Reduction of the Dirac Equation

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A method is given for reducing the Dirac equation for an electron in an external field to small components, leading to reduced equations linear in the time-derivative operator.

THERE exist at present several methods[1-3] for reducing the Dirac equation for an electron in an electromagnetic field ($\mathbf{A}$; V) to "large components." Aside from the Foldy-Wouthuysen transformation,[3] which cannot be exhibited in closed form, all of these lead to equations which are nonlinear in $p_0=i\partial/\partial t$ and hence to nonlinear eigenvalue problems for stationary states.

We should like to point out a reduction method which is in closed form and yields equations linear in p_0. Thus, a linear eigenvalue problem results for a stationary state.

Consider

$$p_0\psi=\mathcal{H}\psi=(\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta m+V_0)\psi, \tag{1}$$

with

$$\boldsymbol{\pi}=\mathbf{p}+e\mathbf{A}, \quad V_0=-eV, \tag{2}$$

and ($\mathbf{A}$; V) independent of t. Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be the projection operators on the positive and negative energy parts, respectively, of the spectrum of $\mathcal{H}$. In operator form,

$$\mathcal{L}_\pm=(1\pm\mathcal{H}\mathcal{E}^{-1})/2, \tag{3}$$

where

$$\mathcal{E}=|\mathcal{H}|=+[\mathcal{H}^2]^{\frac{1}{2}}=m(1+Q)^{\frac{1}{2}} =m(1+\tfrac{1}{2}Q-\tfrac{1}{8}Q^2+\cdots), \tag{4}$$

$$Q=(\mathcal{H}^2-m^2)/m^2.$$

Using the decomposition

$$\psi=\phi+\chi, \quad \phi=\mathcal{L}_+\psi, \quad \chi=\mathcal{L}_-\psi, \tag{5}$$

we get the following equations, equivalent to (1) and (5):

$$p_0\phi=\mathcal{H}\phi, \quad \mathcal{L}_+\phi=\phi, \tag{6a,b}$$

$$p_0\chi=\mathcal{H}\chi, \quad \mathcal{L}_-\chi=\chi. \tag{7a,b}$$

Using (6b), and the decomposition

$$\phi=\phi_+\!+\phi_-, \quad \phi_\pm=[(1\pm\beta)/2]\phi, \tag{8}$$

we find

$$\phi_-=\mathcal{R}\phi_+, \tag{9}$$

where

$$\mathcal{R}=[m+(\mathcal{E}+\mathcal{E}')/2-V_0]^{-1}[\boldsymbol{\alpha}\cdot\boldsymbol{\pi}-(\mathcal{E}-\mathcal{E}')/2], \quad \mathcal{E}'=\beta\mathcal{E}\beta. \tag{10}$$

Multiplication of (6a) by $(1+\beta)/2$, and the use of (9) then gives

$$p_0\phi_+=\mathcal{H}_{\rm red}\phi_+, \tag{11}$$

where

$$\mathcal{H}_{\rm red}=\boldsymbol{\alpha}\cdot\boldsymbol{\pi}\mathcal{R}+V_0+m \tag{12}$$

is an "even" matrix operator since $\mathcal{R}$ is "odd." Equations (7a,b) can be handled in similar fashion. For a bound state of energy $E>0$, we set $\chi=0$ and $p_0=E$ in (11), obtaining the eigenvalue problem

$$E\phi_+=\mathcal{H}_{\rm red}\phi_+, \tag{13}$$

where we may consider ϕ_+ and $\mathcal{H}_{\rm red}$ as a Pauli-type spinor and operator, respectively.

For the hydrogen atom $\mathcal{H}_{\rm red}$ is Hermitian at least to order α^2 Ry, and expansion of (12) in powers of $\mathbf{p}/m$ yields the usual fine-structure operators plus α^4 Ry corrections. In general, $\mathcal{H}_{\rm red}$ cannot be completely Hermitian, since even if $\chi=0$, only $(\phi,\phi)=(\phi_+,\phi_+)+(\phi_-,\phi_-)$ is conserved and not (ϕ_+,ϕ_+) or (ϕ_-,ϕ_-) separately.

The preceding operator formalism, although possibly of no practical advantage for the one-body problem, has been found convenient in the reduction to nonrelativistic form of a Bethe-Salpeter type equation for two electrons in an external field, in a study of the radiative corrections to the helium fine structure.

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[1] See, e.g., L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 321 for the standard method.
[2] B. Kurşunoğlu, Phys. Rev. **101**, 1419 (1956).
[3] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).