Multiple Photon Production by Electron Pair Annihilation in Flight*

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Starting with the Feynman-Dyson S matrix, we have computed the two leading terms in the high-energy expansion of the cross section for multiple photon production by electron pair annihilation in flight. In the extreme relativistic limit, when $\ln(E_+/m)\gg 1$, where E_+ is the positron energy in the laboratory system, our result reduces to that previously obtained by Gupta.

We also calculated the first four terms in the high-energy expansion of the three-photon production cross section. Gupta obtained the leading term only and found a discrepancy between σ_3 and σ_n for $n=3$ by a factor $\frac{3}{2}$. This discrepancy is removed.

For the sake of simplicity, Gupta assumed that the principal contributions to the cross section come from (1) low-energy photons and (2) small scattering angles. We proved that all these assumptions are indeed valid.

The possibility of observing multiple photon production is discussed.

INTRODUCTION

THE first explicit estimate of the high-energy cross section for multiple photon production by electron pair annihilation in flight was given by Gupta.[1] In an independent estimate of the high-energy cross section for three-photon production, Gupta showed that the principal contribution to the cross section arises when at least two of the photons have very small scattering angles. He assumed that the third photon is also likely to be emitted in the forward direction.

When estimating the cross section for multiple photon production, Gupta further assumed (1) that all photons are likely to have small scattering angles, and (2) that in the rest system of the electron pair, all but two photons are likely to be soft.[2]

Gupta's results exhibit a discrepancy by a factor $\frac{3}{2}$ between his general formula for the multiple photon production cross section σ_n, as specialized to $n=3$, and his independently calculated formula for the three-photon production cross section σ_3. He attributed this discrepancy to the approximations made in computing σ_n.

This raises the questions: (a) Does the discrepancy between Gupta's formula and the "correct" formula for σ_n increase, decrease, or remain constant for increasing n? (b) Under what conditions are Gupta's assumptions valid?

We propose to answer these questions by first proving the validity of assumption (2). Then we will make full use of it at the very beginning of the computation of σ_n. The calculations thereby become very much simpler than in Gupta's work.

We also propose to compute the first four terms in the high-energy expansion for the three-photon production cross section.

I. MULTIPLE PHOTON PRODUCTION

(α) Transition Amplitude

Throughout this discussion we employ the system of units in which $\hbar=1$, $c=1$, and a metric tensor which has the nonvanishing components $g_{11}=g_{22}=g_{33}=-g_{00}=1$. The product $a_\mu\gamma^\mu$, between the four-vector a_μ and the Dirac matrices γ_μ is denoted by $\boldsymbol{a}$.

According to energy-momentum conservation, photon production by electron pair annihilation in flight can occur only by the emission of a minimum of two hard photons (k and k'). Considering the production of $n+2$ photons, we will show that the principal contributions to the cross section arise when: (A) There are only two hard photons, the remaining photons ($q_1\cdots q_n$) being soft.[2] (B) In the Feynman diagrams, none of the soft photons is emitted by internal emission, i.e., from the electron line segment between the two hard photons.

For this purpose it is convenient to isolate the two hard photons and to consider a partition of the soft photons into two groups of r and $n-r$ in number. The two hard photons, and the bare electron line segment between them, are isolated in the diagram Fig. 1. The

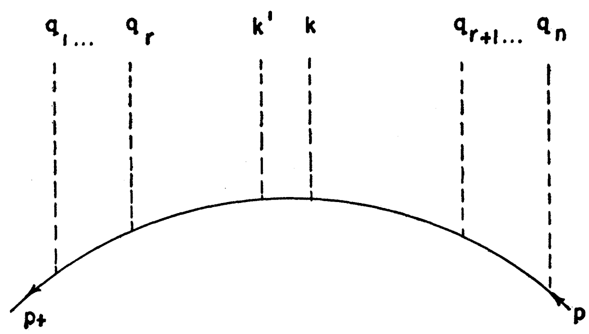

FIG. 1. A Feynman diagram for multiple photon emission by electron-pair annihilation. The energy-momentum vectors q_1, q_2, $\cdots$, q_n represent soft photons. The vectors k' and k represent hard photons.

* Submitted in partial fulfillment of the requirements for the Ph.D. degree at the State University of Iowa, Iowa City, Iowa.
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[1] S. N. Gupta, Phys. Rev. 98, 1502 (1955).
[2] A group of photons whose collective energy is small compared to the electron energy, consists of what we call *soft* photons. At all other energies a photon is called *hard*. This statement refers only to the center-of-mass system of the electron pair.

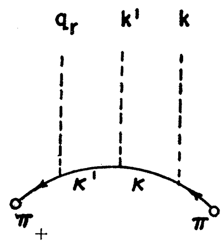

FIG. 2. A part of the diagram illustrated in Fig. 1. The four-vectors π and π_+ are associated with internal lines, and $\pi_+ = p_+ - (q_1 + \cdots q_r)$, $\pi = p - (q_n + \cdots q_{r+1})$, $\kappa = \pi - k$, $\kappa' = \pi_+ - q_r$.

first group of $n-r$ photons is emitted from the directed electron line segment leading into the isolated pair, and the other group of r photons is emitted from the electron line emerging from the isolated pair.

Let us compare the contributions to the transition amplitude from the configuration illustrated in Fig. 1 to one which replaces the soft photon q_r by a hard photon Q_r. Everything else remaining the same, the relevant factors are illustrated in Figs. 2 and 3. The corresponding analytical expressions are

$$(i\pi_+ - m)^{-1} \frac{e(q_r)}{(2\omega_r)^{\frac{1}{2}}} (i\kappa' - m)^{-1} \times \frac{e(k')}{(2\omega')^{\frac{1}{2}}} (i\kappa + m)^{-1} \frac{e(k)}{(2\omega)^{\frac{1}{2}}} (i\pi + m)^{-1}, \quad (1.1)$$

$$(i\pi_+ - m)^{-1} \frac{e(Q_r)}{(2\Omega_r)^{\frac{1}{2}}} (i\kappa'' - m)^{-1} \times \frac{e(k')}{(2\omega')^{\frac{1}{2}}} (i\kappa + m)^{-1} \frac{e(k)}{(2\omega)^{\frac{1}{2}}} (i\pi + m)^{-1}, \quad (1.2)$$

where $e_\mu(k)$ is the polarization four-vector associated with the photon k. The variables ω, ω', ω_r, and Ω_r, are the energy components of k, k', q_r, and Q_r, respectively. All electron line segments are internal, and the associated four-vectors can be expressed in terms of the initial and final variables by

$$\pi = p - (q_n + \cdots q_{r+1}), \quad \pi_+ = p_+ - (q_1 + \cdots q_{r-1}), \quad \kappa = \pi - k, \quad \kappa' = \pi_+ - q_r, \quad \kappa'' = \pi_+ - Q_r. \quad (1.3)$$

The remaining parts of Fig. 1, which are not illustrated in Fig. 2, can be represented by two spinors $\bar{\varphi}$ and χ. If we now introduce the spinors

$$\bar{\psi} = \bar{\varphi}(i\pi_+ - m)^{-1}, \quad (1.4)$$

$$\xi = (i\kappa + m)^{-1} \frac{e(k)}{(2\omega)^{\frac{1}{2}}} (i\pi + m)^{-1}\chi, \quad (1.5)$$

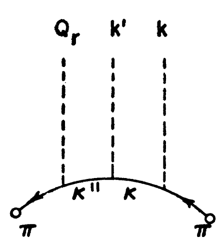

FIG. 3. A part of a Feynman diagram for multiple photon emission by electron-pair annihilation. This is the same as the part illustrated by Fig. 2 except that the soft photon q_r is replaced by a hard photon Q_r, and $\kappa'' = \pi_+ - Q_r$.

we can write for the ratio of the matrix elements corresponding to (1.1) and (1.2)

$$R = \left[\frac{\bar{\psi} e(Q_r)(i\kappa'' + m) e(k')\xi}{\bar{\psi} e(q_r)(i\kappa' + m) e(k')\xi} \right] \left(\frac{\omega_r}{\Omega_r} \right)^{\frac{1}{2}} \left(\frac{\kappa'^2 + m^2}{\kappa''^2 + m^2} \right). \quad (1.6)$$

In the soft photon limit

$$\bar{\psi} \to \bar{\psi}_0(p_+), \quad \xi \to \xi_0(k,p), \quad (1.7)$$

so that R decreases like

$$R = \left(\frac{\bar{\psi}_0 e(Q_r)[i(p_+ - Q_r) + m] e(k')\xi_0}{\bar{\psi}_0 e(q_r)[ip_+ + m] e(k')\xi_0} \right) \times \left(\frac{\omega_r}{\Omega_r} \right)^{\frac{1}{2}} \frac{p_+ \cdot q_r}{p_+ \cdot Q_r}. \quad (1.8)$$

The first factor depends on ω_r only through the polarization vector which hardly affects the magnitude of R. It follows that R decreases like $(\omega_r/\Omega_r)^{\frac{1}{2}}$. The quantity of interest, however, is $(\Omega_r/\omega_r)R$, because the density of final states contributes a factor $(\Omega_r/\omega_r)^2$ to the corresponding ratio of transition *probabilities*, and therefore

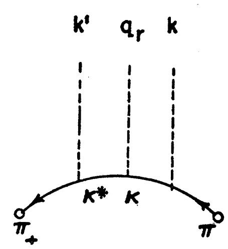

FIG. 4. A part of a Feynman diagram for multiple photon emission by electron-pair annihilation. This is the same as Fig. 2 except that the soft photon q_r and the hard photon k' are permuted, and $\kappa^* = \pi_+ - k'$.

a factor (Ω_r/ω_r) to the ratio of the transition *amplitudes*. It follows that the probability distribution involving three hard photons is smaller than the probability distribution involving only two hard photons. Their ratio is the ratio of soft- to hard-photon energies. Our proof was given for a special case only, namely for the photon in Fig. 1 which is the immediate neighbor of the hard photon k'. To make the proof more general, we only need to prove the second condition (B).

Turning to condition (B), let us compare the contribution to the transition amplitude from the configuration illustrated in Fig. 1 to one which is obtained from it by permuting the hard photon k' with the soft photon q_r. This permutation results in the internal emission (see condition B) of a soft photon. The relevant factor arising from this permutation is illustrated in Fig. 4. The corresponding analytical expression is

$$(i\pi_+ - m)^{-1} \frac{e(k')}{(2\omega')^{\frac{1}{2}}} (i\kappa^* - m)^{-1} \times \frac{e(q_r)}{(2\omega_r)^{\frac{1}{2}}} (i\kappa + m)^{-1} \frac{e(k)}{(2\omega)^{\frac{1}{2}}} (i\pi + m)^{-1}, \quad (1.9)$$

where $\kappa^* = \pi_+ - k'$. We can write for the ratio of the matrix elements corresponding to (1.1) and (1.9)

$$R' = \left[\frac{\bar{\psi}e(k')(i\kappa^* + m)e(q_r)\xi}{\bar{\psi}e(q_r)(i\kappa' + m)e(k')\xi}\right]\left(\frac{\kappa'^2 + m^2}{\kappa^{*2} + m^2}\right). \quad (1.10)$$

In the soft-photon limit, R' decreases like

$$R' = \left(\frac{\bar{\psi}_0 e(k')[i(p_+ - k') + m]e(q_r)\xi_0}{\bar{\psi}_0 e(q_r)[ip_+ + m]e(k')\xi_0}\right) \times \left(\frac{p_+ \cdot q_r}{p_+ \cdot k'}\right). \quad (1.11)$$

The first factor again depends on ω_r through the polarization vector only, which hardly affects the magnitude of R'. It follows that R' decreases like (ω_r/ω'). Since the difference between the matrix elements arises from merely permuting the soft- and hard-photon variables, there is no additional factor arising from the density of final states and R' is actually the quantity of interest. We can conclude, by a suitable generalization of the argument presented above, that for each soft photon emitted by internal emission (see condition B), the corresponding transition amplitude is smaller by the ratio of soft- to hard-photon energies.

The conditions (A) and (B) having thus been corroborated, we now proceed with the calculations of the transition amplitude.

The analytical expression corresponding to the isolated pair, F_2, is the spinor matrix related to the transition amplitude M_2 for two-photon production by

$$M_2(p_+p; kk') = \bar{v}(p_+)F_2(p_+p; kk')u(p), \quad (1.12)$$

where the spinors $u(p)$ and $\bar{v}(p_+)$ are the negatron and positron plane wave amplitudes normalized to

$$\bar{u}(p)u(p) = 1, \quad \bar{v}(p_+)v(p_+) = -1. \quad (1.13)$$

The transition amplitude for $(n+2)$-photon production is in the first approximation

$$\begin{aligned} M_{n+2} \simeq \mathcal{L} \sum_{\langle r\rangle} \sum_{(p)} \bar{v}(p_+) \Bigg\{ \prod_{s=1}^{s=r} \frac{e}{(2\pi)^{\frac{3}{2}}} \frac{e_s}{(2\omega_s)^{\frac{1}{2}}} \\ \times \left[\frac{-i(p_+ - q_1 - \cdots - q_s) - m}{-2p\cdot(q_1 + \cdots + q_s) + (q_1 + \cdots + q_s)^2}\right]\Bigg\} \\ \times F_2(p_+ - q_1 - \cdots q_r, p - q_{r+1} - \cdots q_n; kk') \\ \times \Bigg\{ \prod_{s=r+1}^{s=n} \frac{e}{(2\pi)^{\frac{3}{2}}} \\ \times \left[\frac{i(p - q_s - \cdots - q_n) - m}{-2p\cdot(q_s + \cdots + q_n) + (q_s + \cdots + q_n)^2}\right] \\ \times \frac{e_s}{(2\omega_s)^{\frac{1}{2}}}\Bigg\} u(p). \quad (1.14) \end{aligned}$$

We write the four-vector products $a^\mu b_\mu$ as $a\cdot b$ for simplicity. The symbol $\mathcal{L}$ indicates the soft photon limiting form. There is a sum over the index $\langle r\rangle$ which describes the partition of the soft photons, and a sum over the permutations (p) among them. That the symmetrization of the initial and final states is correctly included in these two sums can be seen by dividing the sum over all the permutations into the following four classes: *Class I* consists of the permutations among the soft photons alone. *Class II* consists of the permutations between the two hard photons alone; they are included in F_2. *Class III* consists of those permutations among soft and hard photons which do not result in internal emission (see condition B). The sum over these permutations can be expressed as the sum over all partitions. *Class IV* consists of the remaining permutations, each of which provides a configuration involving internal emission.

According to statement (B), which was proved above, class IV can be neglected. Class II is included in F_2. Therefore, Eq. (1.14) is correctly symmetrized.

We use the relations

$$\begin{aligned} e_s p_+ &= 2e_s\cdot p_+ - p_+ e_s, \\ p e_s &= 2e_s\cdot p - e_s p, \\ \bar{v}(p_+)(ip_+ - m) &= 0, \\ (ip + m)u(p) &= 0, \end{aligned} \quad (1.15)$$

which we apply in succession in the soft photon limit, and we obtain from (1.14)

$$\begin{aligned} M_{n+2} \simeq \frac{e^n}{(2\pi)^{3n/2}} \frac{1}{(2\omega_1\cdots 2\omega_n)^{\frac{1}{2}}} \sum_{\langle r\rangle} (-1)^{n-r} \sum_{(p)} \\ \times \left[\frac{p_+\cdot e_1 \cdots p_+\cdot e_r}{p_+\cdot q_1 \cdots p_+\cdot(q_1 + \cdots + q_r)} \right. \\ \left. \times \frac{p\cdot e_{r+1}\cdots p\cdot e_n}{p\cdot(q_{r+1} + \cdots q_n)\cdots p\cdot q_n}\right] M_2(p_+p; kk'). \quad (1.16) \end{aligned}$$

At this point we use the identity, easily proved by induction:

$$\sum \frac{1}{y_1(y_1+y_2)\cdots(y_1+\cdots+y_n)} = \frac{1}{y_1 y_2\cdots y_n}, \quad (1.17)$$

where the sum extends over all permutations of the variables, and find

$$\begin{aligned} M_{n+2} \simeq \frac{e^n}{(2\pi)^{3n/2}} \frac{1}{(2\omega_1\cdots 2\omega_n)^{\frac{1}{2}}} \sum_{\langle r\rangle} (-1)^{n-r} \sum_{(p')} \\ \times \left[\left(\frac{p_+\cdot e_1}{p_+\cdot q_1}\right)\cdots\left(\frac{p_+\cdot e_r}{p_+\cdot q_r}\right) \right. \\ \left. \times \left(\frac{p\cdot e_{r+1}}{p\cdot q_{r+1}}\right)\cdots\left(\frac{p\cdot e_n}{p\cdot q_n}\right)\right] M_2(p_+p;\ kk'). \quad (1.18) \end{aligned}$$

Here (p') is any of the remaining permutations of the soft photons excluding those within either group of the partition. Next, we employ another identity, which can also be proved by induction,

$$\sum_{<r>}(-1)^{n-r}\sum_{(p')}a_1^\dagger\cdots a_r^\dagger a_{r+1}\cdots a_n=\prod_{s=1}^{s=n}(a_s^\dagger-a_s),\quad (1.19)$$

and obtain finally

$$M_{n+2}\simeq\left\{\prod_{s=1}^{s=n}\frac{e}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2\omega_s)^{\frac{1}{2}}}\left(\frac{e_s\cdot p_+}{p_+\cdot q_s}-\frac{e_s\cdot p}{p\cdot q_s}\right)\right\}M_2.\quad (1.20)$$

The reduction of M_{n+2} to the particularly simple form (1.20) is not surprising in view of previous work by Glauber,[3] Thirring and Touscheck,[4] and Jauch and Rohrlich.[5] These authors, considering the emission of an additional soft photon during an almost arbitrary scattering process, described by the matrix element M, obtained for the matrix element M_1 describing the same process with one additional soft photon

$$M_1=\frac{e}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2\omega_1)^{\frac{1}{2}}}\left(\frac{p'\cdot e_1}{p'\cdot q_1}-\frac{p\cdot e_1}{p\cdot q_1}\right)M,\quad (1.21)$$

where p' is associated with the scattered electron. We can apply the substitution law[6] to (1.21) in substituting the positron momentum p_+ for $-p'$ in (1.21) and M_2 for M, to obtain

$$M_{1+2}=\frac{e}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2\omega_1)^{\frac{1}{2}}}\left(\frac{p_+\cdot e_1}{p_+\cdot q_1}-\frac{p\cdot e_1}{p\cdot q_1}\right)M_2.$$

If one were to apply an iteration process in this manner, the result would be precisely (1.20).

(β) Cross Section

The transition amplitude and cross section are related by

$$\sigma_{n+2}=\frac{EE_+}{[(p\cdot p_+)^2-m^4]^{\frac{1}{2}}}\int d^3k\int d^3k'\int d^3q_1\cdots\times\int d^3q_nG_{n+2}\delta(p_f-p_i),\quad (1.22)$$

with

$$G_{n+2}=\tfrac{1}{4}(2\pi)^2\sum_{\text{pol}}\sum_{\text{spin}}|M_{n+2}|^2.\quad (1.23)$$

The negatron and positron energies are E and E_+: The initial energy momentum is $p_i=p_++p$ and the final energy momentum is $p_f=k+k'+q_1+\cdots q_n$. Introducing the quantities

$$\Phi_s=-\omega_s^2\left[\frac{2p\cdot p_+}{p\cdot q_sp_+\cdot q_s}+\frac{m^2}{(p\cdot q_s)^2}+\frac{m^2}{(p_+\cdot q_s)^2}\right],\quad (1.24)$$

it follows from (1.20) that

$$G_{n+2}=\frac{\alpha^n}{(2\pi)^{2n}}\left\{\prod_{s=1}^{s=n}\omega_s^{-3}\Phi_s\right\}G_2.\quad (1.25)$$

We evaluate the integral in the c.m. system of the electron pair. Defining

$$\vartheta_s=\measuredangle(\mathbf{p}_+,\mathbf{q}_s),\quad s=1,\cdots,r,$$
$$\vartheta_s=\measuredangle(\mathbf{p},\mathbf{q}_s),\quad s=r+1,\cdots,n,\quad (1.26)$$

we can write

$$\Phi_s=\frac{2(1+\beta^2)}{(1-\beta^2\cos^2\vartheta_s)}-\frac{1}{\gamma^2(1-\beta\cos\vartheta_s)^2}-\frac{1}{\gamma^2(1+\beta\cos\vartheta_s)^2},\quad (1.27)$$

where γm is the electron energy and $\beta=|\mathbf{p}_+|/\gamma m$. We can see by inspection that in the high-energy limit $\gamma\gg1$, the angular distribution has very sharp maxima for $|\vartheta_s|\ll1$ and $|\pi-\vartheta_s|\ll1$. We know this is also the case for the two- hard-photon distribution contained in G_2. This proves the validity of assumption (1) by Gupta (see Introduction). We can also say that both energy and momentum conservation in the soft-photon limit is essentially that between the electron pair and the two hard photons. It follows that

$$\sigma_{n+2}\simeq\sigma_2\prod_{s=1}^{s=n}\frac{\alpha}{(2\pi)^2}\int\frac{d^3q_s}{\omega_s^3}\Phi_s\frac{1}{n!}.\quad (1.28)$$

The remaining integrals are elementary and we find

$$\sigma_{n+2}(\epsilon_2,\epsilon_1)=\sigma_2\frac{[\alpha C\ln(\epsilon_2/\epsilon_1)]^n}{n!}.\quad (1.29)$$

The factor $n!$ arises from the fact that the n soft photons are indistinguishable. The factor C arises from the integral over the angular distribution,

$$C=\frac{1}{2\pi}\int\frac{d\Omega_s}{2\pi}\Phi_s\simeq\frac{2}{\pi}[\ln(2\gamma)^2-1],\quad (1.30)$$

which we have evaluated in the high-energy limit.

The factor $\ln(\epsilon_2/\epsilon_1)$ arises from the integral over the energy distribution

$$\int_{\epsilon_1m}^{\epsilon_2m}\frac{d\omega_s}{\omega_s}=\ln\left(\frac{\epsilon_2}{\epsilon_1}\right).$$

[3] R. H. Glauber, Phys. Rev. **84**, 395 (1951).
[4] W. Thirring and B. Touscheck, Phil. Mag. **42**, 244 (1951).
[5] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta **27**, 613 (1954).
[6] See J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Press, Cambridge, 1955).

We have carefully refrained from specifying the energy limits ϵ_2 and ϵ_1. We wish to make a few remarks concerning these limits. However, we first show that $\omega^{-1}d\omega$ is invariant under Lorentz transformations. This would provide the energy distribution in the laboratory system. Furthermore, it would enable us to discuss the limits in the laboratory system.

The quantity $\omega^{-1}d^3k$ is a well-known invariant. We write $\omega^{-1}d^3k=\omega^{-1}d\omega d\varphi\omega^2\sin\vartheta d\vartheta$. If we choose the polar axis along the direction of motion, then $d\varphi$ is invariant. Thus we need only to prove that $\omega^2\sin\vartheta d\vartheta$ is invariant. According to the transformation equations $x\equiv\omega'/\omega=\gamma(1-\beta\cdot\cos\vartheta)$ and $x^{-1}=\gamma(1+\beta\cdot\cos\vartheta')$ we have $(dx/dx^{-1})=(\omega'/\omega)^2=\sin\vartheta d\vartheta/\sin\vartheta' d\vartheta'$ so that indeed, $\omega^2\sin\vartheta d\vartheta$ is invariant. It follows that $\omega^{-1}d\omega$ is invariant.

As proved by Jauch and Rohrlich,[5] the contributions from real and virtual photon energies below the experimental energy resolution ϵmc^2, cancel to each order of the coupling constant. We thus take for the lower limit $\epsilon_1=\epsilon$.[7] The upper limit $\epsilon_2\ll\gamma_+m$, where γ_+m is the positron energy in the laboratory system. However, we have shown that the principal contributions come from the low-energy distribution. We thus obtain a crude estimate and a cutoff-independent result by using γ_+m for ϵ_2. One can show in a rough way that this leads to an error which amounts to neglecting terms of the relative order $[\ln(\gamma_+/\epsilon)]^{-1}$. We thus obtain

$$\sigma_{n+2}\simeq\sigma_2\frac{[\alpha C\ln(\gamma_+/\epsilon)]^n}{n!}, \tag{1.31}$$

where $C=(2/\pi)[\ln(2\gamma_+)-1]$.

2. THREE-PHOTON PRODUCTION

In this section we wish to make an independent and better calculation of the cross section discussed in (1) for the special case of three-photon production. We do not at the beginning distinguish hard and soft photons, and we regard all three photons on an equal footing. For this reason we make a slight change of notation by using k_1, k_2, and k_3 for the energy-momentum vectors of the three photons.

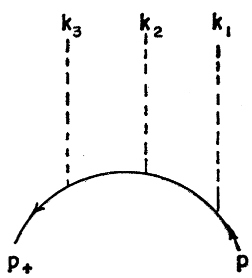

FIG. 5. A Feynman diagram for three-photon emission by electron-pair annihilation.

[7] One should use the lowest photon energy which is observable with the particular apparatus employed. However, this quantity is of the same order of magnitude as ϵmc^2.

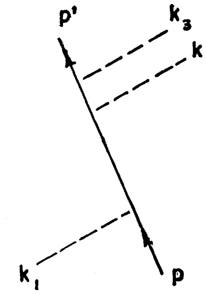

FIG. 6. A Feynman diagram for double Compton scattering.

The cross section for this process is

$$\sigma_3=\int d^3k_1\int d^3k_2\int d^3k_3\frac{EE_+}{[(p\cdot p_+)^2-m^4]^{\frac{1}{2}}}$$
$$\times\frac{(2\pi)^2}{4}\sum_{\text{pol}}\sum_{\text{spin}}|M_3|^2\delta(p+p_+-k_1-k_2-k_3). \tag{2.1}$$

In order to specify the probability amplitude M_3, we introduce the spinor matrix

$$T=-\sum \mathbf{e}(k_2)[i(\mathbf{p}_+-\mathbf{k}_2)-m]^{-1}\mathbf{e}(k_3)$$
$$\times[i(\mathbf{p}-\mathbf{k}_1)+m]^{-1}\mathbf{e}(k_1), \tag{2.2}$$

where the sum extends over all permutations of the three-photon variables. Then we can write

$$M_3=\frac{e^3}{(2\pi)^{7/2}}\frac{m}{(EE_+2\omega_12\omega_22\omega_3)^{\frac{1}{2}}}\bar{v}(p_+)Tu(p), \tag{2.3}$$

which corresponds to the sum over all permutations of the three photons on the Feynman diagram illustrated in Fig. 5.

We have the relation

$$\frac{(2\pi)^2}{4}\sum_{\text{pol}}\sum_{\text{spin}}|M_3|^2=\frac{\alpha r_0^2}{(4\pi)^2}\frac{m^2X}{EE_+\omega_1\omega_2\omega_3}, \tag{2.4}$$

where the quantity

$$X=-\sum_{\text{pol}}\text{Tr}(i\mathbf{p}-m)T^*(i\mathbf{p}_++m)T \tag{2.5}$$

has been computed for the double Compton process by Mandl and Skyrme.[8] A typical diagram for the double Compton process is illustrated in Fig. 6. We make the following substitutions[6]: (a) $p'\to-p_+$; (b) $k_1\to-k_1$; (c) we change the over-all sign of the probability distribution.

By introducing the system

$$\begin{aligned}&a=\sum a_l^{-1}, && b=\sum b_l^{-1}, && c=\sum(a_lb_l)^{-1},\\ &\sigma=\sum a_l, && \sigma'=\sum b_l, && \delta=\sum(a_lb_l),\\ &A=a_1a_2a_3, && B=b_1b_2b_3, && \rho=\sum(a_lb_l^{-1}+a_l^{-1}b_l),\end{aligned} \tag{2.6}$$

[8] F. Mandl and T. H. R. Skyrme, Proc. Roy. Soc. (London) **A215**, 497 (1952).

where $a_l=-p\cdot k_l$, $b=-p_+\cdot k_l$, we can write (using $\sigma=\sigma'$ from energy-momentum conservation)

$$-X=\Big\{2(ab-c)[(a+b)(\sigma+2)-(ab-c)-8-2\rho]$$
$$+[2\rho(\sigma-2)-8]c-2\sigma(a^2+b^2)$$
$$+\frac{4\sigma}{AB}\Big[(A+B)(\sigma+1)+(aA+bB)$$
$$\times\left(\delta\frac{\sigma-1}{\sigma}-2\right)+\sigma^2(1-\delta)+2\delta\Big]\Big\},\quad(2.7)$$

which is wholly unsurveyable and requires a suitable approximation.

In order to see which approximation can be made, we develop (2.3) in further detail. We introduce center-of-mass coordinates and the quantities

$$\vartheta_1=\sphericalangle(\mathbf{p},\mathbf{k}_1),\quad \vartheta_2=\sphericalangle(\mathbf{p}_+,\mathbf{k}_2),\quad \vartheta_3=\sphericalangle(\mathbf{p},\mathbf{k}_3),$$
$$\Gamma(123|pp_+)=-e_2[i(p_+-k_2)+m]\times e_3[i(p-k_1)-m]e_1,$$
$$X_{lsn}=\frac{m^{-4}}{16\omega_l\omega_s}\sum_{\text{pol}}\text{Tr}(ip-m)\times\Gamma(nsl|pp_+)(ip_++m)\Gamma(123|pp_+).\quad(2.8)$$

Noting that

$$\frac{\omega_3}{m}\left|\frac{\partial(\omega_2+\omega_3)}{\partial\omega_2}\right|=\left|2\gamma-\frac{\omega_1}{m}(1+\cos\vartheta_1\cos\vartheta_2+\sin\vartheta_1\sin\vartheta_2\cos|\varphi_2-\varphi_1|)\right|=D^{-1},\quad(2.9)$$

we obtain, after performing the trivial integrations,

$$\sigma_3=\frac{3!\alpha r_0^2}{2\gamma^2\beta}\int\frac{d\omega_1}{m}\int\frac{d\Omega_1}{4\pi}\int\frac{d\Omega_2}{4\pi}\times\frac{\Psi D}{\gamma^2(1-\beta\cos\vartheta_1)\gamma^2(1-\beta\cos\vartheta_2)},\quad(2.10)$$

with

$$\Psi=\Big[\frac{X_{123}}{(1-\beta\cos\vartheta_1)(1-\beta\cos\vartheta_2)}+\frac{X_{312}}{(1-\beta\cos\vartheta_3)(1+\beta\cos\vartheta_1)}+\frac{X_{231}}{(1+\beta\cos\vartheta_2)(1+\beta\cos\vartheta_3)}+\frac{X_{132}}{(1-\beta\cos\vartheta_1)(1+\beta\cos\vartheta_3)}+\frac{X_{213}}{(1+\beta\cos\vartheta_2)(1+\beta\cos\vartheta_1)}+\frac{X_{321}}{(1-\beta\cos\vartheta_3)(1-\beta\cos\vartheta_1)}\Big].\quad(2.11)$$

When $\gamma\gg1$, we see that the integrand of (2.10) has two very sharp maxima for $|\vartheta_1|\ll1$, $|\vartheta_2|\ll1$ and both $|\vartheta_3|\ll1$ and $|\pi-\vartheta_3|\ll1$. The latter is, however, the charge-conjugate case of the former. Thus we return to (2.4) and (2.7), and we can use the approximations

$$\gamma\gg1,\quad|\vartheta_1|\ll1,\quad|\vartheta_2|\ll1,\quad|\vartheta_3|\ll1.\quad(2.12)$$

We must also multiply by a factor 2 to compensate for an equal contribution from the charge conjugate case. Then, with[9]

$$z_l=2\gamma^2(1-\beta\cos\vartheta_l)\simeq1+\gamma^2\vartheta_l^2,\qquad \omega_1+\omega_3\simeq\omega_2\simeq\gamma,\quad(2.13)$$

we have

$$a_1=\omega_1z_1/2\gamma,\quad a_2\simeq2\gamma^2,\quad a_3=\omega_3z_3/2\gamma,\qquad b_1\simeq2\gamma\omega_1,\quad b_2\simeq\omega_2z_2/2\gamma,\quad b_3\simeq2\gamma\omega_3.\quad(2.14)$$

In this approximation, according to (2.14) and (2.6),

$$\sigma,\rho,\delta\gg1,\quad\text{and}\quad aA+bB\ll\sigma^2.\quad(2.15)$$

It follows from (2.15) and (2.7) that

$$X\simeq\Big\{2(ab-c)[2\rho+(ab-c)-\sigma(a+b)]+2\sigma(a^2+b^2-\rho c)+\frac{4\sigma^2}{AB}[\sigma\delta-A-B]\Big\}.\quad(2.16)$$

The system (2.6) reduces to

$$a\simeq2\gamma\left(\frac{1}{\omega_1z_1}+\frac{1}{\omega_3z_3}\right),\quad b\simeq\frac{2\gamma}{z_2}+\frac{1}{2\omega_1\omega_3},\quad c=\sum\frac{1}{\omega_l^2z_l},$$
$$\sigma\simeq2\gamma^2,\quad\rho\simeq4\gamma^2\sum z_l^{-1},\quad\delta=\sum\omega_l^2z_l,$$
$$A\simeq\tfrac{1}{2}\omega_1\omega_3z_1z_3,\quad B\simeq2\gamma^2\omega_1\omega_3z_2.\quad(2.17)$$

There are altogether nine final variables, three momentum components for each photon. Energy-momentum conservation eliminates four of them. The conservation of angular momentum, expressed as the azimuthal symmetry about the polar axis, eliminates one more. There remain therefore altogether four independent variables for which we choose the set

$$\omega_1,\ z_1,\ z_2,\quad\text{and}\quad\varphi=|\varphi_2-\varphi_1|,\quad(2.18)$$

where φ_1 and φ_2 are the azimuthal angles for $\mathbf{k}_1$ and $\mathbf{k}_2$. According to (2.9) and (2.13), $D^{-1}\simeq2(\gamma-\omega_1)$ so that

$$\sigma_3=\int_{\epsilon'/\gamma}^{1-\epsilon'/\gamma}dw_1\int_1^{z_0}dz_1\int_1^{z_0}dz_2\times\int_0^{2\pi}\frac{d\varphi}{2\pi}\sigma_3(w_1,z_1,z_2,\varphi),\quad(2.19)$$

[9] We now let $m=1$.

TABLE I. A summary of results for multiple- and three-photon production cross sections by electron pair annihilation in flight.

Multiple-photon production	Three-photon production
$\sigma_n=\sigma_2\frac{1}{(n-2)!}\left[\frac{2\alpha}{\pi}\ln\left(\frac{\gamma_+}{\epsilon}\right)(\ln(2\gamma_+)-1)\right]^{n-2}$	$\sigma_3=\sigma_2\frac{2\alpha}{\pi}\left[\left(\ln\left(\frac{\gamma_+}{\epsilon}\right)+\frac{1}{2}\right)(\ln(2\gamma_+)-1)+\frac{\pi^2}{6}\right]$
$\ln\left(\frac{\gamma_+}{\epsilon}\right)\gg 1 \quad \gamma_+\gg 1$	$\left(\frac{\gamma_+}{\epsilon}\right)\gg 1 \quad \gamma_+\gg 1$
(Gupta) $\sigma_n=\sigma_2\frac{1}{(n-2)!}\left[\frac{2\alpha}{\pi}\ln\left(\frac{\gamma_+}{\epsilon}\right)\ln(2\gamma_+)\right]^{n-2}$	(Gupta) $\sigma_3=\frac{3}{2}\sigma_2\frac{2\alpha}{\pi}\ln\left(\frac{\gamma_+}{\epsilon}\right)\ln(2\gamma_+)$
$\ln\left(\frac{\gamma_+}{\epsilon}\right)\gg 1 \quad \ln\gamma_+\gg 1$	$\ln\left(\frac{\gamma_+}{\epsilon}\right)\gg 1 \quad \ln\gamma_+\gg 1$

$\gamma_+ mc^2$ = electron energy
ϵmc^2 = experimental energy resolution

with

$$\sigma_3(w_1,z_1,z_2,\varphi)=\frac{\alpha r_0^2}{4\gamma^2}\frac{w_1}{w_3}\times[F(w_1,z_1,z_2,\varphi)+F(w_3,z_3,z_2,\varphi)],$$

$$w_3=1-w_1,\quad w_1=\frac{\omega_1}{\gamma},\quad z_3=\frac{1}{w_3^2}\{w_1^2z_1+z_2-2w_1[1+(z_2-1)^{\frac{1}{2}}(z_1-1)^{\frac{1}{2}}\cos\varphi]\},\tag{2.20}$$

$$F(w_1,z_1,z_2,\varphi)=\frac{1}{z_1z_2}\left(\frac{1}{w_1^2}-1\right)+\frac{1}{z_1z_3}\frac{1}{w_1}\left(2+\frac{1}{w_1}\right)-\frac{2}{z_1z_2}\frac{1}{w_1}\left(\frac{1}{z_1w_1}-\frac{1}{z_2}\right).$$

In the last step we have divided by an additional factor 2, because the two photons (with energy-momenta k_1 and k_3) emitted within the same cone are indistinguishable. The physical property that these two photons are completely equivalent permits us to see readily the following relation (which can be obtained by straightforward manipulation):

$$\left|\frac{\partial(w_1,z_1,z_2,\varphi)}{\partial(w_3,z_3,z_2,\varphi')}\right|=\frac{w_3^2}{w_1^2}.$$

The variables φ and φ' are related to the azimuthal angles φ_1, φ_2, and φ_3 for the vectors $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$, respectively, according to $\varphi'=|\varphi_2-\varphi_3|$, and, as before, $\varphi=|\varphi_2-\varphi_1|$. It follows, therefore, that with

$$\sigma_{31}=\frac{\alpha r_0^2}{2\gamma^2}\int_{\epsilon'/\gamma}^{1-\epsilon'/\gamma}\frac{dw_1}{w_1}\int_1^{z_0}\frac{dz_1}{z_1}\int_1^{z_0}\frac{dz_2}{z_2}\times\left\{1+w_1-\frac{2}{z_1}\left(1+\frac{w_1}{w_3}\right)+\frac{2w_1}{z_2w_3}\right\},\tag{2.21}$$

and

$$\sigma_{32}=\frac{\alpha r_0^2}{2\gamma^2}\int_{\epsilon'/\gamma}^{1-\epsilon'/\gamma}\frac{dw_1}{w_1}(1+w_1-2w_1^2)\times\int_1^{z_0}\frac{dz_1}{z_1}\int_1^{z_0}dz_2\int_0^{2\pi}\frac{d\varphi}{2\pi}\frac{1}{(a-b\cos\varphi)},\tag{2.22}$$

where

$$a=w_1^2z_1+z_2-2w_1,\quad b=2w_1(1-z_1)^{\frac{1}{2}}(1-z_2)^{\frac{1}{2}},$$

we can write

$$\sigma_3=\sigma_{31}+\sigma_{32}.\tag{2.23}$$

We find

$$\sigma_{31}=\frac{\alpha r_0^2}{2\gamma^2}\left\{\ln^2z_0\left[\ln\left(\frac{\gamma}{\epsilon'}\right)+1\right]-2\ln z_0\ln\left(\frac{\gamma}{\epsilon'}\right)\right\},\tag{2.24}$$

$$\sigma_{32}=\frac{\alpha r_0^2}{2\gamma^2}\ln z_0\left\{\ln z_0\ln\left(\frac{\gamma}{\epsilon'}\right)-(\ln z_0)\left(1-\frac{\pi^2}{3}\right)\right\},\tag{2.25}$$

where we have neglected terms of the relative order ϵ'/γ and $1/z_0$.

According to (2.23), (2.24), and (2.25), we obtain

$$\sigma_3=\frac{\alpha r_0^2}{\gamma^2}\ln z_0\left\{\left[\ln\left(\frac{\gamma}{\epsilon'}\right)+\frac{1}{2}\right][\ln z_0-1]+\frac{\pi^2}{6}\right\}.\tag{2.26}$$

By transforming to laboratory coordinates and by estimating the contributions from all angles, we obtain finally

$$\sigma_3=\sigma_2\alpha\{C\ln(\gamma_+/\epsilon)+\tfrac{1}{2}C+\tfrac{1}{6}\pi^2\},\tag{2.27}$$

where as before $C=(2/\pi)[\ln(2\gamma_+)-1]$. In the limit $\ln(\gamma_+/\epsilon)\gg 1$ this result agrees with the multiple-photon-production cross section (1.31) which gives

$$\sigma_{1+2}=\sigma_2\alpha C\ln(\gamma_+/\epsilon).$$

CONCLUSION

A comparison of our results with those of Gupta is shown in Table I. One sees that in the *extreme* relativistic limit ($\ln\gamma_+ \gg 1$), our σ_n reduces to that of Gupta. It is also apparent that when $\ln(\gamma_+/\epsilon) \gg 1$, our σ_3 and σ_n specialized to $n=3$, agree. Gupta's σ_3, which is obtained in the limit $\ln(\gamma_+/\epsilon) \gg 1$, $\ln\gamma_+ \gg 1$, disagrees by a factor $\frac{3}{2}$. The error lies in Gupta's estimate for σ_3, not in σ_n as he asserted.

Some rather interesting photon showers have recently been observed in cosmic rays.[10,11] These showers consist of about 20 high-energy photons within an extremely narrow cone of less than 0.001 radian, and the very conspicuous absence of charged particles. It was first thought that the small value of the fine structure constant α ruled out multiple-photon production by processes in quantum electrodynamics. Gupta conjectured, however, that perhaps the energy dependence is such that for sufficiently high energies, the energy-dependent factor becomes comparable to α^{-n}, where n is the number of photons produced.

In order to investigate this point further we note that the average or most probable number $\bar{n}$ of photons in excess of 2, emitted in the extreme relativistic limit is

$$\bar{n} \sim (2\alpha/\pi) \ln(\gamma_+/\epsilon) \ln(2\gamma_+).$$

This leads to the following table for $\bar{n}$, with γ_+/ϵ taken to be 100:

$2\gamma_+$	10^6	10^7	10^8	10^9
$\bar{n}$	0.296	0.344	0.394	0.444

[10] Schein, Haskin, and Glasser, Phys. Rev. **95**, 855 (1954).
[11] A. DeBenedetti *et al.*, Nuovo cimento **12**, 954 (1954).

We see that it is very unlikely that the Schein event is an electron pair annihilation. One may argue that this is a freak event. Since there have been several such photographs involving about 15 to 20 photons, one wonders why cases involving from 5 to 10 photons have not been observed. We therefore agree with Gupta's conclusion that the Schein event cannot be accounted for by electromagnetic processes.

The table suggests that even three-photon annihilation is unlikely. Although the multiplicity increases with increasing energy, we must remember that σ_2 (and thus σ_n) decreases rapidly with increasing energy. In fact, using the exact formula for σ_3 at energies of the order 10^8 Mev, we find

$$\sigma_2 \sim 3.5 \times 10^{-8} \text{ mb}.$$

$$\sigma_3 \sim 7.4 \times 10^{-8} \text{ mb}.$$

Let us consider smaller energies. Taking ($\gamma_+/\epsilon \sim 100$) we find

$2\gamma_+$	σ_2 mb	σ_3/σ_2
10	32	0.038
10^3	1.5	0.15
10^5	0.025	0.26

Thus, three-photon production appears to be most likely to be observed at relatively low relativistic energies.

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