frac{ce_2^2}{B^2}\right)\ln\left(\frac{2}{\theta_0}\right)\times\left\{\left(1+\frac{e_1}{e_2}\right)N_1\frac{\partial N_2}{\partial X}-\frac{\partial}{\partial X}N_1N_2\right\}.\quad(20)$$

If the particles are identical, higher derivatives in (14) and (15) have to be retained; the result is

$$F_1(1)=\frac{8}{15}\left(\frac{\pi m_1c^2}{kT}\right)^{\frac{1}{2}}\left(\frac{ce_1^2}{B^2}\right)\left(\frac{m_1kTc^2}{e_1^2B^2}\right)\times\ln\left(\frac{2}{\theta_0}\right)N_1^2\frac{\partial}{\partial X}\frac{1}{N_1}\frac{\partial^2N_1}{\partial X^2}.\quad(21)$$

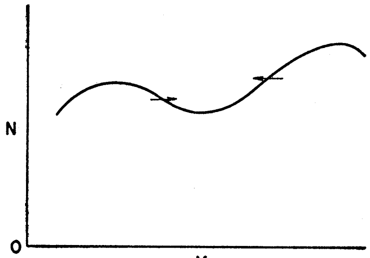

FIG. 2. Diagram illustrating direction of fluxes.

IV. DISCUSSION

Equation (20) agrees with the result obtained by Spitzer.[1] Equation (21) has the same form as the result obtained by Kruskal,[3] who did not attempt to evaluate the coefficient of the derivatives. Equation (21) also agrees with the result obtained by Simon,[4] except for a numerical factor 4/3; the source of this discrepancy is not known.

The additional coefficient $m_1kTc^2/e_1^2B^2$ in Eq. (21) is essentially the square of the Larmor radius of the particles 1. The ratio of the ion flux $F_i(i)$, due to collisions with ions, to the ion flux $F_i(e)$, due to collisions with electrons is

$$F_i(i)/F_i(e)\approx(R_i/D)^2(m_i/m_e)^{\frac{1}{2}},$$

where R_i is the ion Larmor radius, D a characteristic distance in which the density N_i changes substantially, and m_i and m_e are the ion and electron masses, respectively.

For a density distribution of the form shown in Fig. 2, it can be seen that the like-particle flux (21), as well as (20), is such as to fill up the low places and deplete the high places. Thus the result (21) has a measure of reasonableness. However, if N were exponential or linear, there would be no like-particle flux.

In order to calculate the second and third derivative terms exactly, one ought to include the third and fourth moments of the step in Eq. (4). However, these moments add only terms small compared to the logarithm to the previous results. Since θ_0 can be calculated only approximately, it is not justified to carry these extra terms.

In a collision between like particles, Eq. (6) and the conservation of momentum show that the steps taken by the two guiding centers are equal in magnitude and opposite in direction. It might appear that this rigid correlation should be taken into account in the stochastic Eq. (4) by averaging over pairs of steps instead of steps of particles individually. However, it can be shown that this correlation in fact produces no effect, and that Eq. (4) is correct. The simplest way to see this is to imagine that all of the like particles are divided into a large number of subsets, so that collisions between particles in the same subset can be neglected. Equation (4) is then correct for the flux of each one of these subsets, since the colliding partner is in a different subset. On summing over subsets, (4) is correct for the flux of all the like particles simultaneously.

One is tempted, at first sight, to assert that the rigid correlation mentioned in the foregoing paragraph is the cause of the cancellation in the first-order flux due to the like-particle collisions. The fact that the center of gravity of the two guiding centers does not move in the collision may lead one to conclude that there can be no net flux. However, this argument is fallacious. In any random-walk problem where the probabilities of equal steps to the right or left from a given point are equal, the center of gravity of the distribution is conserved, apart from absorption and edge effects; yet the flux need not vanish. The result is not altered if steps to the right and left are imagined to occur in pairs. In our problem, the probabilities of steps to the right and left are not equal, with the result that the first-moment flux does not vanish, but rather accidentally cancels the second-moment flux. If particles with guiding center at a given point could somehow be restricted to collide only with other particles whose guiding centers were at the same point, the first-moment flux would vanish, the net flux would not vanish, and the center of mass would still be conserved.

[3] M. Kruskal (private communication).

[4] A. Simon, Phys. Rev. **100**, 1557 (1955). Simon uses the Chapman-Cowling method.